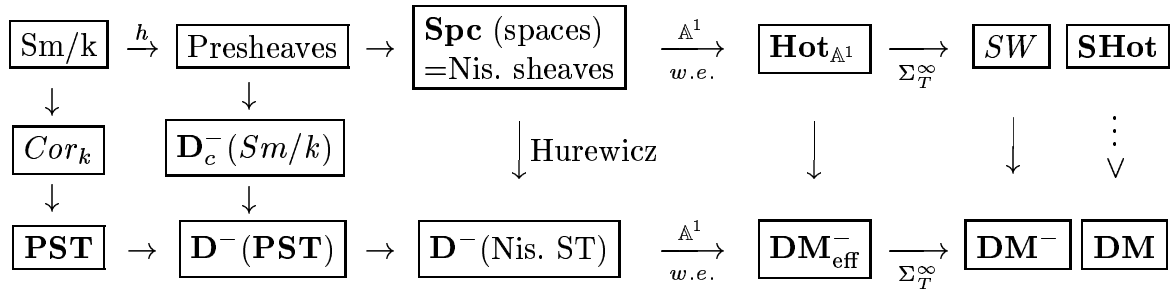
A ROAD MAP OF MOTIVIC HOMOTOPY AND HOMOLOGY THEORY

Charles Weibel

The parallel constructions of Motivic Homotopy and Motivic Homology are based on the construction of stable homotopy and homology in topology. Instead of starting with topological spaces and using the unit interval $[0,1]$ to define homotopy, one starts with smooth schemes over a fixed field k and uses the affine line $\mathbb{A}^1 = \mathrm{Spec}(k[t])$. The constructions are related by certain functors from homotopy to homology which, by analogy, we call Hurewicz functors. Here is the main diagram, or road map.



We begin with a description of the top row of this diagram. These constructions are due to Morel and Voevodsky (see [MV] and [V1]). In the upper left corner we have the category Sm/k of smooth schemes over k. This is the natural home of several important objects, such as the geometric circle $S^1_t = (\mathbb{A}^1 - 0)$ and the geometric n-simplex $\Delta^n = \mathrm{Spec}(k[t_0, \ldots, t_n]/(\sum t_i = 1))$.

Motivic Homotopy Theory

First one embeds Sm/k into the category of presheaves (of sets) on Sm/k, via the Yoneda embedding: to Y we associate the presheaf $h_Y : X \mapsto \mathrm{Hom}_{Sm/k}(X, Y)$. Passing to presheaves enables us to construct arbitrary colimits, including pushouts and quotients. In particular, the simplicial circle $S^1_s = \mathbb{A}^1/\{0,1\}$ and the Tate object $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$ exist as pointed presheaves. The smash product operation and the geometric realization of simplicial presheaves also make sense. (Geometric realization sends the simplicial n-simplex $\Delta[n]$ to Δ^n, and $X \times \Delta[n]$ to $X \times \Delta^n$; see [MV, 2.3.14]).

Moving horizontally, we may sheafify with respect to the Nisnevich topology on Sm/k. (Each h_Y is already a sheaf.) Morel and Voevodsky use the word *space* to mean a sheaf with respect to the Nisnevich topology on Sm/k, so the category **Spc** of "spaces" is just the category of Nisnevich sheaves on Sm/k. Sheafification turns a

presheaf into a space, of course. In the category **Spc**, a smooth scheme X is the colimit of any Zariski open cover, and $X/U \cong V/(V \cap U)$ if $X = U \cup V$. Similarly, X is the colimit of any Nisnevich cover.

A map between two spaces is called an $\mathbb{A}^1$*-weak equivalence* if it belongs to the smallest saturated class of maps which contains all maps $X \times \mathbb{A}^1 \to X$, and which is also closed under isomorphisms, colimits and pushouts along monomorphisms. The homotopy category **Hot** of schemes over k is the category obtained from **Spc** by inverting the $\mathbb{A}^1$-weak equivalences. Morel and Voevodsky showed in [MV] that the $\mathbb{A}^1$-weak equivalences are the weak equivalences of a proper closed model structure on **Spc** whose cofibrations are monomorphisms.

An alternative and historically more familiar approach is to move quickly to the category of simplicial spaces and impose a proper closed model structure, replacing $\mathbb{A}^1$ by the simplicial space $X \mapsto h_{\mathbb{A}^1}(X \times \Delta^\bullet)$, where $\Delta^\bullet$ is the cosimplicial scheme $n \mapsto \Delta^n$. The corresponding homotopy category of simplicial spaces is equivalent to **Hot** by the geometric realization functor described above. (See [MV, 2.3.14]).

