TALBOT: SCISSORS CONGRUENCE AND ALGEBRAIC K-THEORY

INNA ZAKHAREVICH AND JONATHAN A. CAMPBELL

1. Introduction

Euclid's definition of the area of a polygon is that area is the property which is preserved by "decomposition" and "rearrangement." To make this definition precise, we can say that two polygons P and Q are *scissors congruent* if they can be decomposed into subpolygons $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ with $P_i \cong Q_i$ for all i. We can then define "area" to be the invariant of scissors congruence classes, and this in fact agrees with other definitions of area: if two polygons have the same area then they are scissors congruent, and if they are scissors congruent then they have the same area. However, if we wish to define volume in a similar manner, a problem arises. Two polyhedra which are scissors congruent always have the same volume, but the proof of the converse eluded discovery for many years. Gauss showed that if we allow infinitely many "cuts" in our decompositions, then (by calculus) the definition works. However, when restricting to finitely many cuts the proof did not work, and it was conjectured that in fact this was not true. Hilbert's third problem asked to produce two polyhedra of equal volume which are not scissors congruent. In 1901 Dehn showed that a second invariant, now called the Dehn invariant, was preserved under such decompositions, and that this invariant is zero for the cube but nonzero for the regular tetrahedron, thus providing the example Hilbert requested. In 1965 Sydler showed that no other invariants are necessary: if two polyhedra have the same volume and the same Dehn invariant, then they are scissors congruent. Analogs of this problem, in other geometries and in other dimensions, are still open.

By defining the sum of polytopes to be their disjoint union we can define a group structure on scissors congruence classes. Hilbert's third problem can then be reinterpreted as the question of analyzing these *scissors congruence groups*. Dupont [Dup82] showed that Euclidean, sphereical and hyperbolic scissors congruence groups can be described in terms of twisted homology groups of the isometry groups (considered as discrete groups). Thus, for example, the Euclidean scissors congruence group for n-dimensional polytopes can be defined to be $H_0(E(n), \mathcal{P}(E^n, 1))$. Here, $E(n)$ is the n-dimensional Euclidean isometry group, and $\mathcal{P}(E^n, 1)$ is the *Steinberg module* of Euclidean polytopes. This closely relates scissors congruence to algebraic K-theory since algebraic K-theory is also the homology of a Lie group made discrete: $\mathrm{prim}(H_*(GL(k); \mathbf{Q})) \cong \pi_* K(k)_{\mathbf{Q}}$. Although the presence of rationalization may be surprising, it should not be: it is known that Euclidean scissors congruence groups form a rational vector space, and it is conjectured that the corresponding spherical and hyperbolic groups do as well. In a more direct and even stranger connection, Goncharov [Gon99] conjectures that a complex built out of higher Dehn invariants actually computes the weight spaces of rationalized higher algebraic K-theory. Thus computations of scissors congruence groups should be closely related to computations of algebraic K-theory. The first part of this course will focus on learning about these computations and connections.

Considering scissors congruence more abstractly, there is no fundamental reason why the notion of scissors congruence must be restricted to polytopes. In fact, scissors congruence can be considered to be a step in the commonly-used "divide and conquer" approach to mathematical problem-solving: break a problem down into smaller problems, solve each of those, and reassemble a solution. Even when the solution is not unique (such as in the case of using spectral sequences for computations) it can still give valuable information. The question of what it means to "break down" and "reassemble" objects can be abstracted to asking for some kind of combinatorial data about which objects make up other objects. Such data abounds: exact sequences, open covers, abstract blowup squares, etc. The first of these has an associated algebraic K-theory, and the main thesis of scissors congruence K-theory is that all such collections of information should have K-theories. The second and third part of the course will focus on these questions. The second part will discuss how

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to construct scissors congruence invariants using K-theory and various implications of these constructions. The third part dives deeper into the technical underpinnings of K-theory to illustrate how the algebraic approaches of classical K-theory can be expanded to include more combinatorial descriptions.

The last part of the course will use the intuition built in this deep dive to circle back to the original questions of calculations of group homology. Using topological techniques and the understanding of the structure of K-theory, as well as a derived Dehn invariant, we show that Goncharov's mysterious complex is the bottom row of the E^1 page of a spectral sequence which computes group homology with twisted coefficients. With this spectral sequence we can show that volume (which is a homomorphism out of scissors congruence groups) is related to the Borel regulator (a "volume-like" homomorphism out of algebraic K-theory groups).

In the final talk we will describe some open problems and new directions for research.

2. Introductory Talk

Talk 0: Inna+Jonathan: Introduction. A rapid overview of the course: an introduction to classical scissors congruence problems, a perspective on algebraic K-theory as a combinatorial construction describing how certain objects decompose, and the surprising computational connections between them. Includes a discussion of the Goncharov complex and Goncharov's conjectures as motivation for the computational links.

3. Classical Results

Talk 1: Classical scissors congruence, Dehn invariants, Dehn–Sydler. Introduce the problem of scissors congruence, discuss the Dehn invariant and the kernel and cokernel of the Dehn invariant.

Reference: [Jes68]
Secondary: [Dup01, Syd65]

Talk 2: Scissors congruence as homology of Lie groups made discrete. Discuss the relationship between scissors congruence and the homology of Lie groups with discrete topology. Introduce the Tits building and compute its homology.

Reference: [Dup01, Ch. 2, 3]
Secondary: [Sah81, DS82, DPS88]

Talk 3: Rational structures. Classification of translational scissors congruence, rational structure on translational/Euclidean scissors congruence. Remark on open problems about spherical/hyperbolic.

Relate the Euclidean scissors congruence group to homology of $O(n)$ with coefficients in the translational scissors congruence group. Analyze the Adams operations on translational scissors congruence.

