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LECTURE NOTES

Supersymmetry for Mathematicians: An Introduction

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Preface

These notes are essentially the contents of a minicourse I gave at the Courant Institute in the fall of 2002. I have expanded the lectures by discussing spinors at greater length and by including treatments of integration theory and the local Frobenius theorem, but otherwise have not altered the plan of the course. My aim was and is to give an introduction to some of the mathematical aspects of supersymmetry with occasional physical motivation. I do not discuss supergravity.

Not much is original in these notes. I have drawn freely and heavily from the beautiful exposition of P. Deligne and J. Morgan, which is part of the AMS volumes on quantum field theory and strings for mathematicians, and from the books and articles of D. S. Freed and D. A. Leites, all of which and more are referred to in the introduction.

I have profited greatly from the lectures that Professor S. Ferrara gave at UCLA as well as from many extended conversations with him, both at UCLA and at CERN, where I spent a month in 2001. He introduced me to this part of mathematical physics and was a guide and participant on a seminar on supersymmetry that I ran in UCLA in 2000 with Rita Fioresi. I am deeply grateful to him for his unfailing patience and courtesy. I also gave a course in UCLA and a miniworkshop on supersymmetry in 2000 in Genoa, Italy, in the Istituto Nazionale di Fisica Nucleare. I am very grateful to Professors E. Beltrametti and G. Cassinelli, who arranged that visit; to Paolo Aniello, who made notes of my UCLA course; to Ernesto De Vito and Alberto Levrero, whose enthusiasm and energy made the Genoa workshop so memorable; and finally to Lauren Caston, who participated in the Courant course with great energy and enthusiasm. I also wish to thank Alessandro Toigo and Claudio Carmeli of INFN, Genoa, who worked through the entire manuscript and furnished me with a list of errors and misprints in the original version of the notes, and whose infectious enthusiasm lifted my spirits in the last stages of this work. I am very grateful to Julie Honig for her help during all stages of this work. Last, but not least, I wish to record my special thanks to Paul Monsour and Reeva Goldsmith whose tremendous effort in preparing and editing the manuscript has made this book enormously better than what it was when I sent it to them.

The course in the Courant Institute was given at the suggestion of Professor S. R. S. Varadhan. My visit came at a time of mourning and tragedy for him in the aftermath of the 9/11 catastrophe, and I do not know how he found the time and energy to take care of all of us. It was a very special time for us and for him, and in my mind this course and these notes will always appear as a small effort on my part to alleviate the pain and grief by thinking about some beautiful things that are far bigger than ourselves.

V. S. Varadarajan
Pacific Palisades
March 2004

CHAPTER 1

Introduction

1.1. Introductory Remarks on Supersymmetry

The subject of *supersymmetry* (SUSY) is a part of the theory of elementary particles and their interactions and the still unfinished quest of obtaining a unified view of all the elementary forces in a manner compatible with quantum theory and general relativity. Supersymmetry was discovered in the early 1970s and in the intervening years has become a major component of theoretical physics. Its novel mathematical features have led to a deeper understanding of the geometrical structure of spacetime, a theme to which great thinkers like Riemann, Poincaré, Einstein, Weyl, and many others have contributed.

Symmetry has always played a fundamental role in quantum theory: rotational symmetry in the theory of spin, Poincaré symmetry in the classification of elementary particles, and permutation symmetry in the treatment of systems of identical particles. Supersymmetry is a new kind of symmetry that was discovered by physicists in the early 1970s. However, it is different from all other discoveries in physics in the sense that there has been no experimental evidence supporting it so far. Nevertheless, an enormous effort has been expended by many physicists in developing it because of its many unique features and also because of its beauty and coherence.[1] Here are some of its salient features:[2]

- It gives rise to symmetries between bosons and fermions at a fundamental level.
- Supersymmetric quantum field theories have "softer" divergences.
- Supersymmetric string theory (superstrings) offers the best context known so far for constructing unified field theories.

The development of supersymmetry has led to a number of remarkable predictions. One of the most striking of these is that every elementary particle has a SUSY partner of opposite spin parity, i.e., if the particle is a boson (resp., fermion), its partner is a fermion (resp., boson). The partners of electrons, neutrinos, and quarks are called selectrons, sneutrinos, and squarks, the partner of the photon is a fermion named photino, and so on. However, the masses of these partner particles are in the TeV range and so are beyond the reach of currently functioning accelerators (the Fermilab has energies in the 1 TeV range). The new LHC being built at CERN and expected to be operational by 2005 or so will have energies greater than 10 TeV, and it is expected that perhaps some of these SUSY partners may be found among the collisions that will be created there. Also, SUSY predicts a mass for the Higgs particle in the range of about several hundred times the mass of the proton, whereas there are no such bounds for it in the usual standard model.

For the mathematician the attraction of supersymmetry lies above all in the fact that it has provided a new look at geometry, both differential and algebraic, beyond its conventional limits. In fact, supersymmetry has provided a surprising continuation of the long evolution of ideas regarding the concept of space and more generally of what a geometric object should be like, an evolution that started with Riemann and was believed to have ended with the creation of the theory of schemes by Grothendieck. If we mean by a geometrical object something that is built out of local pieces and which in some sense reflects our most fundamental ideas about the structure of space or spacetime, then the most general such object is a *superscheme*, and the symmetries of such an object are *supersymmetries*, which are described by *supergroup schemes*.

1.2. Classical Mechanics and the Electromagnetic and Gravitational Fields

The temporal evolution of a deterministic system is generally described by starting with a set S whose elements are the "states" of the system, and giving a one-parameter group

$$D : t \longmapsto D_t\,, \quad t \in \mathbf{R}\,,$$

of bijections of S. D is called the *dynamical group* and its physical meaning is that if s is the state at time 0, then $D_t[s]$ is the state at time t. Usually S has some additional structure and the D_t would preserve this structure, and so the D_t would be "automorphisms" of S. If thermodynamic considerations are important, then D will be only a semigroup, defined for $t > 0$; the D_t would then typically be only *endomorphisms* of S, i.e., not invertible, so that the dynamics will not be reversible in time. Irreversibility of the dynamics is a consequence of the second law of thermodynamics, which says that the entropy of a system increases with time and so furnishes a direction to the arrow of time. But at the microscopic level all dynamics are time reversible, and so we will always have a dynamical group.

If the system is relativistic, then the reference to time in the above remarks is to the time in the frame of an (inertial) observer. In this case one requires additional data that describe the fact that the description of the system is the same for all observers. This is usually achieved by requiring that the set of states should be the same for all observers, and that there is a "dictionary" that specifies how to go from the description of one observer to the description of another. The dictionary is given by an action of the Poincaré group P on S. If

$$P \times S \longrightarrow S\,, \quad g, s \longmapsto g[s]\,,$$

is the group action, and O, O' are two observers whose coordinate systems are related by $g \in P$, and if $s \in S$ is the state of the system as described by O, then $s' = g[s]$ is the state of the system as described by O'. We shall see examples of this later.

Finally, physical observables are represented by real-valued functions on the set of states and form a real commutative algebra.

Classical Mechanics. In this case S is a *smooth* manifold and the dynamical group comes from a *smooth* action of $\mathbf{R}$ on S. If

$$X_s := X_{D,s} = \left(\frac{d}{dt}\right)_{t=0} (D_t[s]), \quad s \in S,$$

then $X(s \longmapsto X_s)$ is a vector field on S, the *dynamical vector field.* In practice only X is given in physical theories and the construction of the D_t is only implicit. Strictly speaking, for a given X, the D_t are not defined for all t without some further restriction on X (compact support will do, in particular, if S is compact). The D_t are, however, defined uniquely for small time starting from points of S, i.e., we have a *local flow* generated by X. A key property of this local flow is that for any compact set $K \subset S$ there is $\varepsilon > 0$ such that for all points $s \in K$ the flow starting from s at time 0 is defined for all $t \in (-\varepsilon, +\varepsilon)$.

In most cases we have a manifold M, the so-called "configuration space"of the system. For instance, for a system consisting of N point masses moving on some manifold U, U^N is the configuration space. There are then two ways of formulating classical mechanics.

Hamiltonian Mechanics. Here $S = T^*M$, the cotangent bundle of M. S has a canonical 1-form ω which in local coordinates $q_i, p_i (1 \leq i \leq n)$ is $p_1\, dq_1 + \cdots + p_n\, dq_n$. In coordinate-free terms the description of ω is well-known. If $s \in T^*M$ is a cotangent vector at $m \in M$ and π is the projection $T^*M \longrightarrow M$, and if ξ is a tangent vector to T^*M at s, then $\omega(\xi) = \langle d\pi_m(\xi), s\rangle$. Since $d\omega = \sum dp_i \wedge dq_i$ locally, $d\omega$ is nondegenerate, i.e., S is *symplectic*. At each point of S we thus have a nondegenerate bilinear form on the tangent space to S at that point, giving rise to an isomorphism of the tangent and cotangent spaces at that point. Hence there is a natural map from the space of 1-forms on S to the space of vector fields on S, $\lambda \longmapsto \lambda^\sim$.

In local coordinates we have $dp_i^\sim = \partial/\partial q_i$ and $dq_i^\sim = -\partial/\partial p_i$. If H is a real function on S, then we have the vector field $X_H := (dH)^\sim$, which generates a dynamical group (at least for small time locally). Vector fields of this type are called *Hamiltonian*, and H is called the *Hamiltonian* of the dynamical system. In local coordinates (q, p) the equations of motion for a path $x(t \longmapsto x(t))$ are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad 1 \leq i \leq n.$$

Notice that the map

$$H \longmapsto X_H$$

has only the space of constants as its kernel. Thus the dynamics determines the Hamiltonian function up to an additive constant.

The function H is constant on the dynamical trajectories and so is a *preserved quantity*; it is the *energy* of the system. More generally, physical observables are real functions, generally smooth, on T^*M, and form a real commutative algebra. If U is a vector field on M, then one can view U as a function on T^*M that is linear on each cotangent space. These are the so-called *momentum observables*. If (u_t)

is the (local) one-parameter group of diffeomorphisms of M generated by U, then U, viewed as a function on T^*M, is the momentum corresponding to this group of symmetries of M. For $M = \mathbf{R}^N$ we thus have linear and angular momenta, corresponding to the translation and rotation subgroups of diffeomorphisms of M.

More generally, S can be any symplectic manifold and the D_t symplectic diffeomorphisms. Locally the symplectic form can be written as $\sum dp_i \wedge dq_i$ in suitable local coordinates (Darboux's theorem). For a good introduction see the Arnold book.[3]

Lagrangian Mechanics. Here $S = TM$, the tangent bundle of M. Physical observables are the smooth, real-valued functions on S and form a real commutative algebra. The dynamical equations are generated once again by functions L on S, called *Lagrangians.* Let L be a Lagrangian, assumed to be smooth. For any path x defined on $[t_0, t_1]$ with values in S, its *action* is defined as

$$\mathcal{A}[x] = \int_{t_0}^{t_1} L(x(t), \dot{x}(t))dt\,.$$

The dynamical equations are obtained by equating to 0 the variational derivative of this functional for variations of x for which the values at the endpoints t_0, t_1 are fixed. The equations thus obtained are the well-known *Euler-Lagrange equations.* In local coordinates they are

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right), \qquad 1 \le i \le n\,.$$

Heuristically one thinks of the actual path as the one for which the action is a minimum, but the equations express only the fact that that the path is an *extremum*, i.e., a stationary point in the space of paths for the action functional. The variational interpretation of these equations implies at once that the dynamical equations are coordinate independent. Under suitable conditions on L one can get a diffeomorphism of TM with T^*M preserving fibers (but in general not linear on them) and a function H_L on T^*M such that the dynamics on TM generated by L goes over to the dynamics on T^*M generated by H_L under this diffeomorphism (*Legendre transformation*).

Most dynamical systems with finitely many degrees of freedom are subsumed under one of these two models or some variations thereof (holonomic systems); this includes celestial mechanics. The fundamental discoveries go back to Galilei and Newton, but the general coordinate independent treatment was the achievement of Lagrange. The actual solutions of specific problems is another matter; there are still major unsolved problems in this framework.

Electromagnetic Field and Maxwell's Equations. This is a dynamical system with an *infinite* number of degrees of freedom. In general, such systems are difficult to treat because the differential geometry of infinite-dimensional manifolds is not yet in definitive form except in special cases. The theory of electromagnetic fields is one such special case because the theory is *linear.* Its description was the great achievement of Maxwell, who built on the work of Farady. The fundamental objects are the electric field $\mathbf{E} = (E_1, E_2, E_3)$ and the magnetic field $\mathbf{B} = (B_1, B_2, B_3)$, which are functions on space depending on time and so may be viewed as functions on spacetime $\mathbf{R}^4$.

In vacuum, i.e., in regions where there are no sources present, these are governed by Maxwell's equations (in units where c, the velocity of light in vacuum, is 1):

$$\frac{d\mathbf{B}}{dt} = -\nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{B} = 0, \tag{1.1}$$

and

$$\frac{d\mathbf{E}}{dt} = \nabla \times \mathbf{B}, \qquad \nabla \cdot \mathbf{E} = 0. \tag{1.2}$$

Here the operators ∇ refer only to the space variables. Notice that equations (1.1) become equations (1.2) under the duality transformation

$$(\mathbf{E}, \mathbf{B}) \longmapsto (-\mathbf{B}, \mathbf{E}).$$

To describe these equations concisely it is customary to introduce the *electromagnetic tensor* on spacetime given by the 4×4 skew-symmetric matrix

$$\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}.$$

It is actually better to work with the exterior 2-form

$$F = E_1 dt \wedge dx + \cdots - B_1 dy \wedge dz - \cdots$$

where $\cdots$ means cyclic summation in x, y, z. Then it is easily verified that the system of equations (1.1) is equivalent to $dF = 0$.

To describe the duality that takes (1.1) to (1.2) we need some preparation. For any vector space V of dimension n over the reals equipped with a nondegenerate scalar product $(\cdot, \cdot)$ of arbitrary signature, we have nondegenerate scalar products defined on all the exterior powers $\Lambda^r(V) = \Lambda^r$ by

$$(v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r) = \det((v_i, w_j))_{1 \le i,j \le r}.$$

We choose an orientation for V and define $\tau \in \Lambda^n$ by

$$\tau = v_1 \wedge \cdots \wedge v_n$$

where (v_i) is an oriented orthogonal basis for V with $(v_i, v_i) = \pm 1$ for all i; τ is independent of the choice of such a basis. Then the Hodge duality $*$ is a linear isomorphism of Λ^r with Λ^{n-r} defined by

$$a \wedge *b = (a, b)\tau, \quad a, b \in \Lambda^r.$$

If M is a pseudo-Riemannian manifold that is oriented, the above definition gives rise to a $*$-operator smooth with respect to the points of M that maps r-forms to $(n-r)$-forms and is linear over $C^\infty(M)$. In our case we take V to be the dual to

$\mathbf{R}^4$ with the quadratic form $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$, where we write the dual basis as dx^μ. Then for $\tau = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ we have

$$*dx^\mu \wedge dx^\nu = \varepsilon_\mu \varepsilon_\nu \, dx^\rho \wedge dx^\sigma$$

with $(\mu\nu\rho\sigma)$ an even permutation of (0123), the ε_μ being the metric coefficients, being 1 for $\mu = 0$ and -1 for $\mu = 1, 2, 3$. Now we regard $\mathbf{R}^4$ as a pseudo-Riemannian manifold with metric $dt^2 - dx^2 - dy^2 - dz^2$, and extend the $*$-operator defined above to a $*$-operator, linear over $C^\infty(\mathbf{R}^4)$ and taking 2-forms to 2-forms. In particular,

$$*dt \wedge dx = -dy \wedge dz\,, \quad *dy \wedge dz = dt \wedge dx\,,$$

with similar formulae obtained by cyclically permuting x, y, z. Then $*F$ is obtained from F by the duality map $(\mathbf{E}, \mathbf{B}) \longmapsto (-\mathbf{B}, \mathbf{E})$. So the two sets of Maxwell equations are equivalent to

$$dF = 0\,, \quad d * F = 0\,.$$

In this coordinate independent form they make sense on any pseudo-Riemannian manifold of dimension 4. F is the *electromagnetic field.*

The Maxwell equations on $\mathbf{R}^4$, or, more generally, on any convex open set $\Omega \subset \mathbf{R}^4$, can be written in a simpler form. First, all closed forms on Ω are exact, and so we can write $F = dA$ where A is a 1-form. It is called the *four-vector potential.* It is not unique and can be replaced by $A + d\alpha$ where α is a scalar function. The classical viewpoint is that only F is physically significant and the introduction of A is to be thought of merely as a mathematical device. A functional dependent on A will define a physical quantity only if it is unchanged under the map $A \longmapsto A + d\alpha$. This is the *principle of gauge invariance.* The field equations are the Euler-Lagrange equations for the action

$$\mathcal{A}[A] = -\frac{1}{2}\int (dA \wedge *dA) d^4x = \frac{1}{2}\int (E^2 - B^2) dt\, dx\, dy\, dz\,.$$

The Maxwell equations on Ω can now be written in terms of A. Let us take the coordinates as $(x^\mu)(\mu = 0, 1, 2, 3)$ where x^0 denotes the time and the x^i $(i = 1, 2, 3)$ the space coordinates. Then

$$A = \sum_\mu A_\mu \, dx^\mu\,, \quad F = \sum_{\mu<\nu} F_{\mu\nu} \, dx^\mu \wedge dx^\nu\,, \quad F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}\,,$$

with the usual convention that $f_{,\mu} = \partial f / \partial x^\mu$. Then, writing $F^{\mu\nu} = \varepsilon_\mu \varepsilon_\nu F_{\mu\nu}$ with the ε_μ as above, the equation $d * F = 0$ can be checked to be the same as

$$\sum_\nu F^{\mu\nu}_{\ \ ,\nu} = \sum_\nu \frac{\partial F^{\mu\nu}}{\partial x^\nu} = 0\,, \quad \mu = 0, 1, 2, 3.$$

Let us now introduce the *Lorentz divergence* of $f = (f_\mu)$ given by

$$\mathrm{div}_L\, f = \sum_\mu \varepsilon_\mu \frac{\partial f_\mu}{\partial x_\mu}\,.$$

Then, writing

$$\mathcal{D} = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 , \quad \partial_\mu = \frac{\partial}{\partial x^\mu} ,$$

the Maxwell equations become

$$\mathcal{D} A_\mu = (\mathrm{div}_L\, A)_{,\,\mu} , \quad \mu = 0, 1, 2, 3.$$

Now from general theorems of PDE one knows that on any convex open set Ω, any constant-coefficient differential operator $P(D)$ has the property that the map $u \longmapsto P(D)u$ is surjective on $C^\infty(\Omega)$. Hence we can find α such that $\mathcal{D}\alpha = -\mathrm{div}_L\, A$. Changing A to $A + d\alpha$ and writing A in place of $A + d\alpha$, the Maxwell equations are equivalent to

$$\mathcal{D} A_\mu = 0 , \quad \mathrm{div}_L\, A = 0 , \quad \mu = 0, 1, 2, 3.$$

The condition

$$\mathrm{div}_L\, A = 0$$

is called the *Lorentz gauge*. Notice, however, that A is still not unique; one can change A to $A + d\alpha$ where $\mathcal{D}\alpha = 0$ without changing F while still remaining in the Lorentz gauge.