The pointed variation of this construction follows the topological paradigm. That is, we may embed the categories of presheaves, spaces and **Hot** into their corresponding pointed categories by sending X to the union X_+ of X with a disjoint basepoint. It is in the pointed category $\mathbf{Hot}_\bullet$ that we have $T \simeq (\mathbb{P}^1, \infty) \simeq S^1_s \wedge S^1_t$; see [MV, 3.2.15].

The stable homotopy categories SW and **SHot** of T-spectra are obtained from the pointed homotopy category by the process of stabilization with respect to the T-suspension $\Sigma_T X = T \wedge X$. In fact, there are two distinct constructions in play: desuspension and stabilization. They are described in [V1] and [VW].

The *Spanier-Whitehead stable homotopy category* SW is the category obtained from $\mathbf{Hot}_\bullet$ by inverting the T-suspension $X \mapsto T \wedge X$, or equivalently, by inverting both the simplicial suspension $X \mapsto \Sigma_s X = S^1_s \wedge X$ and the geometric t-suspension $X \mapsto \Sigma_t X = S^1_t \wedge X$. Every object of SW is a finite desuspension of some pointed space X, meaning that it is isomorphic to $\Sigma_T^{-n} X$ for some $n \geq 0$. In this category, $\mathrm{Hom}(\Sigma_T^{-m} X, \Sigma_T^{-n} Y)$ is the direct limit of the $\mathrm{Hom}(\Sigma_T^{i-m} X, \Sigma_T^{i-n} Y)$ as $i \to \infty$. As in topology [SW], and pointed out in [V1], this construction has the defect that SW is not closed under coproducts: we need to stabilize by passing to T-spectra.

A *T-spectrum* $\mathbf{E}$ is a sequence of pointed spaces E_n, together with bonding maps $T \wedge E_n \to E_{n+1}$. There is a category of T-spectra; a *morphism* $\mathbf{E} \to \mathbf{F}$ of T-spectra is just a sequence of maps $E_n \to F_n$ which commute with the bonding maps. The stable homotopy category **SHot** of T-spectra is obtained from the category of T-spectra by localizing with respect to stable weak equivalences; see [V1, 5.1]. There is a functor Σ^∞_T from $\mathbf{Hot}_\bullet$ to **SHot** sending a pointed space X to the sequence of spaces $X \wedge T^{\wedge n}$. Clearly, this functor factors through the Spanier-Whitehead category SW. (It is faithful when restricted to spaces of finite type by [V1, 5.3].)

Down the left side of the main diagram we have the category Cor_k of finite correspondences and the category **PST** of presheaves with transfer. These constructions were first defined in [V] and detailed in [MVW].

The additive category Cor_k has the same objects as Sm/k: smooth schemes over k. The group $Cor_k(X, Y)$ of morphisms from X to Y is the free abelian group on

the *elementary correspondences* – subvarieties W of $X \times Y$ whose projection $W \to X$ is finite. Morphisms in Cor_k are called *finite correspondences*. Composition of finite correspondences is given by a classical "pull back, intersect and push forward" construction originally due to Hurwitz, Lefschetz and Severi [Dieu]; the modern details are given in [V] or [MVW].

There is a canonical embedding of the category Sm/k into Cor_k sending a morphism $f : X \to Y$ between smooth schemes to its graph Γ_f, considered as an elementary correspondence from X to Y.

A *presheaf with transfers* F is a contravariant additive functor from Cor_k to abelian groups. That is, it is a presheaf on Sm/k equipped with transfer maps $F(Y) \to F(X)$, one for every elementary correspondence from X to Y, subject to the composition rules in Cor_k. The category Cor_k embeds into the category $\mathbf{PST}$ of presheaves with transfer via the Yoneda embedding: to a smooth scheme Y we associate the presheaf with transfer $\mathbb{Z}_{tr}Y : X \mapsto Cor_k(X, Y)$. Cor_k is not idempotent complete, because if (X, x) is a smooth pointed scheme, then $\mathbb{Z}_{tr}(X, x) = \mathbb{Z}_{tr}X/\mathbb{Z}$ is a direct summand of $\mathbb{Z}_{tr}X$. Important example of presheaves with transfer include $\mathbb{Z} = \mathbb{Z}_{tr}(\operatorname{Spec} k)$ and $\mathbb{Z}_{tr}\mathbb{G}_m = \mathbb{Z}_{tr}(\mathbb{A}^1 - 0, 1)$.