Reference: [Dup01, Ch. 4]

Talk 4: hyperbolic low-dimensional theorems. Low dimensional hyperbolic scissors congruence has a striking relationship with $K_2(\mathbf{C})$. Discuss the Bloch-Wigner sequence and hyperbolic scissors congruence.

Reference: [DS82]
Secondary: [DPS88]

Talk 5: McMullen, Goodwillie. A discussion of the polytope algebra of McMullen, as well as Goodwillie's generalizations to more general isometry groups. The weight filtration and the rational structure.

Reference: [McM89]
Secondary: [Goo17]

4. Scissors congruence as K-theory

Talk 6: Assemblers. Definitions of K-theory of "geometric" objects using assemblers. The cofiber theorem. Application: filtration on total scissors congruence is classical scissors congruence.

Reference: [Zak17a]

Talk 7: Grothendieck ring of varieties. An introduction to the Grothendieck ring of varieties and its properties. Examples of motivic measures. Questions about scissors congruence of varieties, the annihilator of the Lefschetz motive, motivic zeta function. Borisov's result.

Reference: [NS11, Bor18]

Talk 8: Annihilator of the Lefschetz motive. Borisov constructs an element of the annihilator of the Lefschetz motive, and coincidentally finds a negative answer to Larsen–Lunts' question about scissors congruence of varieties. Using higher K-groups of varieties it is possible to show that this will always happen.

Reference: [Zak17b]

Talk 9: SW-categories. A new way of thinking about decomposition of objects using Waldhausen categories: SW-categories. Application: Grothendieck ring of varieties, E_∞ structure on the spectrum.

Reference: [Cam]

Talk 10: Derived motivic measures. Using SW-categories, algebraic motivic measures can now be constructed. Examples: local ζ-functions, Voevodsky realization.

Reference: [CWZ]
Secondary: [BGN21]

5. K-theory as scissors congruence

Talk 11: $Q = +$, $S_\bullet$-construction, $Q = S_\bullet$. A technical dive into the intricacies of bisimplicial sets and different constructions of K-theory.

Reference: [Wei13, IV.7], [Wal85, Section 1.3, Section 1.9]

Talk 12: CGW-categories, ACGW-categories. Quillen's definition of K-theory using the Q-construction does not actually depend on the algebraic nature of abelian categories, but rather on the combinatorial nature of the interactions between kernels and cokernels. Thus his main theorems hold in far more general contexts, such as the category of reduced separated schemes of finite type. Using this insight it is possible to show that the two constructions of the K-theory of varieties given above are equivalent.

Reference: [CZ]

Talk 13: FCGW-categories. A chain complex in an abelian category is a sequence of objects linked by a composition of an epimorphism with a monomorphism, satisfying the condition that the image of the previous monomorphism is contained in the kernel of the next epimorphism. Translating this into the language of CGW-categories gives a natural notion of a category in which it is possible to define chain complexes which is more general than the usual requirements of abelian (or exact) categories. Moreover, in such categories many of the classical theorems about abelian categories hold as well, including the theorem of Gillet–Waldhausen showing that the K-theory of an abelian category is equivalent to that of bounded chain complexes in that category. This leads to a good definition of a category of chain complexes of finite sets.

Reference: [SS21]

Talk 14: Squares K-theory and manifold invariants. With even more generality, it is possible to define algebraic K-theory using only a notion of "square" relations which interact well. With this it is possible to construct higher invariants of other geometric problems, such as SK-invariants of manifolds.

Reference: [HMM$^+$21]

6. Neoclassical scissors congruence

Talk 15: Cathelineau and Milnor K-theory. Cathelineau resolved the tail end of Goncharov's conjectures showing that the homology of the complex is Milnor K-theory. Discuss the Goncharov complex scissors complex and Cathelineau's proof.

Reference: [Cat03, Cat04]
Secondary: [Gon99]

Talk 16: Rognes' Rank Filtration in Algebraic K-theory. Introduce Rognes' filtration and stable buildings. An interesting application is quick proof of the classical Barrat-Priddy-Quillen-Theorem

REFERENCE: [Rog92]
SECONDARY: [Rog00]

Talk 17: Campbell–Zakharevich. To analyze Goncharov's complex involving scissors congruence groups and relate it to K-theory it is necessary to transform it into a statement involving homology of general linear groups. Following Cathelineau's lead using topology instead of algebra, it is possible to show that Goncharov's complex is contained in the bottom row of a spectral sequence involving homology of orthogonal groups with twisted coefficients.

REFERENCE: [CZ21]

7. Conclusion

Talk -1: Inna + Jonathan: Open Problems. Weight filtration. Weight vs rank conjecture. Beilinson-Soule vanishing conjecture. Goncharov's conjectures. Hilbert's generalized third problem in spherical and Euclidean case. Rational vector space structure of $\mathcal{P}(S^n)$ and $\mathcal{P}(\mathcal{H}^n)$.

8. Further reading

Though no talks will be explicitly given on the following papers, they also form parts of the corpus of papers around scissors congruence, K-theory and the rank filtration.

A beautiful introduction to the scissors congruence problem is in Sah's book [Sah79].

Much of the introductory chapters of Dupont's book [Dup01] is drawn from [Dup82].

The first use of the rank filtration is in Quillen's proof of the finite generation of $K_i(\mathcal{O}_K)$ [Qui73]

The relationship between motivic cohomology and scissors congruence first appeared in [BGSV90] and subsequently developed in [Gon99]. See also [Gon93, Gon95] for relationships between scissors congruence-like complexes, K-theory and ζ-functions.

For the first instance of the relationship between the rank filtration and algebraic K-theory see Suslin's paper [Sus84]. For an elementary but highly non-trivial computation of low dimensional K-theory related to weighth spaces see [MS90].

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