In classical electrodynamics it is usually not emphasized that the vector potential A may not always exist on an open set Ω unless the second de Rham cohomology of Ω vanishes, i.e., $H^{2,\mathrm{DR}}(\Omega) = 0$. If this condition is not satisfied, the study of the Maxwell equations have to take into account the *global topology* of Ω. Dirac was the first to treat such situations when he constructed the electrodynamics of a stationary magnetic monopole in a famous paper.[1] Then in 1959 Aharanov and Bohm suggested that there may be quantum electrodynamic effects in a nonsimply connected region even though the electromagnetic field is 0. They suggested that this is due to the fact that although the vector potential is locally zero, because of its multiple-valued nature, the topology of the region is responsible for the physical effects and hence that the vector potential must be regarded as having physical significance. Their suggestion was verified in a beautiful experiment done by Chambers in 1960.[4]

This link between electrodynamics and global topology has proven to be a very fertile one in recent years.

Returning to the convex open Ω above, the invariance of the Maxwell equations under the Poincaré group is manifest. However, we can see this also in the original form involving F:

$$dF = 0 , \quad d * F = 0 .$$

The first equation is invariant under *all* diffeomorphisms. The second is invariant under all diffeomorphisms that leave $*$ invariant, in particular, under diffeomorphisms preserving the metric. So there is invariance under the Poincaré group. But even more is true. It can be shown that diffeomorphisms that change the metric by a positive scalar function also leave the Maxwell equations invariant. These are the *conformal transformations*. Thus the Maxwell equations are invariant under the *conformal group*. This was first noticed by Weyl and was the starting point of his investigations that led to his discovery of gauge theories.

Conformal Invariance of Maxwell's Equations. It may not be out of place to give the simple calculation showing the conformal invariance of the Maxwell equations. It is a question of showing that on a vector space V with a metric g of even dimension $2n$ and of arbitrary signature, the $*$-operators for g and $g' = cg$ ($c > 0$), denoted by $*$ and $*'$, are related on k-forms by

$$(*) \qquad *' = c^{k-n} *$$

so that, when $k = n$, we have

$$*' = * .$$

Thus if M, M' are oriented pseudo-Riemannian manifolds of even dimension $2n$ and $f(M \simeq M')$ is a conformal isomorphism, then for forms F, F' of degree n on M and M', respectively, with $F = f^*(F')$, we have

$$f^*(*F') = *F .$$

So

$$d * F' = 0 \Leftrightarrow d * F = 0 ,$$

which is what we want to show.

To prove $(*)$ let (v_i) be an oriented orthogonal basis of V for g with $g(v_i, v_i) = \pm 1$ and let $\tau = v_1 \wedge \cdots \wedge v_{2n}$. Let $g' = cg$ where $c > 0$. Then $(v_i' = c^{-1/2} v_i)$ is an orthogonal basis for g' with $g'(v_i', v_i') = \pm 1$ and $\tau' = v_1' \wedge \cdots \wedge v_{2n}' = c^{-n}\tau$. Hence if a, b are elements of $\Lambda^k V$, then

$$a \wedge *'b = g'(a, b)\tau' = c^{k-n} g(a, b)\tau = c^{k-n} a \wedge *b$$

so that

$$a \wedge *'b = c^{k-n} a \wedge b .$$

This gives $(*)$ at once.

The fact that the Maxwell equations are not invariant under the Newtonian (Galilean) transformations connecting inertial frames was one of the major aspects of the crisis that erupted in fundamental classical physics towards the end of the nineteenth century. Despite many contributions from Lorentz, Poincaré, and others, the situation remained murky till Einstein clarified the situation completely. His theory of special relativity, special because only inertial frames were taken into account, developed the kinematics of spacetime events on the sole hypothesis that the speed of light does not depend on the motion of the light source. Then spacetime becomes an *affine* space with a distinguished nondegenerate quadratic form of signature $(+, -, -, -)$. The automorphisms of spacetime are then the elements of the Poincaré group and the Maxwell equations are invariant under these. We shall take a more detailed look into these matters later on in this chapter.

Gravitational Field and Einstein Equations. Special relativity was discovered by Einstein in 1905. Immediately afterward Einstein began his quest of freeing relativity from the restriction to inertial frames so that gravitation could be included. The culmination of his efforts was the creation in 1917 of the theory of *general relativity*. Spacetime became a smooth manifold with a pseudo-Riemannian metric $ds^2 = \sum_{\mu\nu} g_{\mu\nu}\, dx^\mu\, dx^\nu$ of signature $(+, -, -, -)$. The most fantastic aspect of the general theory is the fact that *gravitation is now a purely geometric* *phenomenon, a manifestation of the curvature of spacetime.* Einstein interpreted the $g_{\mu\nu}$ as the *gravitational potentials* and showed that in matter-free regions of spacetime they satisfy

$$R_{ij} = 0$$

where R_{ij} are the components of the Ricci tensor. These are the *Einstein equations*. Unlike the Maxwell equations they are *nonlinear* in the $g_{\mu\nu}$. Physicists regard the Einstein theory of gravitation as the most perfect physical theory ever invented.

1.3. Principles of Quantum Mechanics

The beginning of the twentieth century also witnessed the emergence of a second crisis in classical physics. This was in the realm of atomic phenomena when refined spectroscopic measurements led to results that showed that the stability of atoms, and hence of all matter, could not be explained on the basis of classical electrodynamics; indeed, according to classical electrodynamics, a charged particle revolving around a nucleus will radiate and hence continually lose energy, forcing it to revolve in a steadily diminishing radius, so that it will ultimately fall into the nucleus. This crisis was resolved only in 1925 when Heisenberg created quantum mechanics. Shortly thereafter a number of people including Heisenberg, Dirac, and Schrödinger established the fundamental features of this entirely new mechanics, which was more general and more beautiful than classical mechanics and gave a complete and convincing explanation of atomic phenomena.

The most basic feature of atomic physics is that when one makes a measurement of some physical observable in an atomic system, the act of measurement disturbs the system in a manner that is not predictable. This is because the measuring instruments and the quantities to be measured are both of the same small size. Consequently, measurements under the same conditions will not yield the same value. The most fundamental assumption in quantum theory is that we can at least obtain a *probability distribution* for the values of the observable being measured. Although in a completely arbitrary state this probability distribution will not have zero (or at least small) dispersion, in principle one can change the state so that the dispersion is zero (or at least arbitrarily small); this is called *preparation of state*. However, once this is done with respect to a particular observable, some other observables will have probability distributions whose dispersions are not small.

This is a great departure from classical mechanics where, once the state is determined exactly (or nearly exactly), all observables take exact (or nearly exact) values. *In quantum theory there is no state in which all observables will have zero (or arbitrarily small) dispersion.* Nevertheless, the mathematical model is such that the states still evolve causally and deterministically as long as measurements are not made. This mode of interpretation, called the *Copenhagen interpretation* because it was first propounded by the Danish physicist Niels Bohr and the members of his school such as Heisenberg, Pauli, and others, is now universally accepted. One of the triumphs of quantum theory and the Copenhagen interpretation was a convincing explanation of the wave-particle duality of light.

We recall that in Newton's original treatise *Optiks* light was assumed to consist of particles; but later on, in the eighteenth and nineteenth centuries, diffraction experiments pointed unmistakably to the wave nature of light. Quantum theory resolves this difficulty beautifully. It says that light has both particle and wave properties; it is the structure of the act of measurement that determines which aspect will be revealed. In fact, quantum theory goes much further and says that *all matter* has both particle and wave properties. This is an illustration of the famous Bohr principle of *complementarity*. In the remarks below we shall sketch rapidly the mathematical model in which these statements make perfectly good sense. For discussions of much greater depth and scope, one should consult the beautiful books by Dirac, von Neumann, and Weyl.[3]

States, Observables, and Probabilities. In quantum theory states and observables are related in a manner entirely different from that of classical mechanics. The mathematical description of any quantum system is in terms of a *complex separable* Hilbert space $\mathcal{H}$; the states of the system are then the points of the *projective space* $\mathbf{P}(\mathcal{H})$ of $\mathcal{H}$. Recall that if V is any vector space, the projective space $\mathbf{P}(V)$ of V is the set of one-dimensional subspaces (*rays*) of V. Any one-dimensional subspace of $\mathcal{H}$ has a basis vector ψ of norm 1, i.e., a *unit vector*, determined up to a scalar factor of absolute value 1 (called a *phase factor*). So the states are described by unit vectors with the proviso that unit vectors ψ, ψ' describe the same state if and only if $\psi' = c\psi$ where c is a phase factor.

The observables are described by *self-adjoint operators* of $\mathcal{H}$; we use the same letter to denote both the observable and the operator that represents it. If the observable (operator) A has a pure discrete simple spectrum with eigenvalues $a_1, a_2, \ldots,$ and corresponding (unit) eigenvectors $\psi_1, \psi_2, \ldots$, then a measurement of A in the state ψ will yield the value a_i with probability $|(\psi, \psi_i)|^2$. Thus

$$\mathrm{Prob}_\psi(A = a_i) = |(\psi, \psi_i)|^2, \quad i = 1, 2, \ldots.$$

The complex number (ψ, ψ_i) is called the *probability amplitude*, so that quantum probabilities are computed as squares ofabsolute values of complex probability amplitudes. Notice that as (ψ_i) is an orthonormal (ON) basis of $\mathcal{H}$, we must have

$$\sum_i |(\psi, \psi_i)|^2 = 1$$

so that the act of measurement is certain to produce some a_i as the value of A. It follows from many experiments (see von Neumann's discussion of the Compton-Simons scattering experiment,[3] pp. 211–215) that a measurement *made immediately after* always leads to this value a_i, so that we know that the state after the first measurement is ψ_i. In other words, while the state was arbitrary and undetermined before measurement, once we make the measurement and know that the value is a_i, we know that the state of the system has become ψ_i.

This aspect of measurement, called the *collapse of the wave packet*, is also the method of *preparation of states*. We shall elucidate this remarkable aspect of measurement theory a little later, using Schwinger's analysis of Stern-Gerlach experiments. If the Hilbert space is infinite dimensional, self-adjoint operators can have continuous spectra and the probability statements given above have to make use of the more sophisticated spectral theory of such operators.

In the case of an arbitrary self-adjoint operator A, one can associate to it its *spectral measure* P^A, which is a projection-valued measure that replaces the notion of eigenspaces. The relationship between A and P^A is given by

$$A = \int_{-\infty}^{+\infty} \lambda \, dP^A(\lambda) .$$

In this case

$$\mathrm{Prob}_\psi(A \in E) = \left\| P_E^A \psi \right\|^2 = \left(P_E^A \psi, \psi \right), \quad E \subset \mathbf{R} .$$

The operators representing position and momentum are of this type, i.e., have continuous spectra. For the expectation value and dispersion (variance) of A in the state ψ, we have the following formulae:

$$E_\psi(A) = (A\psi, \psi), \quad \mathrm{Var}_\psi(A) = \|(A - mI)\psi\|^2, \quad m = E_\psi(A) .$$

As an extreme example of this principle, the quantity

$$|(\psi, \psi')|^2 \quad (\text{resp., } (\psi, \psi'))$$

is the probability (resp., probability amplitude) that when the system is in the state ψ and a measurement is made to determine if the state is ψ', the state will be found to be ψ'.

The most impressive aspect of the discussion above is that the states are the *points of a projective geometry.* Physicists call this the *principle of superposition of states*. If ψ_i $(i = 1, 2, 3)$ are three states, ψ_3 is a *superposition* of ψ_1 and ψ_2 if and only if $[\psi_3]$ is on the line in the projective space $P(\mathcal{H})$ joining $[\psi_1]$ and $[\psi_2]$ (here $[\psi_i]$ is the point of $\mathbf{P}(\mathcal{H})$ represented by the vector ψ_i). In terms of vectors this is the same as saying that ψ_3 is a linear combination of ψ_1 and ψ_2.

One should contrast this with the description of classical systems, where states are points of a set where no superposition is possible; there one can say that the states are the points of a *Boolean algebra.* The transition

$$\text{Boolean algebra} \longrightarrow \text{projective geometry}$$

is the mathematical essence of the change of description from classical to quantum that allows a mathematically and physically consistent scheme rich enough to model the unique features of quantum theory like the wave-particle duality of all matter, and, more generally, the principle of complementarity.

In classical *statistical mechanics* the states are often *probability measures* on the phase space. However, this is due to the fact that the huge number of degrees of freedom of the system makes it impossible to know the state exactly, and so the probability measures are a reflection of the incomplete knowledge of the actual state. The statistical nature of the description thus derives from parameters which are "hidden."

By contrast, in quantum mechanics the states are already assumed to be determined with maximal precision and the statistical character is entirely intrinsic. The maximally precise states are often called *pure states*, and these are the ones we have called states. In quantum statistical mechanics we encounter states with less than maximal precision, the so-called *mixed states*. These are described by what are called *density operators*, namely, operators D that are bounded, self-adjoint, positive, and of trace 1. If A is an observable, its expectation value in the state D is given by

$$E_D(A) = \mathrm{Tr}(DA) = \mathrm{Tr}(D^{1/2}AD^{1/2}).$$

These mixed states form a *convex* set, whose extreme points are the pure states; in this picture the pure states correspond to the density operators, which are the projection operators $P_{[\psi]}$ on the one-dimensional subspaces of the Hilbert space. However, it should be remembered that the representation of a mixed state as a convex combination of pure states is not always unique, making the physical interpretation of mixtures a very delicate matter.

For a long time after the discovery of quantum mechanics and the Copenhagen interpretation, some people refused to accept them on the grounds that the statistical description in quantum theory is ultimately due to the *incompleteness* of the quantum state, and that a fuller knowledge of the state will remove the probabilities. This is called the *hidden variables interpretation.*

Among the subscribers to this view was Einstein who never reconciled himself to the new quantum theory ("God does not play dice"), although he was one of the central figures in the quantum revolution because of his epoch-making work on the photon as a light quantum. Among his most spectacular attempts to reveal the incomplete nature ofthe quantum mechanical description of nature is the EPR paradox, firstsugge refuted by Niels Bohr convincingly. Nowadays there is no paradox in the EPR experiment; experiments conducted everyday in high-energy physics laboratories confirm convincingly that things happen as quantum theory predicts.

At the mathematical level one can ask the question whether the results of the quantum theory can be explained by a hidden parameter model. The answer is a resounding no. The first such theorem was proven by von Neumann; since then a galaxy of people have examined this question under varying levels of assumptions: Mackey, Gleason, Bell, et al. However, the question is not entirely mathematical. For a discussion of these aspects, see my book as well as the other references contained in the monumental book of Wheeler and Zurek[5] (which has reprints of most of the fundamental articles on the theory of measurement, including a complete extract of von Neumann's treatment of the thermodynamic aspects of measurement from his book.[3]

Stern-Gerlach Experiments and Finite Models. The discussion above is very brief and does not do full justice to the absolutely remarkable nature of the difference between classical and quantum physics. It is therefore reasonable to ask if there is a way to comprehend better these remarkable features, for instance, by a discussion that is closer to the experimental situations but somewhat simpler from a mathematical standpoint. The Hilbert space $\mathcal{H}$ of quantum theory is usually infinite dimensional because many observables of importance (position coordinates, momenta, etc.) have values that form a continuous range, and any discussion of the features of quantum theory rapidly gets lost among technicalities of the mathematical theory.

To illustrate the striking features of quantum theory most simply and elegantly, one should look at *finite models* where $\mathcal{H}$ is finite dimensional. Such models go back to Weyl in the 1930s;[6] they were revived in the 1950s by Schwinger,[7] and resurrected again in the 1990s.[8] For a beautiful treatment of the foundations of quantum mechanics from this point of view, see Schwinger's book, in particular the prologue.[9]

The simplest such situation is the measurement of spin or the magnetic moment of an atom. The original experiments were done by Stern and Gerlach and so such measurements are known as *Stern-Gerlach measurements*. In this experiment silver pellets are heated in an oven to a very high temperature till they are vaporized, and then they are drawn out through an aperture in the oven and refined by passing through several slits. The beam is then passed through a magnetic field and then stopped on a screen. Since the silver atoms have been heated to a high temperature it is natural to assume that their magnetic moments are distributed randomly. So one should expect a continuous distribution of the magnetic moments on the screen; instead one finds that the atoms are concentrated in two sharp piles of moments $+\mu$ and $-\mu$.

This kind of experiment is a typical spin measurement with two values; the measuring apparatus, in this case the magnetic field oriented in a specific direction, measures the magnetic moment along that direction. Of course, the direction of the magnetic field is at one's disposal so that we have an example of a system where all observables have either one or two values. If we decide to stop only the $-$ beam, the $+$ beam will pass through undeflected through a second magnetic field parallel to the first. Then one knows that the atoms in the $+$ beam all have their spins aligned in the given direction.

This is an example of what we defined earlier as preparation of state. Measurements in different directions will then lead to a more or less complete enumeration of the observables of this system. Moreover, when repeated measurements are made, we can see quite explicitly how the measurement changes the state and destroys any previous information that one has accumulated about the state. The fact that one cannot make the dispersions of all the observables simultaneously small is very clearly seen here. This is the heart of the result that the results of quantum theory do not have an interpretation by hidden variables. Indeed, the experiments suggested by Bohm for elucidating the EPR paradox are essentially spin or polarization measurements and use finite models. In fact, one can even show that all states that possess the features of the EPR phenomenon are of the Bohm type or generalizations thereof.[10]

From the mathematical point of view, these spin systems are examples of systems where all observables have at most N values (N is a fixed integer) and generic observables have exactly N values. The Hilbert space can then be taken to be $\mathbf{C}^N$ with the standard scalar product. The observables are then $N \times N$ Hermitian matrices whose spectra are the sets of values of these observables. The determination of states is made by measurements of observables with exactly N distinct values.

If A is a Hermitian matrix with distinct eigenvalues $a_1, \ldots, a_N$ and eigenvectors $\psi_1, \ldots, \psi_N$, and a measurement of A yields a value a_i, then we can say with certainty that the state is ψ_i immediately after measurement, and it will evolve deterministically under the dynamics till another measurement is made. This is the way states are determined in quantum theory, by specifying the values (i.e., quantum numbers) of one or more observables even in more complicated systems.