Motivic Homology Theory

Across the bottom of the main diagram, we find the construction of Voevodsky's triangulated category of motives, $\mathbf{DM}^-$, as described in [V] and [MVW]. Since $\mathbf{PST}$ is an abelian category, we can consider the derived category $\mathbf{D}^-(\mathbf{PST})$ of bounded above cochain complexes of presheaves with transfer. [One may equally think of this as the category $\mathbf{D}_+$ of bounded below chain complexes.] Since $\mathbb{Z}_{tr}(X \times Y) = \mathbb{Z}_{tr}X \otimes \mathbb{Z}_{tr}Y$ in $\mathbf{PST}$, and projective resolutions are bounded above cochain complexes, $\mathbf{D}^-(\mathbf{PST})$ has a well-behaved total tensor product and a tensor triangulated structure; see [MVW, 8A]. Another reason for restricting to bounded above complexes is that the chain complex associated to a simplicial object will be bounded above when indexed as a cochain complex. For example, a basic object is the cochain complex $\mathbb{Z}(1)$; its shift $\mathbb{Z}(1)[1]$ is the complex associated to the simplicial object $X \mapsto \mathbb{Z}_{tr}\mathbb{G}_m(X \times \Delta^\bullet)$ of $\mathbf{PST}$.

If we sheafify with respect to the Nisnevich topology, we get Nisnevich sheaves with transfer, and the derived category $\mathbf{D}^- = \mathbf{D}^-(\text{Nis. ST})$ of bounded above cochain complexes of Nisnevich sheaves with transfer. The tensor product in $\mathbf{D}^-(\mathbf{PST})$ induces a tensor product $\otimes^{\mathbb{L}}_{tr}$ on $\mathbf{D}^-$, making it into a tensor triangulated category (see [MVW, 14.2]).

A morphism in $\mathbf{D}^-$ is an $\mathbb{A}^1$*-weak equivalence* if it belongs to the smallest multiplicative system containing quasi-isomorphisms and the projections $\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_{tr}X$, which is also closed under shifts, direct sums and cones. Inverting the $\mathbb{A}^1$-weak equivalences yields a tensor triangulated category $\mathbf{DM}^-_{\text{eff}}$; see [V] [MVW, 14.1]. It may be identified with the full subcategory of $\mathbf{D}^-$ consisting of complexes with homotopy invariant cohomology sheaves; see [V, 3.2.3] [MVW, 14.10]. Since $\mathbb{Z}(1) \cong \mathbb{Z}_{tr}\mathbb{G}_m[-1]$ in $\mathbf{DM}^-_{\text{eff}}$, the Tate twist $M(1) = M \otimes^{\mathbf{L}}_{tr} \mathbb{Z}(1)$ of M has good properties.

The triangulated category $\mathbf{DM}^-$ is obtained from $\mathbf{DM}^-_{\text{eff}}$ by inverting the Tate twist $M \mapsto M(1)$. Thus every object in $\mathbf{DM}^-$ is isomorphic to $M(-n)$ for some $n \geq 0$ and some M in $\mathbf{DM}^-_{\text{eff}}$. By [V2], $\mathbf{DM}^-_{\text{eff}} \to \mathbf{DM}^-$ is fully faithful.

The *motive* $M(X)$ of a smooth scheme X is defined to be the class in $\mathbf{DM}^-_{\text{eff}}$ (or equivalently, in $\mathbf{DM}^-$) associated to the chain complex which is $\mathbb{Z}_{tr}(X)$, concentrated in degree zero. The category $\mathbf{DM}_{gm}$ of *geometric motives* is the smallest triangulated subcategory containing the $M(X)$ which is closed under summands as well as the Tate twist. (See p. 192 and 3.2.6 of [V]).

Stable Motivic Complexes

The category $\mathbf{DM}^-$ is sufficient for most homological considerations. However, it is not closed under infinite direct sums. In order to correct this, we need a different construction, paralleling the definition of T-spectra in [V1]. (Cf. [Mor]). A version paralleling the definition of symmetric spectra has been given by Spitzweck (see [Sp, 14.7] [Hov, 7.11]).