Suppose B is another Hermitian matrix with eigenvalues b_i and eigenvectors ψ_i'. If A is measured and found to have the value a_i, an immediately following measurement of B will yield the values b_j with probabilities $|(\psi_i, \psi_j')|^2$. Suppose now (this is always possible) we select B so that

$$\left|(\psi_i, \psi_j')\right|^2 = \frac{1}{N}, \quad 1 \le i,\ j \le N .$$

Then we see that in the state where A has a specific value, all values of B are equally likely and so there is minimal information about B. Pairs of observables like A and B with the above property may be called *complementary*. In the continuum limit of this model A and B will (under appropriate conditions) go over to the position and momentum of a particle moving on the real line, and one will obtain the Heisenberg uncertainty principle, namely, that there is no state in which the dispersions of the position and momentum measurements of the particle are *both* arbitrarily small.

In a classical setting, the model for a system all of whose observables have at most N values (with generic ones having N values) is a set X_N with N elements, observables being real functions on X_N. The observables thus form a *real algebra* whose dimension is N. Not so in quantum theory for a similarly defined system: the states are the points of the projective space $\mathbf{P}(\mathbf{C}^N)$ and the observables are $N \times N$ Hermitian matrices that *do not form an algebra*. Rather, they are the *real* elements of a *complex* algebra with an involution $*$ (adjoint), real being defined as being fixed under $*$. The dimension of the space of observables has now become N^2; the extra dimensions are needed to accommodate complementary observables. The complex algebra itself can be interpreted, as Schwinger discovered,[9] in terms of the measurement process, so that it can be legitimately called, following Schwinger, the *measurement algebra*.

Finally, if A and B are two Hermitian matrices, then AB is Hermitian if and only if $AB = BA$, which is equivalent to the existence of an ON basis for $\mathbf{C}^N$ whose elements are simultaneous eigenvectors for both A and B; in the corresponding states both A and B can be measured with zero dispersion. Thus commutativity of observables is equivalent to *simultaneous observability*. In classical mechanics *all observables are simultaneously observable*. This is spectacularly false in quantum theory.

Although the quantum observables do not form an algebra, they are the real elements of a complex algebra. Thus one can say that *the transition from classical to quantum theory is achieved by replacing the commutative algebra of classical observables by a complex algebra with involution whose real elements form the* *space of observables of the quantum system.*[11] By abuse of language we shall refer to this complex algebra itself as the *observable algebra.*

The preceding discussion has captured only the barest essentials of the foundations of quantum theory. However, in order to understand the relation between this new mechanics and classical mechanics, it is essential to encode into the new theory the fact which is characteristic of quantum systems, namely, that they are really microscopic; what this means is that the quantum of action, namely, *Planck's constant* $\hbar$, really defines the boundary between classical and quantum. In situations where we can neglect $\hbar$, quantum theory may be replaced by classical theory. For instance, the commutation rule between position and momentum, namely,

$$[p, q] = -i\hbar$$

goes over to

$$[p, q] = 0$$

when $\hbar$ is 0.

Therefore a really deeper study of quantum foundations must bring in $\hbar$ in such a way that the noncommutative quantum observable algebra depending on $\hbar$, now treated as a *parameter*, goes over in the limit $\hbar \to 0$ to the commutative algebra of classical observables (complexified). Thus *quantization*, by which we mean the transition from a classically described system to a "corresponding quantum system," is viewed as a *deformation* of the classical commutative algebra into a noncommutative quantum algebra. However, one has to go to *infinite-dimensional* algebras to truly exhibit this aspect of quantum theory.[12]

REMARK. Occasionally there arise situations where the projective geometric model given above has to be modified. Typically these are contexts where there are *superselection observables*. These are observables that are simultaneously measurable with *all observables*. (In the usual model above only the constants are simultaneously measurable with every observable.) If all superselection observables have specific values, the states are again points of a projective geometry; the choice of the values for the superselection observables is referred to as a *sector*.

The simplest example of such a situation arises when the Hilbert space $\mathcal{H}$ has a decomposition

$$\mathcal{H} = \bigoplus_j \mathcal{H}_j$$

and only those operators of $\mathcal{H}$ are considered as observables that commute with all the orthogonal projections

$$P_j : \mathcal{H} \longrightarrow \mathcal{H}_j .$$

The *center* of the observable algebra is then generated by the P_j. Any real linear combination of the P_j is then a superselection observable. The states are then rays that *lie in some* $\mathcal{H}_j$. So we can say that the states are points of the *union*

$$\bigcup_j \mathbf{P}(\mathcal{H}_j) .$$

This situation can be generalized. Let us keep to the notation above but require that for each j there is a $*$-algebra $\mathcal{A}_j$ of operators on $\mathcal{H}_j$ which is isomorphic to a full finite-dimensional matrix $*$-algebra such that the observables are those operators that leave the $\mathcal{H}_j$ invariant and whose restrictions to $\mathcal{H}_j$ commute with $\mathcal{A}_j$. It is not difficult to see that we can write

$$\mathcal{H}_j \simeq V_j \otimes \mathcal{K}_j\,, \quad \dim(V_j) < \infty\,,$$

where $\mathcal{A}_j$ acts on the first factor and observables act on the second factor, with $\mathcal{A}_j$ isomorphic to the full $*$-algebra of operators on V_j, so that the observable algebra on $\mathcal{H}_j$ is isomorphic to the full operator algebra on $\mathcal{K}_j$. In this case the states may be identified with the elements of

$$\bigcup_j \mathbf{P}(\mathcal{K}_j)\,.$$

Notice that once again we have a *union* of projective geometries. Thus, between states belonging to different $\mathbf{P}(\mathcal{K}_j)$ *there is no superposition.* The points of $\mathbf{P}(\mathcal{K}_j)$ are the *states in the $\mathcal{A}_j$-sector.*

The above remarks have dealt with only the simplest of situations and do not even go into quantum mechanics. More complicated systems like quantum field theory require vastly more sophisticated mathematical infrastructure.

One final remark may be in order. The profound difference between classical and quantum descriptions of states and observables makes it important to examine whether there is a deeper way of looking at the foundations that will provide a more natural link between these two pictures. This was done for the first time by von Neumann and then, after him, by a whole host of successors.

Let $\mathcal{O}$ be a complex algebra with involution $*$ whose real elements represent the bounded physical observables. Then for any state of the system we may write $\lambda(a)$ for the *expectation value* of the observable a in that state. Then $\lambda(a^n)$ is the expectation value of the observable a^n in the state. Since the moments of a probability distribution with compact support determine it uniquely, it is clear that we may *identify* the state with the corresponding functional

$$\lambda : a \longmapsto \lambda(a)\,.$$

The natural assumptions about λ are that it be linear and positive in the sense that $\lambda(a^2) \geq 0$ for any observable a. Both of these are satisfied by complex linear functions λ on $\mathcal{O}$ with the property that $\lambda(a^*a) \geq 0$.

Such functionals on $\mathcal{O}$ are then called *states*. To obtain states one starts with a *$*$-representation* ρ of $\mathcal{O}$ by operators in a Hilbert space and then define, for some unit vector ψ in the Hilbert space, the state by

$$\lambda(a) = (\rho(a)\psi, \psi)\,.$$

It is a remarkable fact of $*$-representations of algebras with involution that under general circumstances *any* state comes from a pair (ρ, ψ) as above, and that if we require ψ to be *cyclic*, then the pair (ρ, ψ) is unique up to unitary equivalence. Thus the quantum descriptions of states and observables are essentially inevitable; the only extra assumption that is made, which is a natural simplifying one, is that there is a *single* representation, or a *single* Hilbert space, whose vectors represent the states. For more details, see my book.[5]

1.4. Symmetries and Projective Unitary Representations

The notion of a symmetry of a quantum system can be defined in complete generality.

DEFINITION A *symmetry* of a quantum system with $\mathcal{H}$ as its Hilbert space of states is any bijection of $\mathbf{P}(\mathcal{H})$ that preserves $|(\psi, \psi')|^2$.

For any $\psi \in \mathcal{H}$ that is nonzero, let $[\psi]$ be the point of $\mathbf{P}(\mathcal{H})$ it defines and let

$$p([\psi], [\psi']) = |(\psi, \psi')|^2 .$$

Then a symmetry s is a bijection

$$s : \mathbf{P}(\mathcal{H}) \longrightarrow \mathbf{P}(\mathcal{H})$$

such that

$$p(s[\psi], s[\psi']) = p([\psi], [\psi']) , \quad \psi, \psi' \in \mathcal{H} .$$

Suppose U is a unitary (resp., antiunitary) operator of $\mathcal{H}$; this means that U is a linear (resp., antilinear) bijection of $\mathcal{H}$ such that

$$(U\psi, U\psi') = (\psi, \psi') \quad ((U\psi, U\psi') = (\psi', \psi)) .$$

Then

$$[\psi] \longmapsto [U\psi]$$

is a symmetry. We say that the symmetry is *induced* by U; the symmetry is called unitary or antiunitary according as U is unitary or antiunitary. The fundamental theorem on which the entire theory of symmetries is based is the following:[13]

THEOREM 1.4.1 (Wigner) *Every symmetry is induced by a unitary or antiunitary operator of $\mathcal{H}$, which moreover is determined uniquely up to multiplication by a phase factor. The symmetries form a group and the unitary ones a normal subgroup of index* 2.

This theorem goes to the heart of why quantum theory is *linear*. The ultimate reason is the superposition principle or the fact that the states form the points of a projective geometry, so that the automorphisms of the set of states arise from linear or conjugate linear transformations. Recently people have been exploring the possibility of *nonlinear extensions of quantum mechanics*. Of course, such extensions cannot be made arbitrarily and must pay attention to the remarkable structure of quantum mechanics. Some of these attempts are very interesting.[14]

Let us return to Wigner's theorem and some of its consequences. Clearly the square of a symmetry is always unitary. The simplest and most typical example of an antiunitary symmetry is the map

$$f \longmapsto f^{\mathrm{conj}} , \quad f \in L^2(\mathbf{R}) .$$

Suppose that G is a group which acts as a group of symmetries and that G is generated by squares. Then every element of G acts as a unitary symmetry. Now, if G is a Lie group, it is known that the connected component of G is generated by elements of the form $\exp X$ where X lies in the Lie algebra of G. Because $\exp X = (\exp X/2)^2$, it follows that every element of the connected component of G acts as a unitary symmetry. We thus have the corollary:

COROLLARY 1.4.2 *If G is a connected Lie group and $\lambda : g \longmapsto \lambda(g)$ $(g \in G)$ is a homomorphism of G into the group of symmetries of $\mathcal{H}$, then for each g there is a unitary operator $L(g)$ of $\mathcal{H}$ such that $\lambda(g)$ is induced by $L(g)$.*

If one makes the choice of $L(g)$ for each $g \in G$ in some manner, one obtains a map

$$L : g \longmapsto L(g)\,, \quad g \in G\,,$$

which cannot in general be expected to be a unitary representation of G in $\mathcal{H}$. Recall here that to say that a map of a topological group G into $\mathcal{U}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ is a representation is to require that L be a continuous homomorphism of G into the unitary group $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$ equipped with its strong operator topology. The continuity is already implied (when G and $\mathcal{H}$ are separable) by the much weaker and almost always fulfilled condition that the maps $\varphi, \psi \mapsto (L(g)\varphi, \psi)$ are Borel.

In the case above, we may obviously assume that $L(1) = 1$; because $\lambda(gh) = \lambda(g)\lambda(h)$, we have

$$L(g)L(h) = m(g,h)L(gh)\,, \quad |m(g,h)| = 1\,.$$

Now, although $L(g)$ is not uniquely determined by $\lambda(g)$, its image $L^{\sim}(g)$ in the projective unitary group $\mathcal{U}(\mathcal{H})/\mathbf{C}^{\times}1$ is well-defined. We shall always assume that the action of G is such that the map $L^{\sim}$ is continuous. The continuity of $L^{\sim}$, and hence the continuity of the action of G, is guaranteed as soon as the maps $g \longmapsto |(L(g)\varphi, \psi)|$ are Borel. Given such a continuous action, one can always choose the $L(g)$ such that $g \longmapsto L(g)$ from G to $\mathcal{U}(\mathcal{H})$ is *Borel*. L is then called a *projective unitary representation* of G in $\mathcal{H}$. In this case the function m above is Borel. Thus symmetry actions correspond to projective unitary representations of G. The function m is called the *multiplier* of L; since we can change $L(g)$ to $c(g)L(g)$ for each g, c being a Borel map of G into the unit circle, m is only significant up to multiplication by a function $c(g)c(h)/c(gh)$, and L will be called *unitarizable* if we can choose c so that cL is a unitary representation in the usual sense.

If $G^{\sim}$ is a locally compact second countable topological group, $C \subset G^{\sim}$ is a closed normal subgroup, and $G = G^{\sim}/C$, then any unitary representation of $G^{\sim}$ that takes elements of C into scalars (scalar on C) gives rise to a projective unitary representation of G because for any $g \in G$ all the unitaries of elements above g differ only by scalars. If C is *central*, i.e., if the elements of C commute with all elements of $G^{\sim}$, and if the original representation of $G^{\sim}$ is irreducible, then by Schur's lemma the representation is scalar on C and so we have a projective unitary representation of G. $G^{\sim}$ is called a *central extension* of G if $G = G^{\sim}/C$ where C is central. It is a very general theorem that for any locally compact second countable group G every projective unitary representation arises only in this manner, C

being taken as the circle group, although $G^\sim$ will in general depend on the given projective representation of G.

Suppose G is a connected Lie group and $G^\sim$ is its simply connected covering group with a given covering map $G^\sim \longrightarrow G$. The kernel F of this map is a *discrete* central subgroup of $G^\sim$; it is the fundamental group of G. Although every irreducible unitary representation of $G^\sim$ defines a projective unitary representation of G, not every projective unitary representation of G can be obtained in this manner; in general, there will be irreducible projective unitary representations of G that are not unitarizable even after being lifted to $G^\sim$. However, in many cases we can construct a *universal central extension* $G^\sim$ such that *all* projective irreducible representations of G are induced as above by unitary representations of $G^\sim$.

This situation is in stark contrast with what happens for finite-dimensional representations, unitary or not. A projective finite-dimensional representation of a Lie group G is a smooth morphism of G into the projective group of some vector space, i.e., into some $\mathrm{PGL}(N, \mathbf{C})$. It can then be shown that the lift of this map to $G^\sim$ is renormalizable to an ordinary representation, which will be unique up to multiplication by a character of $G^\sim$, i.e., a morphism of $G^\sim$ into $\mathbf{C}^\times$. To see this, observe that $\mathfrak{gl}(N, \mathbf{C}) = \mathfrak{sl}(N, \mathbf{C}) \oplus \mathbf{C}I$ so that $\mathrm{Lie}(\mathrm{PGL}(N, \mathbf{C})) \simeq \mathfrak{sl}(N, \mathbf{C})$; thus $G \longrightarrow \mathrm{PGL}(N, \mathbf{C})$ defines $\mathfrak{g} \longrightarrow \mathfrak{sl}(N, \mathbf{C})$ and hence also $G^\sim \longrightarrow \mathrm{SL}(N, \mathbf{C})$.

Projective representations of finite groups go back to Schur. The theory for Lie groups was begun by Weyl but was worked out in a definitive manner by Bargmann for Lie groups and Mackey for general locally compact second countable groups.[5,15]

We shall now give some examples that have importance in physics to illustrate some of these remarks.

$G = \mathbf{R}$ or the circle group $\mathbb{S}^1$. A projective unitary representation of $\mathbb{S}^1$ is also one for $\mathbf{R}$, and so we can restrict ourselves to $G = \mathbf{R}$. In this case *any projective unitary representation can be renormalized to be a unitary representation.* In particular, the *dynamical evolution*, which is governed by a projective unitary representation D of $\mathbf{R}$, is given by an ordinary unitary representation of $\mathbf{R}$; by Stone's theorem we then have

$$D : t \longmapsto e^{itH}, \quad t \in \mathbf{R},$$

where H is a self-adjoint operator. Since

$$e^{it(H+k)} = e^{itk} e^{itH}$$

where k is a real constant, the change $H \longmapsto H + kI$ does not change the corresponding projective representation and so does not change the dynamics. However, this is the extent of the ambiguity. H is the *energy* of the system (recall that self-adjoint operators correspond to observables). Exactly as in Hamiltonian mechanics, dynamical systems are generated by the energy observables, and the observable is determined by the dynamics up to an additive constant.

$G = S^1 = \mathbf{R}/\mathbf{Z}$. The unitary operator e^{iH} induces the identity symmetry and so is a phase e^{ik}; thus $e^{i(H-k)}$ is the identity so that $t \longmapsto e^{it(H-k)}$ defines a unitary representation of S^1 that induces the given projective unitary representation of S^1.

$G = \mathbf{R}^2$. It is no longer true that all projective unitary representations of G are unitarizable. Indeed, the commutation rules of Heisenberg, as generalized by Weyl, give rise to an *infinite-dimensional irreducible* projective unitary representation of G. Since irreducible unitary representations of an abelian group are of dimension 1, such a projective unitary representation cannot be unitarized. Let $\mathcal{H} = L^2(\mathbf{R})$. Let Q, P be the position and momentum operators, i.e.,

$$(Qf)(x) = xf(x), \quad (Pf)(x) = -i\frac{df}{dx}.$$

Both of these are *unbounded*, and so one has to exercise care in thinking of them as self-adjoint operators. The way to do this is to pass to the unitary groups generated by them. Let

$$U(a) : f(x) \longmapsto e^{iax}f(x), \quad V(b) : f(x) \longmapsto f(x+b), \quad a, b \in \mathbf{R}.$$

These are both one-parameter unitary groups, and so by Stone's theorem they can be written as

$$U(a) = e^{iaQ'}, \quad V(b) = e^{ibP'}, \quad a, b \in \mathbf{R},$$

where Q', P' are self-adjoint; we define $Q = Q'$, $P = P'$. A simple calculation shows that

$$U(a)V(b) = e^{-iab}V(b)U(a).$$

So, if

$$W(a, b) = e^{iab/2}U(a)V(b)$$

(the exponential factor is harmless and is useful below), then we have:

$$W(a, b)W(a', b') = e^{i(a'b-ab')/2}W(a + a', b + b'),$$

showing that W is a projective unitary representation of $\mathbf{R}^2$. If a bounded operator A commutes with W, its commutativity with U implies that A is multiplication by a bounded function f, and then its commutativity with V implies that f is invariant under translation, so that f is constant; i.e., A is a scalar. So W is irreducible.

The multiplier of W arises directly out of the symplectic structure of $\mathbf{R}^2$ regarded as the *classical phase space* of a particle moving on $\mathbf{R}$. Thus *quantization* may be viewed as passing from the phase space to a projective unitary representation canonically associated to the symplectic structure of the phase space. This was Weyl's point of view.

$G = \mathrm{SO}(3)$, $G^\sim = \mathrm{SU}(2)$. Rotational symmetry is of great importance in the study of atomic spectra. $G^\sim = \mathrm{SU}(2)$ operates on the space of 3×3 Hermitian matrices of trace 0 by $g, h \longmapsto ghg^{-1}$. The Hermitian matrices of trace 0 can be written as

$$h = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}.$$

Since

$$\det(h) = -\left(x_1^2 + x_2^2 + x_3^2\right)$$

is preserved, the action of any element of SU(2) lies in O(3) and so we have a map $G^\sim \longrightarrow \mathrm{O}(3)$. Its kernel is easily checked to be $\{\pm 1\}$. Since $G^\sim$ is connected, its image is actually in SO(3), and because the kernel of the map has dimension 0, the image of SU(2) is also of dimension 3. Because SO(3) also has dimension 3, the map is surjective. We thus have an exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3) \longrightarrow 1\,.$$

Now SU(2) consists of all matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a\bar{a} + b\bar{b} = 1\,,$$

and so topologically $\mathrm{SU}(2) \simeq \mathbb{S}^3$. Thus SU(2) is simply connected, and the above exact sequence describes the universal covering of SO(3). If we omit, in the description of elements of SU(2), the determinant condition, we get the quaternion algebra by the identification

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \longmapsto a + bj\,, \quad i^2 = -1,\ j^2 = -1,\ ij = -ji\,,$$

so that SU(2) may be viewed as the group of elements of unit norm of the quaternion algebra. For dimensions $N > 3$ a similar description of the universal covering group of $\mathrm{SO}(N)$ is possible; the universal covering groups are the *spin groups* $\mathrm{Spin}(N)$, and they appear as the unit groups of the *Clifford algebras* that generalize quaternion algebras.

$G = \mathrm{SO}(1,3)^0$, $G^\sim = \mathrm{SL}(2, \mathbf{C})$. G is the *connected Lorentz group*, namely, the component of the identity element of the group $\mathrm{O}(1,3)$ of all nonsingular matrices g of order 4 preserving

$$x_0^2 - x_1^2 - x_2^2 - x_3^2\,.$$

Also, $\mathrm{SL}(2, \mathbf{C})$ must be viewed as the *real* Lie group underlying the complex Lie group $\mathrm{SL}(2, \mathbf{C})$ so that its real dimension is 6 which is double its complex dimension, which is 3; we shall omit the subscript $\mathbf{R}$ if it is clear that we are dealing with the real Lie group. We have the action $g, h \longmapsto ghg^*$ of $G^\sim$ on the space of 2×2 Hermitian matrices identified with $\mathbf{R}^4$ by writing them in the form

$$h = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}.$$

The action preserves

$$\det(h) = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

and so maps $G^\sim$ into $\mathrm{O}(1,3)$. It is not difficult to check using polar decomposition that $G^\sim$ is connected and simply connected and the kernel of the map $G^\sim \longrightarrow G$ is (± 1). As in the unitary case, as $\dim G = \dim \mathrm{SO}(1,3)^0 = 6$, we have the exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SL}(2, \mathbf{C}) \longrightarrow \mathrm{SO}(1,3)^0 \longrightarrow 1.$$

Representations of SU(2) and SL(2, C). Any irreducible projective unitary representation of SO(3) is finite dimensional and arises from an ordinary irreducible representation of SU(2) via the covering map SU(2) $\longrightarrow$ SO(3). The general representation of SU(2) is parametrized by a half-integer $j \in \frac{1}{2}\mathbf{Z}$ and is of dimension $2j + 1$. It is the representation obtained on the space of homogeneous polynomials p in z_1, z_2 of degree $2j$ from the natural action of SU(2) on $\mathbf{C}^2$. It is usually denoted by D^j. The representation $D^{1/2}$ is the basic one. The parameter j is called the *spin* of the representation. The element -1 of SU(2) goes over to $(-1)^{2j}$, and so the representations of SO(3) are those for which j is itself an integer. These are the odd-dimensional ones. For applications one needs the formula

$$D^j \otimes D^k = D^{|j-k|} \oplus D^{|j-k|+1} \oplus \cdots \oplus D^{j+k}.$$

This is the so-called *Clebsch-Gordan formula.*

Let us go back to the context of the Stern-Gerlach experiment in which atoms are subjected to a magnetic field. The experiment is clearly covariant under SO(3), and the mathematical description of the covariance must be through a projective unitary representation of SO(3). But the measurements of the magnetic moment are all two-valued, and so the Hilbert space must be of dimension 2. So the representation must be $D^{1/2}$. Notice that *the use of projective representations is essential since* SO(3) *has no ordinary representation in dimension* 2 *other than the direct sum of two trivial representations, which obviously cannot be the one we are looking for.* The space of $D^{1/2}$ is to be viewed as an *internal space* of the particle. It is to be thought of as being attached to the particle and so should move with the particle. In the above discussion the symmetry action of SU(2) is *global* in the sense that it does not depend on where the particle is.

In the 1950s the physicists Yang and Mills introduced a deep generalization of this global symmetry that they called *local symmetry.* Here the element of SU(2) that describes the internal symmetry is *allowed to depend on the spacetime point where the particle is located.* These local symmetries are then described by *functions on spacetime with values in* SU(2); they are called *gauge symmetries*, and the group of all such (smooth) functions is called the *gauge group.* The fact that the internal vector space varies with the point of spacetime means that we have a *vector bundle* on spacetime. Thus the natural context for treating gauge theories is a vector bundle on spacetime.

Internal characteristics of particles are pervasive in high-energy physics. They go under names such as spin, isospin, charm, color, flavor, etc. In gauge theories the goal is to work with equations that are gauge invariant, i.e., invariant under the group of gauge symmetries. Since the gauge group is infinite dimensional, this is a vast generalization of classical theory. Actually, the idea of a vector space attached to points of the spacetime manifold originated with Weyl in the context of his unification of electromagnetism and gravitation. Weyl wrote down the gauge-invariant coupled equations of electromagnetism and gravitation. The vector bundle in Weyl's case is a *line bundle*, and so the gauge group is the group of smooth functions on spacetime with values in the unit circle, hence an abelian group. The Yang-Mills equations, however, involve a nonabelian gauge group.[16]

Suppose now $G = \mathrm{SL}(2, \mathbf{C})$. We must remember that we have to regard this as a topological rather than a complex analytic group, or, what comes to the same thing, view it as a real Lie group. So to make matters precise we usually write this group as $\mathrm{SL}(2, \mathbf{C})_{\mathbf{R}}$, omitting the subscript when there is no ambiguity. Notice first of all that the representations D^j defined earlier by the action of SU(2) on the space of homogeneous polynomials in z_1, z_2 of degree $2j$ actually make sense for the *complex* group $\mathrm{SL}(2, \mathbf{C})$; we denote these by $D^{j,0}$ and note that the representing matrices (for instance, with respect to the basis $(z_1^r z_2^{2j-r})$) have entries that are *polynomials* in the entries a, b, c, d of the element of $\mathrm{SL}(2, \mathbf{C})$. They are thus *algebraic* or *holomorphic* representations. If C is the complex conjugation on the space of polynomials, then $D^{0,j} := C D^{j,0} C^{-1}$ is again a representation of $\mathrm{SL}(2, \mathbf{C})$ but with *antiholomorphic* matrix entries. It turns out that the representations

$$D^{j,k} := D^{j,0} \otimes D^{0,k}$$

are still *irreducible* and that they are *precisely all the finite-dimensional irreducible representations of* $\mathrm{SL}(2, \mathbf{C})_{\mathbf{R}}$. None of them except the trivial representation $D^{0,0}$ is unitary. This construction is typical; if G is a complex connected Lie group and $G_{\mathbf{R}}$ is G treated as a real Lie group, then the irreducible finite-dimensional representations of $G_{\mathbf{R}}$ are precisely the ones

$$D \otimes \overline{E}$$

where D, E are holomorphic irreducible representations of the complex group G. In our case the restriction of $D^{0,k}$ to SU(2) is still D^k, and so the restriction of $D^{j,k}$ to SU(2) is $D^j \otimes D^k$, whose decomposition is given by the Clebsch-Gordan formula.

In the Clebsch-Gordan formula the type $D^{|j-k|}$ is *minimal*; such minimal types exist canonically and uniquely in the tensor product of two irreducible representations of any complex semisimple Lie group.[17]

1.5. Poincaré Symmetry and Particle Classification

Special relativity was discovered by Einstein in 1905. Working in virtual isolation as a clerk in the Swiss patent office in Berne, Switzerland, he wrote one of the most famous and influential papers in the entire history of science with the deceptive title *On the electrodynamics of moving bodies*, and thereby changed forever our conceptions of space and time. Using beautiful but mathematically very elementary arguments, he demolished the assumptions of Newton and his successors that space and time were absolute. He showed rather that time flows differently for different observers, that moving clocks are slower, and that events that are simultaneous for one observer are not in general simultaneous for another. By making the fundamental assumption that the speed of light in vacuum is constant in all (inertial) frames of reference (i.e., independent of the speed of the source of light), he showed that the change of coordinates between two inertial observers has the form

$$x' = Lx + u\,, \quad x, u \in \mathbf{R}^4\,,$$

where L is a 4×4 real invertible matrix that preserves the quadratic form

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 ;$$

here $x^0 = ct$ where t is the time coordinate, $x^i (i = 1, 2, 3)$ are the space coordinates, and c is the speed of light in a vacuum; if the units are chosen so that the speed of light in vacuum is 1, then x^0 is the time coordinate itself. Such L are called *Lorentz transformations* and form a group denoted by $O(1, 3)$. The fact that the distinction between space and time, which had been a part of all of our thinking for centuries, is *dependent on the observer*, follows from these formulae. It also follows that no particle can travel with a speed greater than the speed of light in vacuum. The transformations for changing from one inertial frame to another given above form the so-called *inhomogeneous Lorentz group*; this is the set of pairs (u, L) with multiplication defined by

$$(u, L)(u', L') = (u + Lu', LL') .$$

It is the *semidirect product* $\mathbf{R}^4 \times' O(1, 3)$. The term *Poincaré group* is usually reserved for $\mathbf{R}^4 \times' SL(2, \mathbf{C})_{\mathbf{R}}$ where $SL(2, \mathbf{C})_{\mathbf{R}}$ is viewed as a covering group of $SO(1, 3)^0$ acting on $\mathbf{R}^4$ through the covering map. $SO(1, 3)^0$ itself is the group of Lorentz matrices $L = (h_{\mu\nu})$ such that $\det(L) = 1$ and $h_{00} > 0$ (since $h_{00}^2 - h_{01}^2 - h_{02}^2 - h_{03}^2 = 1$, $|h_{00}| \geq 1$ always and so, on $SO(1, 3)^0$, it is ≥ 1).

It may be of interest to add a few remarks to this brief discussion of special relativity. The idea that an observer can describe the surrounding world by four coordinates is the starting point of the mathematical treatment of phenomena. This description applies especially convincingly in the part of the world that is close to the observer. Already Kepler and Copernicus had realized that the laws governing the planetary movements take a simple form only when viewed against the background of the distant fixed stars. This meant that a special class of coordinate frames were singled out, namely those in which the distant stars appear to be fixed or moving with uniform speed (certainly not rotating as they would be if seen from the frame of the rotating earth). These are the so-called *inertial frames*, the ones in which Galilei's law of inertia holds, namely, objects (such as the distant stars) on which no forces are acting are at rest or are in uniform motion in a straight line. Nowadays such frames are commonplace, for instance, the frame of a rocket ship that is moving outside the earth's gravitational field so that all objects inside it are weightless, and Galilei's law of inertia is satisfied for all objects in it. Observers defining such frames are called inertial also. If now two inertial observers observe the world, the change of coordinates between their respective frames must be such that the linear character of the trajectories of objects moving uniformly without acceleration must not change. It is a consequence of results in projective geometry that such a transformation has to be *affine*, i.e., of the form

$$x' = Lx + u$$

where u refers to spacetime translation and is a vector in $\mathbf{R}^4$ and L is a real 4×4 invertible matrix. Thus *spacetime is an affine manifold.* It is important to remember that *the affine nature of spacetime is already true without any assumptions on speeds of signals.*

For Newton, space and time were absolute, and the space part, consisting of events that are simultaneous, formed a euclidean space of dimension 3. Thus Newtonian space time is *layered* by equal-time slices. The group of transformations between Newtonian (or Galilean) inertial frames is then the 10-parameter *Galilean group*, which respects this layering and in which L above is restricted to the group generated by spatial orthogonal transformations and *boosts*. Boosts refer to the transformations linking an observer to another who is moving with uniform velocity with respect to the first. They are of the form

$$(x^0)' = x^0, \quad \xi' = \xi + x^0 v$$

where ξ refers to the space coordinates, and v is the velocity vector.

However, in the last years of the nineteenth century there already appeared cracks in the structure of the Newtonian view of the world. The *Michelson-Morley experiment*, designed to discover the relative velocity of the earth in the ether, came up with the result that the relative velocity was 0. Many different mechanistic hypotheses were put forward to reconcile this with known theories, such as the *Lorentz-Fitzgerald contraction*, which asserted that all objects contracted in the ratio

$$1 : \sqrt{1 - \frac{v^2}{c^2}}$$

along the direction of motion, v being the speed of motion and c the constant velocity of light in vacuum. On the other hand, Poincaré observed that the Maxwell equations are *not* invariant under the Galilean group, but rather light behaves as if its invariance group is really the inhomogeneous Lorentz group. So a number of people sensed that some drastic changes were necessary in order to get a consistent picture of the physical world that would include electromagnetic phenomena. It was Einstein who took the decisive step; with a few simple strokes he painted a coherent picture of space and time that has been vindicated by countless experiments over the past century. Indeed, the experiments in high-energy laboratories confirm every day the assumptions of Einstein. He banished the ether, abandoned mechanistic assumptions to "justify" physical laws, and ushered in the era in which the role of the physicist was limited to building mathematical models that explain and predict phenomena. The revolution in thinking that he started was an absolutely essential prerequisite for the second great revolution in twentieth-century science, namely, quantum theory.

Spacetime with the affine structure given by $\mathbf{R}^4$ and equipped with the basic quadratic form

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

is called *Minkowski spacetime* because its clear mathematical formulation as well as a coordinate-independent treatment of electrodynamics in it was first given by Minkowski in a famous talk.[18] At each point of Minkowski spacetime, the future is represented by all the points in a cone with vertex at that point, the so-called *forward light cone* which is the set of all points that can be reached by a signal emitted at that point. (In Galilean spacetime the future is the half-space of points whose time coordinate is greater than the time coordinate of the given point.) One can show (this is a beautiful result of A. D. Aleksandrov[18]) that any bijection of Minkowski spacetime which preserves this cone structure is necessarily affine

$$x' = a(Lx + u)$$

where a is a nonzero constant, L a Lorentz transformation, and $u \in \mathbf{R}^4$. The constant a cannot be asserted to be 1 if one uses only light signals in analyzing the structure of spacetime; indeed, one cannot pin down the basic quadratic form except up to a multiplicative scalar, because the points reached from the origin by light signals satisfy the equation $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0$, which is unaltered by scalar multiplication. But if we consider material particles as well, one can show that the quadratic form is determined absolutely. Thus the transformation between two inertial observers O, O' where O' is moving uniformly in the positive direction of the x-axis of O with velocity $v > 0$ is given in the units where $c = 1$ by

$$x^{0'} = \frac{1}{\sqrt{1-v^2}}(x^0 - vx^1), \quad x^{1'} = \frac{1}{\sqrt{1-v^2}}(-vx^1 + x^1).$$

To get the formula in the usual units one must replace v by v/c and $x^{0'}, x^0$ by ct', ct. It follows from this that the constant a in the earlier formula must be 1. If the direction of motion of O' is arbitrary, the transformation formula is more complicated; it was first obtained by Herglotz. All the remarkable properties of moving observers such as time dilation, space contraction, relativistic composition formula for velocities, and so on, can be derived from the above formula.[19]

The fact that in the treatment of light the quadratic form is determined only up to a scalar means that description of radiation phenomena must be invariant under the much larger *conformal group*. Globally it is nothing more than adding the dilations to the Poincaré group; but conformal transformations can be fully treated only after compactifying spacetime, and then the conformal group becomes $SO(2, 4)$. We shall discuss this later on. The reader can in the meantime think of the corresponding situation in complex geometry where the group of transformations $z \longmapsto az + b$ of $\mathbf{C}$ enlarges to the group of the *fractional linear transformations* $z \longmapsto (az + b)/(cz + d)$ of the extended complex plane $\mathbf{C} \cup \infty$.

Let us now return to quantum theory. To describe a quantum system in a manner compatible with special relativity means that we must have a projective unitary representation of the Poincaré group on the Hilbert space $\mathcal{H}$ of the system. It was proven by Wigner in a famous paper in 1939[20] that *any projective unitary representation of the Poincaré group is unitarizable*. It was also in the same paper that he classified the *physically relevant* irreducible unitary representations of the Poincaré group. If G is a semidirect product $\mathbf{R}^N \times' H$ where H is a simply connected semisimple group acting on $\mathbf{R}^N$ in such a manner that there are no nonzero skew-symmetric invariant bilinear forms (if the action of H on $\mathbf{C}^N$ is irreducible and admits a nonzero symmetric invariant bilinear form then this condition is satisfied), then all projective representations of G are unitarizable; Wigner's theorem is a special case of this (see my book[5]). However, there are groups for which the unitarizability theorem is not true, such as the additive groups of vector spaces of dimension ≥ 2, and, more significantly, the simply connected covering group of the Galilean group. Indeed, for a given simply connected Lie group to have the property that all projective unitaries are unitarizable, the second cohomology of the group with coefficients in the circle group must vanish.

It follows from these remarks that relativistic invariance of a quantum system is encoded by a unitary representation of the Poincaré group. It is natural to postulate that if the system is that of an elementary particle, then the corresponding representation should be irreducible. Thus, a classification of irreducible unitary representations of the Poincaré group will yield a classification of elementary particles that is compatible with special relativity. We shall now describe how the irreducible unitary representations of the Poincaré group are constructed.

Before taking this up I should point out that physicists do not describe symmetries as we have done, by using unitary representations explicitly. Most of the time the Hilbert spaces they work with contain only the most important states of the system, for instance, those that are obtained by repeated application of certain key operators (creation, annihilation) on certain key states (vacuum); this core is usually invariant under the operators of the Lie algebra of the symmetry group and so only these Lie algebra operators are specified. In certain cases the symmetry group, or rather its Lie algebra, is infinite dimensional, such as the Virasoro or affine Lie algebras; in this case there is no natural Lie group and the symmetry is only infinitesimal.