A *stable motivic complex* $E_\bullet$ is a sequence $E_0, E_1, \dots$ of bounded above complexes of Nisnevich sheaves with transfer, together with structure maps $E_n(1)[2] \to E_{n+1}$. The stable motivic complex $\Sigma^\infty_T X$ associated to a bounded above complex of Nisnevich sheaves X is the sequence $X_n = X(n)[2n]$ with structure maps $X(n)(1)[2] \cong X(n+1)$.

A *function* from $E_\bullet$ to $F_\bullet$ is a sequence of maps $E_n \to F_n$ commuting with the structure maps. Each stable motivic complex defines a cohomological functor $\{E^i\}$ from $\mathbf{DM}_{gm}$ to abelian groups by:

$$E^i(M) = \operatorname*{colim}_{n\to\infty} \operatorname{Hom}_{\mathbf{DM}^-}(M(n)[2n], E_{n+i}).$$

A function $E_\bullet \to F_\bullet$ is called a *stable weak equivalence* if the corresponding natural transformations of functors $E^i \to F^i$ are isomorphisms for all i. A useful example of a stable weak equivalance is any function for which the $E_n \to F_n$ are $\mathbb{A}^1$-weak equivalences for $n \gg 0$. In particular, Σ^∞_T sends $\mathbb{A}^1$-weak equivalences to stable weak equivalences.

Definition. The category $\mathbf{DM}$ is the localization of the category of stable motivic complexes and functions, with respect to the class of stable weak equivalences.

By construction, there is a canonical functor $\Sigma^\infty_T : \mathbf{DM}^-_{\text{eff}} \to \mathbf{DM}$. In fact, $\mathbf{DM}$ is a triangulated category and Σ^∞_T is a triangulated functor. Indeed, the mapping cone of a function $f : E_\bullet \to F_\bullet$ is also a stable motivic complex, and $E_\bullet \to F_\bullet \to \operatorname{cone}(f) \to E_\bullet[1]$ is the prototype of a distinguished triangle.

The T-suspension $\Sigma_T E_\bullet$ and T-desuspension $\Sigma^{-1}_T E_\bullet$ are obtained by shifting the sequence one to the left and right, respectively. They are inverse operations in $\mathbf{DM}$; $E_\bullet = \Sigma^{-1}_T \Sigma_T E_\bullet$ and $\Sigma_T \Sigma^{-1}_T E_\bullet \to E_\bullet$ is a stable weak equivalence. The Tate twist is invertible in $\mathbf{DM}$, because Σ_T is, so Σ^∞_T induces a triangulated functor $\mathbf{DM}^- \to \mathbf{DM}$.

We could have also obtained $\mathbf{DM}$ by restricting the definition of stable motivic complex to negatively graded complexes. To see this, let τ denote the (good) truncation to cohomological degrees ≤ 0. Then the truncation $\tau E_\bullet = \{\tau E_n\}$ is also a stable motivic complex, and the canonical map $\tau E_\bullet \to E_\bullet$ is a stable weak equivalence; this fact uses the fact that for each geometric motive M the $M(n)[2n]$ are eventually negatively graded. The truncation allows us to see that the direct sum of a family $E^\alpha_\bullet$ exists and equals the sequence of the $\oplus_\alpha \tau(E^\alpha_n)$.

Étale analogue. It is clear that we can repeat the above construction with the étale topology, and with $\mathbb{Z}/m$ coefficients, assuming $1/m \in k$. Remark 9.6 and theorem 9.32 in [MVW] shows that $\mathbf{DM}_{et}(k;\mathbb{Z}/m)$ is just the full derived category $\mathbf{D}(G,\mathbb{Z}/m)$ of profinite $\mathbb{Z}/m[G]$-modules, where G is the absolute Galois group of k.

The Hurewicz functor

Having described the outer part of the main diagram, we now need to fill in the middle. The category $\mathbf{D}_c^-(Sm/k) = \mathbf{D}^-(\mathbf{Ab}^{Sm^{\mathrm{op}}})$ is the derived category of bounded above cochain complexes of presheaves of abelian groups on Sm/k, *i.e.*, complexes of objects in the abelian category $\mathbf{Ab}^{Sm^{\mathrm{op}}}$ of presheaves of abelian groups on Sm/k.

Consider the free functor $X \mapsto \mathbb{Z}X$ from presheaves of sets to the category $\mathbf{Ab}^{Sm^{\mathrm{op}}}$ of presheaves of abelian groups; by definition, $\mathbb{Z}X(U)$ is the free abelian group on the set $X(U)$. Composing with the canonical embedding of $\mathbf{Ab}^{Sm^{\mathrm{op}}}$ into its derived category $\mathbf{D}^-(\mathbf{Ab}^{Sm^{\mathrm{op}}})$ gives the first vertical map in the second column of the main diagram.