Let P be the Poincaré group,

$$P = \mathbf{R}^4 \times' \mathrm{SL}(2, \mathbf{C}) .$$

Wigner's determination of the irreducible unitary representations of P in 1939 was extended in great depth to a vast class of locally compact semidirect product groups by Mackey (the "Mackey machine"). But the basic ideas already go back to Frobenius, who was a great pioneer in the theory of representations of finite groups. Let

$$G = A \times' H$$

where A is abelian, and H acts on A through automorphisms. The irreducible representations are then classified by a very general scheme, depending on two "parameters" O, σ where O is an orbit of the (dual) action of H on the character group $\widehat{A}$ of A, and σ is an irreducible unitary representation of the stability subgroup in H of a point $\chi \in O$. In order to motivate the Mackey theory better, I shall first discuss the case when G is finite where there are no technical complications.

Let then G be finite and L an irreducible unitary representation of G. We identify A and H with subgroups of G by the maps $a \longmapsto (a, 1)$ and $h \longmapsto (1, h)$. A will then be normal in G and H will act on it by $h, a \longmapsto hah^{-1}$. The restrictions of L to A and H are unitary representations of these groups, which we write as U and V. Thus

$$L(ah) = U(a)V(h) , \quad V(h)U(a)V(h)^{-1} = U(hah^{-1}) , \quad a \in A, \ h \in H .$$

Conversely, if we start with unitary representations U, V of A and H in the same Hilbert space such that

$$(*) \qquad V(h)U(a)V(h)^{-1} = U(hah^{-1}) , \quad a \in A, \ h \in H ,$$

then

$$L : ah \longmapsto U(a)V(h)$$

is a unitary representation of G. Thus we must try to build pairs (U, V) satisfying $(*)$ that are irreducible.

Let $\mathcal{H}$ be the (finite-dimensional) Hilbert space of U, V. Since the $U(a)(a \in A)$ is a set of commuting unitaries, there is an ON basis in which all the $U(a)$ are diagonal. If v is a basis vector, $U(a)v = \chi(a)v$, and it is clear that $\chi \in \widehat{A}$, where, as we mentioned earlier, $\widehat{A}$ is the group of characters of A. So we can write

$$\mathcal{H} = \bigoplus_{\chi \in F} \mathcal{H}_\chi , \quad U(a)v = \chi(a)v , \quad v \in \mathcal{H}_\chi ,$$

where F is a subset of $\widehat{A}$ and the $\mathcal{H}_\chi \neq 0$. The action of H on A gives rise to the dual action of H on $\widehat{A}$ given by $h, \chi \longmapsto h \cdot \chi$ where $(h \cdot \chi)(a) = \chi(h^{-1}ah)$. Since $U(a)V(h)v = V(h)U(h^{-1}ah)v$, it follows that each $V(h)(h \in H)$ moves $\mathcal{H}_\chi$ into $\mathcal{H}_{h \cdot \chi}$. This shows that F is stable under H and that if O is an orbit contained in F, the space $\bigoplus_{\chi \in O} \mathcal{H}_\chi$ is invariant under both A and H and so invariant under (U, V). Since (U, V) is irreducible, this means that F is a single orbit, say O. Let us fix a $\chi \in O$ and let H_χ be the subgroup of all $h \in H$ such that $h \cdot \chi = \chi$. Since $V(h)$ takes $\mathcal{H}_\chi$ to $\mathcal{H}_{h \cdot \chi}$, we see that $\mathcal{H}_\chi$ is stable under H_χ and so defines a unitary representation σ of H_χ. If W is a subspace of $\mathcal{H}_\chi$ invariant under σ, it is clear that $S[W] := \bigoplus_{h \in H} L(h)[W]$ is stable under V. If $W' \perp W$ is another σ-invariant subspace of $\mathcal{H}_\chi$, then $S[W] \perp S[W']$; indeed, if $hH_\chi \neq h'H_\chi$, then $V(h)[W]$ and $V(h')[W']$ are orthogonal because they belong to different $\mathcal{H}_\xi$, while for $hH_\chi = h'H_\chi$ they are orthogonal because $V(h')[W'] = V(h)[W'] \perp V(h)[W]$ from the unitarity of $V(h)$. These remarks prove that σ is irreducible. We have thus defined O, σ corresponding to L.

Notice that we can think of $\mathcal{H}$ as the collection of vector spaces $(\mathcal{H}_\xi)$ parametrized by $\xi \in O$, i.e., as a *vector bundle* over O. A *section* of the bundle is a family $(v(\xi))$ where $v(\xi) \in \mathcal{H}_\xi$ for all $\xi \in O$. Under componentwise addition these sections form a vector space that is isomorphic to $\mathcal{H}$ by the map $(v(\xi)) \mapsto \sum_\xi v(\xi)$. The action of V on $\mathcal{H}$ takes $\mathcal{H}_\xi$ to $\mathcal{H}_{h \cdot \xi}$ and so can be viewed as an action of H on the bundle compatible with its action on O. The stabilizer H_χ then acts on $\mathcal{H}_\chi$. Thus irreducible pairs (U, V) are completely determined by such vector bundles on the orbits of H. We call them H-bundles.

Suppose that we have two H-bundles $(\mathcal{H}_\xi)$ and $(\mathcal{K}_\xi)$ on O such that the representations of H_χ on $\mathcal{H}_\chi$ and $\mathcal{K}_\chi$ are equivalent by an isomorphism $v \longmapsto v'$. We claim that we can extend this to an isomorphism of the bundles that commutes with the action of H. In fact, there is just one possible way to define the extension: it should take $h[v]$ to $h[v']$ for $h \in H$ and $v \in \mathcal{H}_\chi$. So the claim will be proved if we show that this is well-defined. But suppose that $h[v] = h_1[v_1]$ where $h, h_1 \in H$ and $v, v_1 \in \mathcal{H}_\chi$. Then $h \cdot \chi = h_1 \cdot \chi$ and so $h_1 = hk$ for some $k \in H_\chi$. But then $h[v] = h_1[v_1] = hk[v_1]$ so that $v = k[v_1]$, and so

$$h_1[v_1'] = hk[v_1'] = h[(k[v_1])'] = h[v'] .$$

It only remains to show that any pair (O, σ) gives rise to an H-bundle for which these are the corresponding objects. We define the vector bundle $\mathcal{V}$ over O as the quotient of the trivial bundle $H \times \mathcal{H}_\chi$ on H by a natural equivalence relation that will make the quotient a vector bundle over O. More precisely,

$$\mathcal{V} = H \times \mathcal{H}_\chi / \sim$$
$$(h, v) \sim (h', v') \iff h' = hk,\ v' = \sigma(k)^{-1} v \quad \text{for some } k \in H_\chi,$$

Note that $(h, v) \longmapsto hH_\chi$ gives a well-defined map of $\mathcal{V}$ to H/H_χ and allows us to view $\mathcal{V}$ as a vector bundle over H/H_χ. The map

$$h, (h', v) \longmapsto (hh', v)$$

then defines an action of H on $\mathcal{V}$ and converts $\mathcal{V}$ into an H-bundle.

The subgroup H_χ is called the *little group*. Thus the irreducible representations of G correspond bijectively (up to equivalence, of course) to H-bundles $\mathcal{V}$ on orbits O of H in $\hat{A}$ such that the action of the little group H_χ at a point $\chi \in O$ on the fiber at χ is irreducible. The scalar product on the fiber vector space at χ that makes the representation σ of H_χ unitary can then be transported by H to get a covariant family of scalar products $((\cdot)_\xi)$ on the fibers of $\mathcal{V}$. $\mathcal{V}$ is thus a *unitary H-bundle*. The sections of $\mathcal{V}$ then form a Hilbert space for the scalar product

$$(s, t) = \sum_{\xi \in O} (s(\xi), t(\xi))_\xi \,.$$

The representation L of G is then given by

$$L(ah) = U(a)V(h)\,,$$
$$(U(a)s)(\xi) = \xi(a)s(\xi)\,, \qquad \xi \in O\,,$$
$$(V(h)s)(\xi) = h\big[s(h^{-1}(\xi))\big]\,, \quad \xi \in O\,.$$

The vector bundle on $O \simeq H/H_\chi$ is determined as soon as σ is given. Indeed, we can replace H_χ by any subgroup H_0 of H. Thus, given any subgroup H_0 of H and a vector space F that is an H_0-module, there is a vector bundle $\mathcal{V}$ on H/H_0 that is an H-bundle whose fiber at the coset H_0 is F. The H-action on $\mathcal{V}$ gives rise to a representation of H on the space of sections of $\mathcal{V}$. If F has a unitary structure, then $\mathcal{V}$ becomes a unitary bundle and the representation of H is unitary. This is called the *representation of H induced by σ*. We shall now give a definition of it without the intervention of the vector bundle; this will be useful later on in situations where the vector bundles are not so easily constructed. Recall that we have defined $\mathcal{V}$ as the set of equivalence classes (h, v) with $h \in H$, $v \in F$. A section is then a map of H into F,

$$s : h \longmapsto s(h)\,, \quad h \in H,\ s(h) \in F\,,$$

such that

$$s(hk) = \sigma(k)^{-1} s(h)\,, \quad k \in H_0,\ h \in H\,,$$

the corresponding section being

$$hH_0 \longmapsto \big[(h, s(h))\big]$$

where $[(h, v)]$ represents the equivalence class of (h, v). A simple calculation shows that the action of $h \in H$ on the section becomes the action

$$s \longmapsto s', \quad s'(h') = s(h^{-1}h'), \quad h' \in H.$$

Thus the space of sections is identified with the space F^σ of functions s from H to F satisfying

$$s(hk) = \sigma(k)^{-1}s(h), \quad h \in H,\ k \in H_0,$$

and the representation $V = V^\sigma$ is just left translation:

$$(V(h)s)(h') = s(h^{-1}h'), \quad h, h' \in H.$$

The defining condition for F^σ is on the right, and so the action from the left does not disturb it. The representation is unitary (if F is unitary) for the scalar product

$$(s, t) = \sum_{h \in H/H_0} (s(h), t(h)).$$

The sum is over the coset space H/H_0 since $(s(h), t(h))$ is really a function on H/H_0.

Apart from technical measure-theoretic points, the theory is the same when G is locally compact and second countable. The second countability is strictly speaking not necessary but is satisfied in all applications, and so there is no sense in not imposing it. In this case the dual group $\widehat{A}$ is also locally compact abelian and second countable, and the action of H on $\widehat{A}$ is continuous. What has to be faced, however, is that there are in general continuum many orbits of H in $\widehat{A}$, and the *space of orbits* may not have good properties. As a result, we can only say that while a given orbit and an irreducible representation of the little group of a point on that orbit still define an irreducible unitary representation of G, there will be still others if the orbit space is not nice in a measure-theoretic sense. So there has to be an additional condition of regularity.

What do we mean by the requirement that the orbit space is nice? Let X be a second countable, locally compact Hausdorff space on which a second countable, locally compact group L acts continuously. Both X and L are separable metric and have their σ-algebras (Borel structures) of Borel sets. These σ-algebras are *standard* in the sense that X and L, equipped with their Borel structures, are Borel-isomorphic to the Borel space of the real line or the set of integers. We now introduce the space Y of the L-orbits in X and the natural map $\pi : X \longrightarrow Y$ that sends any point to the orbit containing it. We can equip Y with the σ-algebra of sets with the property that their preimages are Borel in X. One way to formulate the niceness of Y is to require that Y with this Borel structure is standard. A more heuristic idea is to require that we can enumerate the orbits in some way, namely, that there is a Borel set in X that meets each orbit exactly once. The central theorem in the subject is a remarkable criterion due to Effros for the space of orbits to be nice in any one of these senses. We shall formulate it by first giving a definition. The action of L on X is said to be *regular* if the orbits of L in X are all locally closed. Recall here that a subset Z of a topological space Y is *locally closed* if the following equivalent conditions are satisfied:

(i) Z is open in its closure.
(ii) $Z = C \cap U$ where C is closed and U is open in Y.
(iii) For any $z \in Z$ there is an open neighborhood V of z in Y such that $Z \cap V$ is closed in V.

The significance of the condition of regularity is contained in the following theorem, which is a special case of Effros's work on group actions in the Polonais category.[21]

THEOREM 1.5.1 (Effros) *Let X be a locally compact, second countable Hausdorff space and L a locally compact, second countable group acting continuously on X. Then the following are equivalent*:

(i) *All L-orbits in X are locally closed.*

(ii) *There exists a Borel set $E \subset X$ that meets each L-orbit exactly once.*

(iii) *If $Y = L \setminus X$ is equipped with the quotient topology, the Borel structure of Y consists of all sets $F \subset Y$ whose preimages in X are Borel, and the space Y with this Borel structure is standard.*

(iv) *Every invariant measure on X that is ergodic for L is supported on an orbit and is the unique (up to a normalizing scalar) invariant measure on this orbit.*

REMARK. Conditions (ii) through (iv) are the ones that say that the space of orbits is nice in a measure-theoretic sense. The real depth of the Effros theorem is that this property of niceness of the orbit space, which is *global*, is equivalent to condition (i), which is essentially *local*; it can be verified by looking at each orbit without worrying about the others. If the group L is *compact*, then *all* orbits are closed and so the semidirect product is always regular. The action of $\mathbf{Q}$ on $\mathbf{R}$ by $q, r \longmapsto q + r$ is not regular as Lebesgue first pointed out; indeed, any set meeting each orbit exactly once is not even Lebesgue measurable. There are many other examples of this kind.

To relate this definition of regularity in our setup, we shall say that the *semidirect product $G = A \times' H$ is regular* if the action of H on $\widehat{A}$ is regular. In order to state the main result elegantly we need the concept of *induced representations* in this general context. Let H be a locally compact, second countable group and H_0 a closed subgroup; let $X = H/H_0$. For simplicity we shall assume that H/H_0 has an H-invariant measure, although everything goes through with suitable modifications in the general case. Given a unitary representation σ of H_0 in a Hilbert space F (with norm $|\cdot|$ and scalar product $(\cdot,\cdot)$) we define F^σ to be the space of all (Borel) functions (up to equality almost everywhere) s from H to F such that

$$s(hk) = \sigma(k)^{-1} s(h)$$

for each $k \in H_0$ for almost all $h \in H$, and

$$\|s\|^2 = \int_{H/H_0} |s(h)|^2 \, d\bar{h} < \infty$$

where $d\overline{h}$ is the invariant measure on H/H_0. Under the scalar product

$$(s|t) = \int_{H/H_0} (s(h), t(h))d\overline{h}$$

F^σ is a Hilbert space. If

$$(V^\sigma(h)s)(h') = s(h^{-1}h'), \quad h, h' \in H,$$

then V^σ is a unitary representation of H; it is the representation *induced* by σ.

Under additional assumptions it is possible to exhibit a more geometric definition of the induced representation. Let H be a Lie group and let σ be a unitary *finite-dimensional* representation of H_0 in F. Then one can construct a smooth vector bundle on X with fibers isomorphic to F in the same way as we did in the finite case. The fact that the action of H on X has local sections implies that we have a smooth vector bundle $\mathcal{V}^\sigma$ on H/H_0 admitting an action $h, u \longmapsto h[u]$ of H such that the action of H_0 on the fiber at H_0 is just σ. Using the scalar products on F we can define the structure of a unitary bundle on $\mathcal{V}$. If we assume that X has an H-invariant measure, then we can define the notion of square integrability of sections and form F^σ, the Hilbert space of square integrable sections of $\mathcal{V}^\sigma$. Let us define

$$(V^\sigma(h)s)(x) = h\big[s(h^{-1}(x))\big], \quad h \in H, \ s \in F^\sigma, \ x \in \frac{H}{H_0}.$$

Then V^σ is the induced representation we defined earlier.

THEOREM 1.5.2 (Mackey) *Let $G = A \times' H$ and let O be an orbit of H in $\widehat{A}$. Fix a point χ in O and let H_χ be the stabilizer of χ in H. Let σ be a unitary irreducible representation of H_χ, and let $V = V^\sigma$ be the induced representation of H. For any $a \in A$ let $U(a)$ be the unitary operator on F^σ defined by*

$$(U(a)s)(h) = (h \cdot \chi)(a)s(h), \quad s \in F^\sigma, \ h \in H, \ a \in A.$$

If we define

$$L(ah) = U(a)V^\sigma(h), \quad a \in A, \ h \in H,$$

then $L = L_{O,\sigma}$ is a unitary representation of G that is irreducible. If G is a regular semidirect product, then every irreducible unitary representation of G is of this form. The choice of a different χ in O leads to equivalent representations. Finally,

$$L_{O,\sigma} \simeq L_{O',\sigma'} \iff O = O', \ \sigma \simeq \sigma'$$

(for the same choice of χ).

The subgroup H_χ is called the *little group* at χ.

REMARK. Suppose that G is not a regular semidirect product. Then by the theorem of Effros, there is an H-invariant measure on $\widehat{A}$ that is ergodic but gives measure 0 to all orbits. Let μ be such a measure and let $\mathcal{H} = L^2(\mu)$. Define U, V and L by $L(ah) = U(a)V(h)$ and

$$(U(a)f)(\xi) = \xi(a)f(\xi), \quad (V(h)f)(\xi) = f(h^{-1} \cdot \xi), \quad f \in \mathcal{H}, \ a \in A, h \in H.$$

Then L is an irreducible unitary representation of G that does not arise from any orbit.

The Poincaré Group. Here $A = \mathbf{R}^4$ with coordinates (x_μ), H is the group $\mathrm{SO}(1,3)^0$, and

$$P = \mathbf{R}^4 \times' \mathrm{SO}(1,3)^0 ,$$

the Poincaré group. We identify $\widehat{A}$ with a copy of $\mathbf{R}^4$, which we write as $\mathbf{P}^4$ with coordinates (p_μ), by the map $p = (p_\mu) \longmapsto \chi_p$ where $\chi_p(x) = e^{i\langle x,p\rangle}$, with

$$\langle x, p\rangle = x_0 p_0 - x_1 p_1 - x_2 p_2 - x_3 p_3 .$$

$\mathbf{P}^4$ is the *momentum space*. The dual action of $\mathrm{O}(1,3)$ on $\mathbf{P}^4$ is then the same as its action on $\mathbf{R}^4$. There is one invariant, namely, the quadratic form

$$p_0^2 - p_1^2 - p_2^2 - p_3^2$$

and so the level sets of this form are certainly invariant and fill up $\mathbf{P}^4$. The orbits are obtained by splitting these level sets.

The Orbits $X_m^\pm$. The sets $X_m^\pm$ are defined by

$$X_m^\pm = \left\{ p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2,\ p_0 >< 0 \right\}, \quad m > 0 .$$

These are hyperboloids inside the forward or the backward cone at the origin. Note that $p_0^2 = p_1^2 + p_2^2 + p_3^2 + m^2 > m^2$ on the orbits and so p_0 is either greater than m or less than $-m$ on any of these orbits. The point $(m, 0, 0, 0)$ is the rest frame of a particle of mass m since all the momenta are 0. The little group at $(m, 0, 0, 0)$ is the preimage of $\mathrm{SO}(3)$ in $\mathrm{SL}(2, \mathbf{C})$ and so is $\mathrm{SU}(2)$. The representations of $\mathrm{SU}(2)$ are the D^j ($j \in \frac{1}{2}\mathbf{Z}$), and the corresponding representations of P are denoted by $L_{m,j}^\pm$. There is an antiunitary isomorphism of $L_{m,j}^+$ with $L_{m,j}^-$, allowing the interpretation of the representations defined by the latter as the antiparticle with opposite charge. We write $L_{m,j}$ for the representation $L_{m,j}^+$. It describes a *massive particle*, of mass m and spin j (and, by convention, of negative charge). The representation $L_{m,1/2}$ describes any massive particle of spin $\frac{1}{2}$ such as the *electron*. We note also that there is an invariant measure on the orbit. There are several ways of seeing this. The simplest is to note that in the region $F = \{p_0^2 - p_1^2 - p_2^2 - p_3^2 > 0\}$ the change of coordinates

$$q_0 = p_0^2 - p_1^2 - p_2^2 - p_3^2 > 0 , \quad q_i = p_i,\ i = 1, 2, 3,$$

is a diffeomorphism and we have

$$d^4 p = \frac{1}{2(q_0 + q_1^2 + q_2^2 + q_3^2)^{1/2}}\, d^4 q .$$

Since q_0 is invariant under $\mathrm{SO}(1,3)^0$ we see that for any $m > 0$ the measure

$$d\mu_m^+ = \frac{d^3 p}{2(m^2 + p_1^2 + p_2^2 + p_3^2)^{1/2}}$$

is an invariant measure on X_m^+ where we use the p_i ($i = 1, 2, 3$) as the coordinates for X_m^+ through the map

$$(p_0, p_1, p_2, p_3) \longmapsto (p_1, p_2, p_3) ,$$

which is a diffeomorphism of X_m^+ with $\mathbf{R}^3$.

The Orbits $X_0^{\pm}$. The sets $X_0^{\pm}$ are defined by

$$X_0^{\pm} = \left\{p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0,\ p_0 >< 0\right\}.$$

We determine the little group at $(1, 0, 0, 1)$ (as before, we ignore the orbit where $p_0 < 0$). The points of X_0^+ represent particles traveling with the speed of light. Classically the only such particles are the *photons*. There is no frame where such particles are at rest, contrary to the case of the massive particles. We choose for convenience the point $(1, 0, 0, 1)$. In our identification of $\mathbf{P}^4$ with 2×2 Hermitian matrices, it corresponds to the Hermitian matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

which goes into

$$\begin{pmatrix} 2a\bar{a} & 2a\bar{c} \\ 2\bar{a}c & 2c\bar{c} \end{pmatrix}$$

under the action of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

So the little group is the group of all matrices

$$e_{a,b} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \qquad a, b \in \mathbf{C},\ |a| = 1.$$

This is also a semidirect product of the group of all elements $e_{1,b}$ that is isomorphic to $\mathbf{C}$, and the group of all elements $e_{a,0}$ that is isomorphic to the circle group S; the action defining the semidirect product is

$$a, b \longmapsto a^2 b.$$

So the little group at $(1, 0, 0, 1)$ is the 2-fold cover of the euclidean motion group of the plane, the plane being identified with $\mathbf{C}$. The only *finite-dimensional* unitary irreducible representations of the little group are

$$\sigma_n : \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \longmapsto a^n, \qquad n \in \mathbf{Z}.$$

The corresponding representations are denoted by $L_{0,n}$. The representations

$$L'_{0,n} = L_{0,n} \oplus L_{0,-n}, \qquad n > 0,$$

are called *representations with helicity* $|n|$; they are not irreducible. The representation $L'_{0,2}$ describes the *photon*. The orbit X_0^+ also has an invariant measure (seen by letting $m \to 0^+$ in the formula for the invariant measure on X_m^+), namely,

$$d\mu_0^+ = \frac{d^3 p}{2(p_1^2 + p_2^2 + p_3^2)^{1/2}}$$

is an invariant measure on X_0^+ where we use the p_i $(i = 1, 2, 3)$ as the coordinates for X_0^+ through the map

$$(p_0, p_1, p_2, p_3) \longmapsto (p_1, p_2, p_3),$$

which is a diffeomorphism of X_0^+ with $\mathbf{R}^3 \setminus \{0\}$.

The Orbits Y_m. These are defined by

$$Y_m = \{p_0^2 - p_1^2 - p_2^2 - p_3^2 = -m^2\}, \quad m > 0.$$

The little groups are not compact and these are unphysical, as we shall explain a little later.

The Orbit (0). The orbit is the single point 0, the origin of $\mathbf{P}^4$. The little group is all of SL(2, **C**), and the corresponding representations are just the irreducible unitary representations of SL(2, **C**) viewed as representations of P via the map $P \longrightarrow P/\mathbf{R}^4 \simeq \mathrm{SL}(2, \mathbf{C})$. These are also unphysical except for the trivial one-dimensional representation that models the vacuum.

Let O denote any one of these orbits and H_0 the little group at the point described above of the orbit in question (base point). We shall presently construct smooth vector bundles V over O that are SL(2, **C**)-bundles, namely, which admit an action by SL(2, **C**), written $h, v \mapsto h[v]$, compatible with its action on the orbit, such that the action of the little group H_0 on the fiber at the corresponding base point is a specified irreducible unitary representation of the little group. Let μ be an invariant measure on O. Since the representation of the little group is unitary, the scalar product on the fiber at the base point can be transported by the action of H to scalar products $((\cdot, \cdot)_p, |\cdot|_p)$ on the fibers at all points of O that vary covariantly under the action of H. V thus becomes a unitary bundle. Then the Hilbert space of the representation is the space of sections s such that

$$\|s\|^2 = \int_O |s(p)|_p^2 \, d\mu(p) < \infty$$

and the representation L of the Poincaré group is given as follows:

$$(L(a)s)(p) = e^{i\langle a,p\rangle} s(p), \quad a \in \mathbf{R}^4, \ p \in O,$$

$$(L(h)s)(p) = h[s(h^{-1}p)], \quad h \in \mathrm{SL}(2, \mathbf{C}), \ p \in O.$$

In this model spacetime translations act as multiplication operators. If e_μ is the vector in spacetime with components $\delta_{\mu\nu}$, then

$$(L(te_\mu)s)(p) = e^{itp_\mu} s(p)$$

so that the momentum operators are multiplications:

$$P_\mu : s \longmapsto p_\mu s.$$

We have

$$P_0^2 - P_1^2 - P_2^2 - P_3^2 = m^2,$$

which is the relativistic energy momentum relation. Thus the parameter m may be identified with the *mass* of the particle. This identification makes clear why we excluded the orbits Y_m; they lead to particles with *imaginary* mass. The representations corresponding to the orbit (0) are such that the spacetime translations act trivially in them. So the energy is 0 and the only representation of this type that is physical is the trivial 1-dimensional representation, which represents the *vacuum*.

There is, however, a more serious omission in our discussion for the case $m = 0$. We have considered only the characters σ_n of the little group H_0. This group is a semidirect product

$$P_0 = \mathbf{C} \times' S$$

where S is the circle group acting on $\mathbf{C}$ by

$$a[b] = a^2 b, \quad |a| = 1, \ b \in \mathbf{C}.$$

The Mackey theory can be used to determine all of its unitary representations. The orbits are the circles $|b| = \beta^2$ for $\beta \geq 0$. The orbit $\beta = 0$ corresponds to the representations σ_n, these being the characters of S viewed as representations of H_0. At the points β the little group is (± 1), which has two characters, the trivial one and the one that takes -1 to -1. So there are two irreducible unitaries $\lambda_{\beta,\pm}$ for each $\beta > 0$. Associated to these we have representations $L_{0,\beta,\pm}$, which define particles of mass 0 and infinite helicity, i.e., possessing an internal space of *infinite* dimension. These also have to be excluded because of their unphysical nature.

Representations of the Poincaré Group of Minkowski Space of Arbitrary Dimension. The theory described above goes over with virtually no changes to the case of the Minkowski space $V = \mathbf{R}^{1,D-1}$ of dimension D. Thus

$$G = \mathbf{R}^{1,D-1} \times' H, \quad H = \mathrm{Spin}(1, D-1)$$

where the spin group $\mathrm{Spin}(1, D-1)$ is the universal (= 2-fold) cover of $\mathrm{SO}(1, D-1)^0$ (see Chapter 5 for notation and results on real spin groups). The orbits are classified as before. For the orbits $X_m^{\pm}$ the little group at $(m, 0, \dots, 0)$ is $\mathrm{Spin}(D-1)$. The orbits have the invariant measure

$$d\mu_m^+ = \frac{d^{D-1}p}{(m^2 + p_1^2 + \cdots + p_{D-1}^2)^{1/2}}.$$

The orbits $X_0^{\pm}$ require a little more care because our earlier description of the little groups for the case $D = 4$ used the special model of Hermitian matrices for spacetime.

We write $(e_\mu)_{0 \leq \mu \leq D-1}$ for the standard basis of $V = \mathbf{R}^{1,D-1}$, with $(e_0, e_0) = -(e_j, e_j) = 1 (1 \leq j \leq D-1)$. We wish to determine the little group at the point $q = e_0 + e_{D-1}$. Let ℓ be the line $\mathbf{R}{\cdot}q$, and let H_q be the little group at q, the subgroup of H fixing q. We write H_q' for the stabilizer of q in the group $V \times' \mathrm{SO}(1, D-1)^0$ so that H_q is the lift of H_q' inside G. Clearly, H_q fixes $\ell^\perp$ and so we have the H_q-invariant flag

$$\ell \subset \ell^\perp \subset V.$$

Now ℓ is the radical of the restriction of the metric to $\ell^\perp$, and so the induced metric on $E := \ell^\perp/\ell$ is *strictly negative definite*. We shall now show that there is a natural map

$$H_q' \simeq E \times' \mathrm{SO}(E).$$

Let $h \in H_q'$. Then h induces an element $h^\sim$ of $\mathrm{O}(E)$. We claim first that $h^\sim \in \mathrm{SO}(E)$ and that h induces the identity on $V/\ell^\perp$. Since $\det(h) = 1$ and $\det(h^\sim) =$

± 1, we see that h induces ± 1 on $V/\ell^{\perp}$ and so it is enough to prove that h induces $+1$ on $V/\ell^{\perp}$. Now $e_0 \notin \ell^{\perp}$ and $h{\cdot}e_0 = ae_0 + u$ where $a = \pm 1$ and $u \in \ell^{\perp}$. Then $(e_0, q) = (h{\cdot}e_0, q) = a(q, e_0)$ so that $a = 1$. Since $h{\cdot}e_0 - e_0 \in \ell^{\perp}$, its image in E is well-defined; we write $t(h)$ for it. We thus have a map

$$H'_q \longrightarrow E \times' \mathrm{SO}(E)\,, \quad h \longmapsto (t(h), h^{\sim})\,.$$

It is easy to check that this is a morphism of Lie groups. We assert that this map is injective. Suppose that h is in the kernel of this map so that $h{\cdot}u = u + a(h)q$ for all $u \in \ell^{\perp}$ and $h{\cdot}e_0 = e_0 + b(h)q$. Then $(e_0, e_0) = (h{\cdot}e_0, h{\cdot}e_0) = (e_0, e_0) + 2b(h)(q, e_0)$, giving $b(h) = 0$. Also, $(u, e_0) = (h{\cdot}u, h{\cdot}e_0) = (u, e_0) + a(h)(q, e_0)$, giving $a(h) = 0$. Thus $h = 1$. A simple calculation with the Lie algebra shows that $\mathrm{Lie}(H'_q)$ has the same dimension as $E \times' \mathrm{SO}(E)$. Therefore the map above is an isomorphism of H'_q with $E \times' \mathrm{SO}(E)$.

Let H_q be the stabilizer of q in $V \times' \mathrm{Spin}(1, D-1)$. We shall show that H_q is connected if $D \geq 4$. For this we need to use the theory of Clifford algebras (see Chapter 5). Let $x = e_1 e_2$ and $a_t = \exp tx$. Since $(e_1, e_1) = (e_2, e_2) = -1$, we have $x^2 = -1$ and so $a_t = \cos t{\cdot}1 + \sin t{\cdot}x$. It is obvious that a_t fixes q and so lies in H_q for all t. But for $t = \pi$ we have $a_\pi = -1$. Thus H_q^0 contains the kernel of the map from $\mathrm{Spin}(1, D-1)$ to $\mathrm{SO}(1, D-1)^0$, proving that $H_q = H_q^0$. Thus finally

$$H_q = H_q^0 \simeq E \times' \mathrm{Spin}(E)\,.$$

We have thus shown that for $D \geq 4$, the little group of any point q of X_0^+ is the 2-fold cover of the euclidean motion group of $\ell^{\perp}/\ell$ where $\ell = \mathbf{R}q$, exactly as in the case $D = 4$.

1.6. Vector Bundles and Wave Equations: The Maxwell, Dirac, and Weyl Equations

Two things remain to be done. The first is to construct the representations explicitly by describing the corresponding vector bundles. This will give a description of the states in what is called the *momentum picture*, in which the momentum operators are diagonalized. The physicists also frequently use a description where the states are represented by functions on spacetime and the spacetime group acts naturally on them. Indeed such descriptions are very useful when treating interactions of the particles with other systems such as an external electromagnetic field. In the spacetime picture the states will be formally singled out by a *wave equation*. This description can be obtained from the momentum space representation by taking *Fourier transforms*. Throughout this section Fourier transforms are taken with respect to the Lorentz-invariant scalar product

$$\langle x, p\rangle = \sum \varepsilon_\mu x_\mu p_\mu$$

so that

$$\widehat{u}(x) = \int e^{-i\langle x, p\rangle} u(p) d^4 p\,.$$

In particular, multiplication by p_μ goes over to $i\varepsilon_\mu \partial_\mu$:

$$p_\mu \longrightarrow i\varepsilon_\mu \partial_\mu , \quad \partial_\mu = \frac{\partial}{\partial x_\mu} .$$

Klein-Gordon Equation. As our first example, let us take the simplest particle, one of mass $m \geq 0$ and spin 0. It could be charged or neutral. Here there is no internal space and so the bundle is trivial; the Hilbert space is

$$\mathcal{H}_m^{\pm} = L^2(X_m^{\pm}, \mu_m^{\pm})$$

where $\mu_m^{\pm}$ is the invariant measure on $X_m^{\pm}$. The action of the Poincaré group is as follows:

$$(L(a)f)(p) = e^{i\langle a,p\rangle} f(p) , \quad a \in \mathbf{R}^4 ,$$
$$(L(h)f)(p) = f(h^{-1}p) , \quad h \in \mathrm{SL}(2, \mathbf{C}) .$$

To take Fourier transforms of the f, we view them as distributions on $\mathbf{R}^4$,

$$f d\mu_m^{\pm} : \varphi \longmapsto \int_{\mathbf{P}^4} f\varphi \, d\mu_m^{\pm} , \quad \varphi \in \mathcal{D}(\mathbf{P}^4)$$

where $\mathcal{D}(\mathbf{P}^4)$ is the space of smooth, compactly supported functions on $\mathbf{P}^4$. It is not difficult to show that these distributions, which are actually complex measures, are *tempered*. Indeed, this follows from the easily established fact that the $\mu_m^{\pm}$-measure of a ball of radius R grows at most like a power of R, actually like R^3 in this case. Since the $f d\mu_m^{\pm}$ live on $X_m^{\pm}$, it is immediate that they satisfy the equation

$$\left(p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2\right) \cdot \left(f d\mu_m^{\pm}\right) = 0 .$$

Taking Fourier transforms and writing $\psi = \widehat{f d\mu_m}$, we have

$$\left(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 + m^2\right)\psi = 0 ,$$

which is the so-called *Klein-Gordon* (K-G) *equation*. One can say that the states of the scalar massive particle of mass $m > 0$ are the *tempered* solutions of the K-G equation. On the other hand, if we are given a tempered solution ψ of the K-G equation, it is not difficult to see that $\psi = \widehat{u}$, where u is a distribution that *lives on* X_m. Whether the support of u is confined to one of the two orbits $X_m^{\pm}$ cannot be easily decided in terms of ψ alone. At the same time, from the formula for the action of the spacetime translations we see that the energy operator P_0 is multiplication by p_0, and so the spectrum of P_0 is $\geq m$ on $\mathcal{H}_m^+$ and $\leq -m$ on $\mathcal{H}_m^-$ (the so-called negative energy states). Nowadays, following Dirac (see below), the space $\mathcal{H}_m^-$ is viewed as *antiparticle* charged oppositely to the original particle described by $\mathcal{H}_m^+$. We can combine the two Hilbert spaces $\mathcal{H}_m^{\pm}$ into one,

$$\mathcal{H}_m = \mathcal{H}_m^+ \oplus \mathcal{H}_m^- = L^2(X_m, \mu_m) ,$$

where μ_m is the measure on X_m coinciding with $\mu_m^{\pm}$ on $X_m^{\pm}$, and allow the full symmetry group

$$\mathbf{R}^4 \times' \mathrm{O}(1, 3)$$

to act on $\mathcal{H}_m$. Thus the K-G spinless particle-antiparticle of mass m has this complete symmetry group and the distributions $\psi = \widehat{u}$ $(u \in \mathcal{H}_m)$ satisfy the K-G equation. For any tempered solution ψ we have $\psi = \widehat{u}$ where u lives on X_m; but to define an actual state u must be a measure on X_m absolutely continuous with respect to μ_m and $du/\mu_m \in f \in L^2(X_m, \mu_m)$, the L^2-norm of this derivative being the norm of the state.

Dirac Equation. During the early stages of development of relativistic quantum mechanics the K-G equation was the only equation that described relativistic particles. But Dirac was dissatisfied with this picture. For various reasons connected with difficulties in defining probability densities and currents, he felt that the wave equation should be of the *first order* in time, and hence, as time and space coordinates are to be treated on the same footing, *it should be of the first order in all variables*. He therefore looked for an equation of the form

$$i\Big(\sum_\mu \gamma_\mu \partial_\mu\Big)\psi = m\psi\,.$$

Of course, the K-G equation was not to be abandoned; it was required to follow as a consequence of this equation. Dirac therefore assumed that

$$\Big(\sum_\mu \gamma_\mu \partial_\mu\Big)^2 = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2\,.$$

In this way the differential operator he was looking for would be a sort of *square root* of the K-G operator. Dirac's assumption leads to the relations

$$\gamma_\mu^2 = \varepsilon_\mu\,, \qquad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0\,, \qquad \mu \neq \nu\,,$$

where

$$\varepsilon_\mu = \begin{cases} 1 & \text{if } \mu = 0 \\ -1 & \text{if } \mu = 1, 2, 3. \end{cases}$$

It is now clear that the γ_μ cannot be scalars. Dirac discovered that there is a solution with 4×4 matrices and that this solution is unique up to a similarity. But then the operator

$$D = i\sum_\mu \gamma_\mu \partial_\mu$$

has to operate on *vector functions with four components* so that the Dirac particle automatically has an internal space of dimension 4! D is the famous *Dirac operator*.