Note that the terminal presheaf $*$ is sent to the presheaf $U \mapsto \mathbb{Z}$. If $(X,*)$ is a pointed presheaf of sets, $\mathbb{Z}(X,*)(U)$ is defined to be the cokernel of the canonical basepoint map $\mathbb{Z} \xrightarrow{*} \mathbb{Z}X(U)$; these definitions are consistent since $\mathbb{Z}(X_+,*) = \mathbb{Z}X$.

To map downward to $\mathbf{D}^-(\mathbf{PST})$, we use the fact the abelian category $\mathbf{Ab}^{Sm^{\mathrm{op}}}$ has enough projectives: every representable presheaf $\mathbb{Z}h_Y$ is projective and every projective is a summand of a direct sum of representable presheaves. Since we may replace any bounded above complex by a projective resolution, the derived category $\mathbf{D}^-(\mathbf{Ab}^{Sm^{\mathrm{op}}})$ is equivalent to the triangulated category $\mathbf{K}$ of bounded above complexes of projective presheaves (and chain homotopy equivalence classes of maps). By the Yoneda lemma, a map $\mathbb{Z}h_X \to \mathbb{Z}h_Y$ is the same as an element of $\mathbb{Z}h_Y(X)$, *i.e.*, a $\mathbb{Z}$-linear combination of elements of $\mathrm{Hom}_{Sm/k}(X,Y)$. It follows that the canonical embedding $Sm/k \to Cor_k$ induces a functor

$$\mathbf{D}^-(\mathbf{Ab}^{Sm^{\mathrm{op}}}) \cong \mathbf{K} \to \mathbf{D}^-(\mathbf{PST}),$$

sending $\mathbb{Z}h_Y$ to $\mathbb{Z}_{tr}Y$, considered as a complex in degree zero.

Because the vertical functor from presheaves of sets to $\mathbf{D}^-(Cor_k)$ sends the presheaf h_Y to $\mathbb{Z}_{tr}Y$, the left rectangle in the main diagram commutes. The composite map sends the Tate object T to the complex $\mathbb{Z}_{tr}\mathbb{G}_m \to \mathbb{Z}_{tr}(\mathbb{A}^1,1)$, and sends the simplicial circle S^1_s to the complex $\mathbb{Z} \to \mathbb{Z}_{tr}(\mathbb{A}^1,1)$; these complexes are concentrated in cohomological degrees -1 and 0, and are $\mathbb{A}^1$-weak equivalent to $\mathbb{Z}_{tr}\mathbb{G}_m[1]$ and $\mathbb{Z}[1]$, respectively.

Sheafifying with respect to the Nisnevich topology immediately yields a functor H from spaces to $\mathbf{D}^-(\text{Nis. ST})$. We shall call it the *Hurewicz functor*, because

$$X(S^i) = \mathrm{Hom}(h_{S^i}, X) \xrightarrow{H} \mathrm{Hom}_{\mathbf{D}^-}(\mathbb{Z}_{tr}(S^i), HX) \to \mathrm{Hom}_{\mathbf{DM}^-_{\mathrm{eff}}}(\mathbb{Z}[i], HX)$$

is the analogue of the Hurewicz map $\pi_i(X) \to H_i(X)$ in ordinary homotopy theory; it is obtained by replacing a space by a nicer space, abelianizing and passing to the associated complex (which is well defined up to quasi-isomorphism). The birth certificate of the Hurewicz functor ensures that the second rectangle commutes in the main diagram.

By construction, the Hurewicz functor sends $X \times \mathbb{A}^1 \to X$ to $\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_{tr}X$, and hence sends $\mathbb{A}^1$-weak equivalences of spaces to $\mathbb{A}^1$-weak equivalences of complexes.

Therefore it preserves localization, *i.e.*, it induces a Hurewicz functor from the homotopy category **Hot** to $\mathbf{DM}^-_{\rm eff}$. As noted above, we have $H(S^1_s) \cong \mathbb{Z}[1]$ and $H(T) \cong \mathbb{Z}(1)[2]$.