We shall follow this procedure of Dirac in constructing the vector bundle on the *full* orbit X_m. We look for objects γ_μ such that

$$\Big(\sum_\mu \gamma_\mu p_\mu\Big)^2 = \sum \varepsilon_\mu p_\mu^2 = \sum_\mu p_\mu p^\mu\,, \qquad p^\mu = \varepsilon_\mu p_\mu\,,$$

giving the relations

$$\gamma_\mu^2 = \varepsilon_\mu\,, \qquad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0\,, \qquad \mu \neq \nu\,.$$

We consider the algebra $\mathcal{C}$ generated by the γ_μ with the relations above. It is called the *Clifford algebra*, which is a generalization of the quaternion algebra. It is of dimension 16 and is isomorphic to a full matrix algebra (this will follow from our explicit formula for the γ's below). Hence it has a *unique* irreducible representation in dimension 4; any representation of $\mathcal{C}$ is a direct sum of copies of this representation. The uniqueness of the 4-dimensional representation means that if γ_μ, γ'_μ are 4×4 matrices satisfying the above relations, there is a 4×4 invertible matrix S such that

$$\gamma'_\mu = S\gamma_\mu S^{-1}$$

for all μ. S is unique up to a scalar multiplier because if S' is another such, then $S'S^{-1}$ commutes with all the γ's and so must be a scalar by Schur's lemma. As a useful application of this principle, we note that given a set (γ_μ), the matrices $(-\gamma_\mu)$ also satisfy the same relations and so there is $S \in \mathrm{GL}(4, \mathbf{C})$ such that

$$-\gamma_\mu = S\gamma_\mu S^{-1}.$$

Because $\gamma_0^2 = 1$ and γ_0 and $-\gamma_0$ are similar, we see that γ_0 has the eigenvalues ± 1 with eigenspaces of dimension 2 each. The same is true of $i\gamma_j$ $(j = 1, 2, 3)$. The γ_μ are the famous *Dirac gamma matrices*. They are a part of a whole yoga of *spinor calculus* (see Chapter 5).

At the risk of being pedantic, let us write Λ for the covering morphism from $\mathrm{SL}(2, \mathbf{C})$ onto $\mathrm{SO}(1, 3)^0$. Consider now a variable point $p = (p_\mu)$. Fix a set of 4×4 gamma matrices γ_μ. Write $p^\mu = \varepsilon_\mu p_\mu$. If $h = (h_{\mu\nu}) \in \mathrm{O}(1, 3)$ and $q = hp$, we have

$$\Big(\sum_\mu p_\mu\gamma_\mu\Big)^2 = \sum_\mu p_\mu p^\mu = \sum_\mu q_\mu q^\mu = \Big(\sum_\mu q_\mu\gamma_\mu\Big)^2 = \Big(\sum_\mu p_\nu\gamma'_\nu\Big)^2$$

where

$$\gamma'_\nu = \sum_\mu h_{\mu\nu}\gamma_\mu.$$

Thus the γ'_μ also satisfy the basic relations and hence there is $S(h) \in \mathrm{GL}(4, \mathbf{C})$ such that

$$S(h)\gamma_\mu S(h)^{-1} = \sum_\nu \gamma_\nu h_{\nu\mu}$$

or, equivalently,

$$S(h)(p\cdot\gamma)S(h)^{-1} = (hp)\cdot\gamma, \quad p\cdot\gamma = \sum_\mu p_\mu\gamma_\mu.$$

From the uniqueness up to a scalar of $S(h)$ and the calculation

$$S(k)S(h)\gamma_\mu S(h)^{-1}S(k)^{-1} = \sum_\rho \gamma_\rho k_{\rho\nu}h_{\nu\mu} = S(kh)\gamma_\mu S(kh)^{-1},$$

we see that $S(k)S(h)$ and $S(kh)$ differ by a scalar. So S defines a homomorphism of $\mathrm{O}(1, 3)$ into the projective group $\mathrm{PGL}(4, \mathbf{C})$. We shall show presently that its restriction to $\mathrm{O}(1, 3)^0$ comes from a representation of $\mathrm{SL}(2, \mathbf{C})$ and that this representation is unique.

For this we shall exhibit a set of γ's and compute the representation S explicitly. Since we want to involve the full symmetry group $O(1,3)$ rather than its connected component, we shall first enlarge $SL(2, \mathbf{C})$ to a group $O(1,3)^{\sim}$ so $SL(2,\mathbf{C})$ is the connected component of $O(1,3)^{\sim}$ and we have a natural 2-fold covering map Λ from $O(1,3)^{\sim}$ to $O(1,3)$. To do this, notice that $O(1,3)$ is the semidirect product

$$O(1,3) = O(1,3)^0 \times' I$$

where

$$I \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2 = \{1, I_s, I_t, I_{st}\},$$

the I's being the inversions in space, time, and spacetime. Since $SL(2, \mathbf{C})$ is simply connected, we can view I (uniquely) as acting on it compatibly with the covering map onto $O(1,3)^0$. This means that for any inversion I_r $(r = s, t, st)$, $g \longmapsto I_r[g]$ is an automorphism of $SL(2, \mathbf{C})$ such that $\Lambda(I_r[g]) = I_r\Lambda(g)I_r$. We can then define

$$O(1,3)^{\sim} = SL(2, \mathbf{C}) \times' I$$

and get a 2-fold cover

$$\Lambda : O(1,3)^{\sim} \longrightarrow O(1,3).$$

Let us introduce the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$\sigma_j^2 = 1, \quad \sigma_j\sigma_k + \sigma_k\sigma_j = 0, \ j \neq k.$$

If we then take

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3,$$

where 1 refers to the 2×2 identity matrix, then we have a set of γ's satisfying the relations we need. It is easy to check that the γ_μ act irreducibly.

Let us write $\mathbf{p} = (p_1, p_2, p_3)$ and $p = (p_0, \mathbf{p})$ and let $\mathbf{s} = (\sigma_1, \sigma_2, \sigma_3)$. Then, writing $\mathbf{p}\cdot\mathbf{s} = p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3$ we have

$$p\cdot\gamma = \sum_\mu p_\mu\gamma_\mu = \begin{pmatrix} 0 & p_0 1 + \mathbf{p}\cdot\mathbf{s} \\ p_0 1 - \mathbf{p}\cdot\mathbf{s} & 0 \end{pmatrix}.$$

On the other hand,

$$p_0 1 + \mathbf{p}\cdot\mathbf{s} = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}$$

so that, with $*$ denoting adjoints,

$$g(p_0 1 + \mathbf{p}\cdot\mathbf{s})g^* = q_0 1 + \mathbf{q}\cdot\mathbf{s}, \quad q = \Lambda(g)p, \quad g \in SL(2, \mathbf{C}).$$

Now $\det(p_0 1 + \mathbf{p}\cdot\mathbf{s}) = p^2$ where $p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2$ and so

$$(p_0 1 + \mathbf{p}\cdot\mathbf{s})^{-1} = (p^2)^{-1}(p_0 1 - \mathbf{p}\cdot\mathbf{s})$$

from which we get

$$g^{*-1}(p_0 1 - \mathbf{p}\cdot\mathbf{s})g^{-1} = q_0 1 - \mathbf{q}\cdot\mathbf{s}.$$

From this we get at once that

$$\begin{pmatrix} g & 0 \\ 0 & g^{*-1} \end{pmatrix} \left(\sum_\mu p_\mu \gamma_\mu\right) \begin{pmatrix} g^{-1} & 0 \\ 0 & g^{*} \end{pmatrix} = \sum_\mu q_\mu \gamma_\mu , \quad q = \Lambda(g)p .$$

Since this is precisely the defining relation for $S(g)$, we get

$$S(g) = \begin{pmatrix} g & 0 \\ 0 & g^{*-1} \end{pmatrix} .$$

We would like to extend S to include the inversions also. A simple calculation shows that we can take

$$S(I_s) = \pm \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} , \quad S(I_t) = \pm \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix} , \quad S(I_{st}) = \pm \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} .$$

The uniqueness of S follows from the fact that SL(2, $\mathbf{C}$) has only the trivial representation in dimension 1. Notice that with any choices $S(I_s)S(I_t) = -S(I_t)S(I_s)$ so that these choices always define the unique irreducible projective representation of $I \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$ in dimension 2 tensored by $\mathbf{C}^2$. A simple calculation shows that

$$S(I_r[g]) = S(I_r)S(g)S(I_r)$$

since both sides are examples of a representation S' satisfying the relations

$$S'(g)(p\cdot\gamma)S'(g)^{-1} = (\Lambda(I_r[g])p)\cdot\gamma .$$

If we define

$$S(I_r g) = S(I_r)S(g) , \quad r = s, t, st, g \in \mathrm{SL}(2, \mathbf{C}) ,$$

we see that S is a double-valued representation of $\mathrm{O}(1,3)^{\sim}$ that restricts on SL(2, $\mathbf{C}$) to a representation. We have thus proven the following:

LEMMA 1.6.1 *Let γ_μ be defined as above. Then there is a double-valued* (d.v.) *representation S of* $\mathrm{O}(1,3)^{\sim}$ *in dimension* 4 *restricting to a representation on* SL(2, $\mathbf{C}$) *such that*

$$S(h)(p\cdot\gamma)S(h)^{-1} = \sum_\mu q_\mu \gamma_\mu , \quad q = \Lambda(h)p .$$

The restriction of S to SL(2, $\mathbf{C}$) *is unique and is given by*

$$S(g) = \begin{pmatrix} g & 0 \\ 0 & g^{*-1} \end{pmatrix} .$$

The d.v. representation S defines a d.v. action of $\mathrm{O}(1,3)^{\sim}$ on the trivial bundle

$$T = X \times \mathbf{C}^4 , \quad X = X_m^+ ,$$

by

$$g, (p, v) \longmapsto (\Lambda(g)p, S(g)v) .$$

Define now

$$\mathbf{D}_m(p) = \{v \in \mathbf{C}^4 \mid (p\cdot\gamma)v = mv\} .$$

If $p = \Lambda(g)p^0$ where p^0 is the base point with coordinates $(m, 0, 0, 0)$, we have $S(g)(m\gamma_0)S(g)^{-1} = \sum_\mu p_\mu \gamma_\mu$. Hence $\sum p_\mu \gamma_\mu$ is semisimple for all (p_μ) with

$\sum_\mu p_\mu p^\mu = m^2 > 0$ and its eigenspaces for the eigenvalues $\pm m$ are of dimension 2. In particular, all the spaces $\mathbf{D}_m(p)$ have dimension 2 and

$$S(g)[\mathbf{D}_m(p)] = \mathbf{D}_m(\Lambda(g)p) .$$

This shows that the spaces $\mathbf{D}_m(p)$ define a *subbundle* $\mathbf{D}_m$ of T of rank 2, stable under the d.v. action of $\mathrm{O}(1,3)^\sim$ given by

$$(p, v) \longmapsto (\Lambda(g)p, S(h)v) , \quad h \in \mathrm{O}(1,3)^\sim .$$

One may call $\mathbf{D}_m$ the *Dirac bundle* on X_m.

The stabilizer of $\pm p^0 = (\pm m, 0, 0, 0)$ within $\mathrm{SL}(2, \mathbf{C})$ is $\mathrm{SU}(2)$, and it acts by

$$\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \quad g \in \mathrm{SU}(2) .$$

It commutes with γ_0 and so leaves invariant the spaces $\mathbf{D}_m(\pm p^0)$ where it acts like the representation **2**. The standard scalar product on $\mathbf{C}^4$ is invariant under $\mathrm{SU}(2)$ and so induces an invariant scalar product on $\mathbf{D}_m(\pm p^0)$. The inversions I_r either preserve the spaces and are unitary on them ($r = st$) or exchange them in a unitary manner ($r = s, t$). We may then transport this scalar product to all the fibers $\mathbf{D}_m(\pm p)$ on X_m covariantly. We thus obtain a Hermitian bundle on X_m on which the action of $\mathrm{SL}(2, \mathbf{C})$ is unitary. The inversions preserve this Hermitian structure and so the action of the entire group $\mathrm{O}(1,3)^\sim$ is unitary.

The Hilbert space of square integrable sections of the bundle $\mathbf{D}_m$ then carries a projective unitary representation of $\mathrm{O}(1,3)^\sim$ whose restriction to $\mathrm{SL}(2, \mathbf{C})$ is

$$L_{m,1/2} := L^+_{m,1/2} \oplus L^-_{m,1/2} .$$

Identifying sections s with measures $s\,d\mu_m$ and taking Fourier transforms, we get the *Dirac equation*

$$i\Big(\sum_\mu \varepsilon_\mu \gamma_\mu \partial_\mu\Big)\psi = m\psi \quad \text{or} \quad i\Big(\sum_\mu \gamma^\mu \partial_\mu\Big)\psi = m\psi .$$

As before, we shall regard $\mathcal{H}_m$ as describing the particle-antiparticle of mass m.

Write any section ψ of T in the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_j : X_m \longrightarrow \mathbf{C}^2 .$$

Since

$$(p_0 1 + \mathbf{p}\cdot\mathbf{s})(p_0 1 - \mathbf{p}\cdot\mathbf{s}) = p^2 1 ,$$

it follows that

$$\begin{pmatrix} \psi_1(p) \\ \psi_2(p) \end{pmatrix} \in \mathbf{D}_m(p) \Leftrightarrow \psi_2(p) = m^{-1}(p_0 1 - \mathbf{p}\cdot\mathbf{s})\psi_1(p) .$$

Hence

$$\begin{pmatrix} v_1 \\ v \end{pmatrix} \longmapsto v_1$$

gives a bundle isomorphism of $\mathbf{D}_m$ with the *trivial bundle* $V_m = X_m \times \mathbf{C}^2$ in such a manner that the action of the Poincaré group on $\mathbf{D}_m$ goes over to the action L'_m on V_m defined by

$$\big(L'_m(u, g)\psi_1\big)(p) = e^{i\langle u,p\rangle} g\psi_1(\Lambda(g)^{-1}p) .$$

The spinor field ψ_1, which is a section of the SL(2, $\mathbf{C}$)-bundle V_m, is usually called a 2-*component spinor*. It was first treated systematically by van der Waerden.

Holes and Antimatter. Let us go back to the description of the states of the electron by the Dirac wave equation

$$i\sum_{\mu} \gamma^{\mu}\partial_{\mu}\psi = m\psi .$$

The Hilbert space $\mathcal{H}_m$ carries a (projective) action of the full group of automorphisms of Minkowski spacetime. Now $\mathcal{H}_m = \mathcal{H}_m^+ \oplus \mathcal{H}_m^-$, and it is clear as in the case of the K-G equation that the spectrum of the energy operator P_0, which is multiplication by p_0, is greater than 0 on $\mathcal{H}_m^+$ and less than 0 on $\mathcal{H}_m^-$. The states in $\mathcal{H}_m^{\pm}$ are usually called the *positive and negative energy states*. As long as the electron is free, its state will be in $\mathcal{H}^+$, but as soon as it is placed in a magnetic field, transitions to negative energy states cannot be excluded. That this does not happen was a big problem to be solved at the time Dirac proposed his equation. It was in order to explain the meaning of the negative energy states that Dirac invented his *hole theory*, which asserts that all the negative energy states are occupied, and transition to them is possible only when one of these states becomes available for occupation as a *hole*. The holes were then interpreted by him as positive energy particles of charge opposite to that of the electron. This led him to predict the existence of a new particle, the *positron*. Shortly after Dirac made his prediction, the positron was discovered by Anderson. Eventually, with the discovery of the antiproton and other antiparticles, it became clear that all particles have their *antiparticles*, which are constituents of *antimatter*. (However, the overwhelming preponderance of matter over antimatter in the universe probably depends on conditions that were prevalent in the early evolution of the universe.) The discovery of antimatter is regarded by physicists as one of the greatest achievements of physics of all time, and consequently the stature of Dirac in the physics pantheon rivals that of Newton and Einstein.

As an interesting historical footnote, when Dirac proposed that particles of positive charge should correspond to the holes, he thought that these should be protons, which were the only particles of positive charge known at that time (circa 1929); it was Weyl who pointed out that *symmetry requirements force the hole to have the same mass as the electron and so the new particle cannot be the proton but a positively charged particle with the same mass as the electron*, nowadays called the *positron*. Eventually this prediction of Dirac was confirmed when Anderson exhibited the track of a positron. In retrospect one knows that at the time of Anderson's discovery, Blackett apparently had three tracks of the positron in his experiments but was hesitant to announce them because he felt more evidence was necessary. Anderson at Caltech had only one track and had no hesitation in announcing it!

We shall now construct the bundles for the representations $L^{\pm}_{0,N}$, the zero mass equations.

Maxwell Equation for the Photon. We consider first the case $N = 2$. We start with the *tangent bundle* F of the cone X^{+}_{0}. The action of the Lorentz group on the cone lifts to an action on F. The tangent space at $(1, 0, 0, 1)$ consists of all $(\xi_0, \xi_1, \xi_2, \xi_0)$. The ambient metric on this space is $-(\xi_1^2 + \xi_2^2)$, which is ≤ 0 but degenerate, and the null vectors are multiples of $(1, 0, 0, 1)$. In the basis

$$v_0 = (1, 0, 0, 1)\,, \quad v_1 = (0, 1, 0, 0)\,, \quad v_2 = (0, 0, 1, 0)\,,$$

the action of the little group at $p^0 = (1, 0, 0, 1)$ is

$$\begin{pmatrix} e^{i\theta} & b \\ 0 & e^{-i\theta} \end{pmatrix} : v_0 \mapsto v_0, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Let R be the subbundle of F whose fiber at p is the line $\mathbf{R}p$; this is the line bundle whose fiber at p is the space of null vectors at p for the induced metric on the tangent space F_p at p. Let F^+ be the quotient bundle F/R. The metric on the fibers of F then descends to a *positive definite* metric on the fibers of F^+ and the representation carried by the square integrable (with respect to μ_0^+) sections of F^+ is $L_{0,2} \oplus L_{0,-2}$. We may regard the sections of F^+ as vectors $a = (a_\mu)$ with four components satisfying

$$\sum \varepsilon_\mu p_\mu a_\mu = 0$$

but identifying sections $a = (a_\mu)$ and $a' = (a'_\mu)$ by

$$a \sim a' \iff p \wedge (a - a') = 0\,.$$

Taking Fourier transforms and writing $A_\mu = \varepsilon_\mu \widehat{a_\mu}$, we get

$$\mathfrak{D} A_\mu = 0\,, \qquad \operatorname{div}_L A = 0\,,$$

with

$$A \sim A' \iff d(A - A') = 0\,.$$

These are just the Maxwell equations in the Lorentz gauge. It is thus natural to call $L_{0,2} \oplus L_{0,-2}$ the *photon representation.* Thus *the one-particle photon equations are already the Maxwell equations.* However, one must remember that the Maxwell equations deal with real vector potentials and the photon equations deal with complex potentials. But because the tangent bundle is real, the real sections define a real form of $L_{0,2} \oplus L_{0,-2}$, and so our identification of the two equations is quite reasonable. The helicity of the photon is ± 1 and the two values correspond to left and right circular polarizations.