The extension of $H : \mathbf{Hot} \to \mathbf{DM}^-_{\rm eff}$ to a functor $SW \to \mathbf{DM}^-$ is a formal consequence of the definitions and the following lemma, which shows that H sends $T \wedge X$ to $\mathbb{Z}_{tr}(T) \otimes^{\mathbb{L}}_{tr} H(X) \cong H(X)(1)[2]$.

Lemma 1. *The functor $H : \mathbf{Hot} \to \mathbf{DM}^-_{\rm eff}$ sends the smash product $X \wedge Y$ of spaces to the total tensor product $H(X) \otimes^{\mathbb{L}}_{tr} H(Y)$ in $\mathbf{DM}^-_{\rm eff}$.*

Proof. If $\tilde{X}$ and $\tilde{Y}$ are unpointed presheaves, then $\mathbb{Z}(\tilde{X} \times \tilde{Y}) = \mathbb{Z}\tilde{X} \otimes \mathbb{Z}\tilde{Y}$ in $\mathbf{Ab}^{Sm^{\rm op}}$ (where $\otimes$ is the presheaf tensor product). From this, a simple calculation shows that if X and Y are pointed presheaves then $\mathbb{Z}(X \wedge Y) = \mathbb{Z}X \otimes \mathbb{Z}Y$. The same is true in $\mathbf{D}^-(\mathbf{Ab}^{Sm^{\rm op}})$: if $P \to \mathbb{Z}X$ and $Q \to \mathbb{Z}Y$ are projective resolutions, then $P \otimes Q \to \mathbb{Z}(X \wedge Y)$ is also a projective resolution (since projectives are flat in $\mathbf{Ab}^{Sm^{\rm op}}$).

Applying the additive functor $\mathbf{K} \to \mathbf{D}^-(\mathbf{PST})$ (induced from $Sm/k \to Cor_k$) sends P, Q and $P \otimes Q$ to $H(X)$, $H(Y)$ and $H(X \wedge Y)$, respectively. Since it sends $\mathbb{Z}(h_{U \times V}) = \mathbb{Z}(h_U) \otimes \mathbb{Z}(h_V)$ to $\mathbb{Z}_{tr}(U \times V) = \mathbb{Z}_{tr}(U) \otimes_{tr} \mathbb{Z}_{tr}(V)$, it also sends $P \otimes Q$ to $H(X) \otimes^{\mathbb{L}}_{tr} H(Y)$, whence the result.

The fact that H extends to a functor from **SHot** to **DM** follows easily from the following lemma. This makes the final rectangle in the main diagram commute, completing our description of the road map.

Lemma 2. *For each T-spectrum $\mathbf{E}$, the stable motivic complex $H(\mathbf{E}) = \{H(E_n)\}$ is well defined up to stable weak equivalence. For each morphism $f : \mathbf{E} \to \mathbf{F}$ there is a well defined morphism $H(\mathbf{E}) \to H(\mathbf{F})$ in $\mathbf{DM}$.*

To see this, choose a projective resolutions $P_n \to \mathbb{Z}E_n$ and maps $\mathbb{Z}T \otimes P_n \to P_{n+1}$; these choices are unique up to chain homotopy equivalence. Applying $Sm/k \to Cor_k$ yields the structure maps $H(E_n)(1)[2] \to H(E_{n+1})$ turning $H(E_0), H(E_1), \dots$ into a stable motivic complex $H(\mathbf{E})$; a different family of choices would yield a stably weak equivalent complex.

Similarly, a choice of $Q_n \to \mathbb{Z}F_n$, etc. yields a stable motivic complex $H(\mathbf{F})$. A morphism f yields a family of maps $f_n : P_n \to Q_n$ and $H(f_n) : H(E_n) \to H(F_n)$, well defined up to chain homotopy equivalence. This may not be a function because the two maps from $H(E_n)(1)[2]$ to $H(F_{n+1})$ are only chain homotopic. To repair this, we replace Q_n by the mapping cylinder $\mathrm{cyl}(f_n)$ and use the structure maps from $\mathbb{Z}T \otimes \mathrm{cyl}(f_n) \cong \mathrm{cyl}(\mathbb{Z}T \otimes f_n)$ to $\mathrm{cyl}(f_{n+1})$. Then the desired morphism in **DM** is the composite of the function $H(\mathbf{E}) \to \mathrm{cyl}(f)$ and the inverse of the stable weak equivalence $H(\mathbf{F}) \to \mathrm{cyl}(f)$.

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Rutgers University

E-mail address: `weibel@math.rutgers.edu`