The fact that the equations describing the photon are the same as Maxwell's equation is very important. In Dirac's theory of radiation he quantized the classical wave equation of Maxwell and found that the states of the (free) quantized electromagnetic field thus obtained were the same as one would obtain by treating a system of photons with Bose-Einstein statistics, i.e., by replacing the one-photon Hilbert space by the *symmetric algebra* over it (see Section 7 below). This was then interpreted by him as an expression of the wave-particle duality of light. Since the Maxwell equations are already the equations describing a free photon, the process of going from a single photon to a system of several photons was called *the second quantization.*

Weyl Equation for the Neutrino. One can make $m = 0$ in the Dirac bundle and get the bundle N on the light cone. However, more care is necessary because for $p \in X_0 = X_0^+ \cup X_0^-$ the operator $p{\cdot}\gamma$ is nilpotent: $(p{\cdot}\gamma)^2 = 0$. Let

$$N(p) = \{v \in \mathbf{C}^4 \mid (p{\cdot}\gamma)v = 0\}.$$

For $p \in X_0$, $p{\cdot}\gamma$ is conjugate by an element of $\mathrm{SL}(2, \mathbf{C})$ to $\pm(\gamma_0 + \gamma_3)$. But $(\gamma_0+\gamma_3)^2 = 0$, and its null space is spanned by e_0, e_3 in $\mathbf{C}^4$. Hence $\dim(N(p)) = 2$. Thus the $N(p)$ define a smooth subbundle N of $X_0 \times \mathbf{C}^4$ stable under $\mathrm{O}(1, 3)^{\sim}$.

For $p \in X_0$ we have

$$\begin{pmatrix} v_- \\ v_+ \end{pmatrix} \in N(p) \Leftrightarrow (p_0 1 \pm \mathbf{p}{\cdot}\mathbf{s})v_\pm = 0\,, \qquad v_\pm \in \mathbf{C}^2\,.$$

We write

$$\ell_\pm(p) = \{v \in \mathbf{C}^2 \mid (p_0 1 \pm \mathbf{p}{\cdot}\mathbf{s})v = 0\}\,.$$

Since we have, for $g \in \mathrm{SL}(2, \mathbf{C})$,

$$\begin{aligned} g(p_0 1 + \mathbf{p}{\cdot}\mathbf{s})g^* &= q_0 1 + \mathbf{q}{\cdot}\mathbf{s}\,, \\ g*^{-1}(p_0 1 - \mathbf{p}{\cdot}\mathbf{s})g^{-1} &= q_0 1 - \mathbf{q}{\cdot}\mathbf{s}\,, \end{aligned} \qquad q = \Lambda(g)p\,,$$

it follows that

$$v \in \ell_-(p) \Longleftrightarrow gv \in \ell_-(\Lambda(g)p)\,, \qquad v \in \ell_+(p) \Longleftrightarrow g^{*^{-1}}v \in \ell_+(\Lambda(g)p)\,.$$

This shows that the $\ell_\pm(p)$ define *line bundles* $\mathbf{W}_{0,\pm}$ that are homogeneous for the action of $\mathrm{SL}(2, \mathbf{C})$ defined by

$$(\mathbf{W}_{0,-}) : (p, v) \longmapsto (\Lambda(g)p, gv)\,, \qquad (\mathbf{W}_{0,+}) : (p, v) \longmapsto (\Lambda(g)p, g^{*^{-1}}v)\,.$$

We then have an isomorphism

$$\ell_-(p) \oplus \ell_+(p) \simeq N(p)\,, \qquad (u, v) \longmapsto \begin{pmatrix} u \\ v \end{pmatrix},$$

which gives an $\mathrm{SL}(2, \mathbf{C})$-equivariant bundle isomorphism

$$\mathbf{W}_{0,-} \oplus \mathbf{W}_{0,+} \simeq N\,.$$

We shall identify $\mathbf{W}_{0,\pm}$ as subbundles of N and denote their restrictions to $X_0^\pm$ by $\mathbf{W}_{0,\pm}^\pm$. The bundles $\mathbf{W}_{0,\pm}^\pm$ may be appropriately called the *Weyl bundles* since the equations satisfied by the Fourier transforms of their sections were first discovered by Weyl and proposed by him as the equations that the neutrinos should satisfy.

Let us compute the little group actions at $\pm p^0 = \pm(1, 0, 0, 1)$. The little group at p^0 is

$$\begin{pmatrix} a & b \\ 0 & \overline{a} \end{pmatrix}, \qquad |a| = 1\,.$$

Further, $\ell_\pm(p^0)$ are spanned by e_3 and e_0, respectively, and the actions are easily computed to be

$$\begin{pmatrix} a & b \\ 0 & \bar{a} \end{pmatrix} : v \longmapsto a^{\mp} v\,.$$

So the representations defined by the $W^{\pm}_{0,\pm}$, the restrictions of $W_{0,\pm}$ to $X^{\pm}_0$, are $L^{\pm}_{0,\mp 1}$. The calculations are the same at $-p^0$. The restriction to the one-dimensional spaces $\ell_\pm(\pm p^0)$ of the standard norm in $\mathbf{C}^4$ transported by the group action now gives the invariant Hermitian structures on the Weyl bundles that is invariant under the action of the Poincaré group.

It must be noticed that the *Weyl bundles are invariant under spacetime inversion but not invariant under the action of either space or time inversions.* In fact, we have

$$I_s, I_t : \mathbf{W}^{\pm}_{0,\pm} \longrightarrow \mathbf{W}^{\pm}_{0,\mp}\,, \qquad I_{st} : \mathbf{W}^{\pm}_{0,\pm} \longrightarrow \mathbf{W}^{\pm}_{0,\pm}\,.$$

Let us now take a closer look at the elements of $\ell_\pm(p)$. We have

$$u \in \ell_+(p) \Longleftrightarrow (\mathbf{p}\cdot\mathbf{s})u = -p_0 u\,.$$

For $p_0 > 0$ or $p_0 < 0$, respectively, we have $p_0 = \pm|\mathbf{p}|$ and so we have

$$u \in \ell_+(p) \Longleftrightarrow (\mathbf{p}{\cdot}\mathbf{s})u = \begin{cases} -|\mathbf{p}|u & \text{if } p_0 > 0 \\ +|\mathbf{p}|u & \text{if } p_0 < 0, \end{cases}$$

showing that the direction of the spin is *antiparallel to the momentum* for $p_0 > 0$ and *parallel to the momentum* for $p_0 < 0$. Similarly, for $u \in \ell_+(p)$, we have that the spin and momentum are parallel for $p_0 > 0$ and antiparallel for $p_0 < 0$. Let us refer to the case where the spin and momentum are antiparallel (resp., parallel) as *left-handed* (resp., *right-handed*). It follows that the bundles $\mathbf{W}^{+}_{0,+}$ and $\mathbf{W}^{-}_{0,+}$ represent, respectively, the left-handed neutrinos and right-handed antineutrinos, while $\mathbf{W}^{+}_{0,-}$ and $\mathbf{W}^{-}_{0,-}$ represent, respectively, the right-handed neutrinos and left-handed antineutrinos.

By taking Fourier transforms of sections of these bundles, we get the *2-component Weyl equations for the neutrino-antineutrino pairs*, namely,

$$(\partial_0 - \nabla{\cdot}\mathbf{s})\psi_+ = 0$$

for the wave functions of the left-neutrino–right-antineutrino pairs and

$$(\partial_0 + \nabla{\cdot}\mathbf{s})\psi_- = 0$$

for the wave functions of the right-neutrino–left-antineutrino pairs. Under space inversion the two equations are interchanged.

Weyl proposed these 2-component equations for the zero mass spin $\frac{1}{2}$ particles in 1929. At that time they were rejected by Pauli because of their lack of invariance with respect to space inversion. Indeed, it was always a basic principle that the wave equations should be invariant under *all* Lorentz transformations, not just those in the connected component. In particular, invariance under space inversion, also called *parity conservation*, was demanded. In the mid 1950s, in experiments performed by Wu following a famous suggestion by Yang and Lee that the neutrinos did not have the parity conservation property, it was found that the neutrinos emitted during beta decay had a preferred orientation. Experimental evidence further indicated that the spin is always antiparallel to the momentum for the neutrinos so that the neutrinos are *always left-handed.* After Wu's experiment, Landau and Salam proposed that the Weyl equation, namely,

$$(\partial_0 - \nabla\cdot\mathbf{s})\psi_\pm = 0\,,$$

for the left-handed neutrino–right-handed antineutrino pairs be restored as the equation satisfied by the neutrino. It is this equation that now governs massless particles, not only in Minkowski spacetime but also in curved spacetime.

It is clear from the entire discussion that in the course of quantization, classical particles acquire internal spaces and symmetries (little groups). Thus classically only the photons travel with the speed of light, but quantum theory allows many more, such as the neutrinos (although there are some recent indications that the neutrinos have a very small but positive mass).

L. Schwartz's Direct Approach to Wave Equations and Hilbert Spaces of Distributions on Spacetime. The method of first getting the bundles in momentum space and then obtaining the wave equations by Fourier transforms that we have followed above is indirect. It is natural to ask if one can construct the wave equations and the Hilbert spaces directly on spacetime. This was carried out by L. Schwartz in a beautiful memoir.[22] Schwartz determined all Hilbert subspaces $\mathcal{H}$ of the space $\mathcal{D}'(M)$ of distributions on Minkowski spacetime M, with scalar or vector values such that

- the natural inclusion $\mathcal{H} \hookrightarrow \mathcal{D}'(M)$ is continuous and
- the natural action of the Poincaré group on $\mathcal{D}'(M)$ leaves $\mathcal{H}$ invariant and induces a unitary representation on it.

Not surprisingly, his classification is the same as the Wigner one. However, by focusing attention on distributions on spacetime, his analysis reveals how restrictive the requirements of Hilbert structure, unitarity, and Poincaré invariance are. For instance, translation invariance already implies that all elements of $\mathcal{H}$ are tempered.

The analysis of Schwartz does not exhaust the subject of wave equations. Indeed, the same representation is obtained by wave equations that look very different formally, and the different versions are important in interpretation. One can formulate the general notion of a relativistic wave equation and try to classify them. Many people have worked on this problem, and the results in some sense are still not definitive. For a discussion of these aspects, see the book by Velo and Wightman.[23]

1.7. Bosons and Fermions

The concept of bosons and fermions arises when one wishes to treat a system of *identical particles* quantum mechanically. If $\mathcal{S}_i$ $(1 \le i \le N)$ are quantum systems, then the natural way to represent the states of $\mathcal{S}$, the system composed of the $\mathcal{S}_i$, is to take its Hilbert space as $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ where $\mathcal{H}_i$ is the Hilbert space of $\mathcal{S}_i$. Thus if $\mathcal{S}_i$ is in the state ψ_i, then $\mathcal{S}$ is in the state $\psi_1 \otimes \cdots \otimes \psi_N$. However, if $\mathcal{S}_i = \mathcal{S}_0$ is the system of a particle such as the electron or the photon, the quantum-theoretic description of $\mathcal{S}$ must take into account the purely quantum-theoretic phenomenon that the *particles are indistinguishable*. For instance, the theory must allow for the *Pauli exclusion principle*, according to which two electrons cannot occupy the same state. It was found that the correct way to describe an N-electron system is to use the space $\Lambda^N(\mathcal{K})$ of *antisymmetric tensors* in $\mathcal{K}^{\otimes N}$, $\mathcal{K}$ being the space of states of a single electron. Similarly, in dealing with a system of N photons the correct space was found to be $S^N(\mathcal{K})$, the space of *symmetric tensors* in $\mathcal{K}^{\otimes N}$ where $\mathcal{K}$ is now the space of states of a single photon. Let P^a, P^s be the orthogonal projection from the full-tensor product onto the space of antisymmetric and symmetric tensors. If ψ is in $\mathcal{K}$, $P^a(\psi \otimes \cdots \otimes \psi)$ is the state in which all the electrons are in the state ψ, and because this is 0 for $N \geq 2$, we see that this model is compatible with the exclusion principle. But for photons $\psi \otimes \cdots \otimes \psi$ is already symmetric and nonzero and represents the state where all the photons are in the state ψ. There is nothing that forbids this, and in fact this possibility is crucial in the construction of the laser.

Experimentally it has been found that all particles belong to one of these two categories, namely, those whose N-particle systems are modeled by $\Lambda^N(\mathcal{K})$, and those whose N-particle systems are modeled by $S^N(\mathcal{K})$. The former type of particles are called *fermions* after the great Italian physicist E. Fermi, and the latter kind *bosons*, after the great Indian physicist S. N. Bose.

Let us now look more closely into the mathematical description of systems of identical particles without assuming anything except the indistinguishability of the particles. Let $\mathcal{K}$ be the Hilbert space of states of a single particle. If there are N particles, then, to start with, the Hilbert space of states of the N-particle system may be taken as $\mathcal{H}_N = \mathcal{K}^{\otimes N}$. This space carries an obvious action of the group S_N, the group of permutations of $\{1, \dots, N\}$. The indistinguishability of the particles may then be expressed by saying that the observable algebra is the centralizer of S_N, the algebra $\mathcal{O}$ of all bounded operators commuting with S_N.

We shall now decompose $\mathcal{H}_N$ with respect to the action of S_N. For any irreducible representation π of S_N of dimension $d(\pi)$, let P_π be the operator

$$P_\pi = \frac{d(\pi)}{N!} \sum_{s \in S_N} \chi_\pi(s)^{\mathrm{conj}} s$$

where we write s for the operator corresponding to s and χ_π is the character of π. It is easy to see that P_π is a projection, and in fact, it is the projection on the span of all subspaces that transform according to π under the action of S_N. Let

$$\mathcal{H}_N[\pi] = P_\pi \mathcal{H}_N .$$

If M is any subspace of $\mathcal{H}_N$ transforming according to π and $L \in \mathcal{O}$, then $L[M]$ is either 0 or transforms according to π and so $\mathcal{H}_N[\pi]$ is stable under L. Thus any element of the observable algebra $\mathcal{O}$ commutes with each P_π. We now have a decomposition

$$\mathcal{H}_N[\pi] \simeq V[\pi] \otimes \mathcal{K}_\pi$$

where:

(i) $V[\pi]$ is a model for π.

(ii) An operator of $\mathcal{H}_N[\pi]$ lies in $\mathcal{O}$ if and only if it is of the form $1 \otimes A$ where A is an operator of $\mathcal{K}_\pi$.

Hence the observable algebra $\mathcal{O}$ has the decomposition

$$\mathcal{O} = \bigoplus_\pi (1 \otimes \mathcal{O}_\pi)$$

where $\mathcal{O}_\pi$ is the full algebra of all bounded operators on $\mathcal{K}_\pi$. This is a situation that we have discussed earlier. After that discussion it is clear that the states may now be identified with

$$\bigcup_\pi \mathbf{P}(\mathcal{K}_\pi)\,.$$

We thus have *superselection sectors* corresponding to the various π. There will be no superposition between states belonging to different sectors. For fixed π if we take the Hilbert space $\mathcal{K}_\pi$ as the Hilbert space of states, we get a model for treating N identical particles *obeying* π*-statistics*.

The group S_N has two basic representations: the trivial one and the alternating one, the latter being the representation in dimension 1 that sends each permutation s to its signature $\mathrm{sgn}(s)$. We then get the two projections

$$\frac{1}{N!}\sum_s s\,, \quad \frac{1}{N!}\sum_s \mathrm{sgn}(s).$$

The corresponding spaces $\mathcal{H}_N[\pi]$ are, respectively,

$$S^N(\mathcal{K})\,, \quad \Lambda^N(\mathcal{K})\,,$$

where $S^N(\mathcal{K})$ is the space of *symmetric tensors* and $\Lambda^N(\mathcal{K})$ is the space of *antisymmetric tensors*. In physics only these two types of statistics have been encountered. Particles for which the states are represented by $S^N(\mathcal{K})$, the bosons, are said to obey the *Bose-Einstein statistics*, while particles for which the states are represented by $\Lambda^N(\mathcal{K})$, the fermions, are said to obey the *Fermi-Dirac statistics*.

The essential question at this stage is the following: can one tell, from the properties of a *single* particle, the type of statistics obeyed by a system consisting of several particles of the same type? It turns out, and this is a consequence of special relativity, that the statistics are completely determined by the spin of the particle. This is the so-called *spin-statistics theorem* in relativistic quantum field theory; it says that particles with half-integral spin are fermions and obey the statistics corresponding to the signature representation (Fermi-Dirac statistics), while particles with integral spin are bosons and obey the statistics corresponding to the trivial representation (Bose-Einstein statistics). Thus for a system of N particles with half-integral spin, we use $\Lambda^N(\mathcal{K})$ as the Hilbert space of states and for a system of N particles with integral spin, we use $S^N(\mathcal{K})$ as the Hilbert space of states. This distinction is of crucial importance in the theory of superconductivity; properties of bulk matter differ spectacularly depending on whether we are dealing with matter formed of particles of integral or half-integral spin.

1.8. Supersymmetry as the Symmetry of a $\mathbf{Z}_2$-Graded Geometry

In a quantum field theory that contains interacting particles of both spin parities, the Hilbert space $\mathcal{K}$ of 1-particle states has a decomposition

$$\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$$

where $\mathcal{K}_0$ (resp., $\mathcal{K}_1$) is the space of states where there is one boson (resp., one fermion). The N-particle space is then

$$\mathcal{H}_N = \bigoplus_{1 \le d \le N} S^d(\mathcal{K}_0) \otimes \Lambda^{N-d}(\mathcal{K}_1) .$$

The full Hilbert space in which the particle number is not fixed is then

$$\mathcal{H} = S(\mathcal{K}_0) \otimes \Lambda(\mathcal{K}_1) .$$

People slowly realized that it would be advantageous to have a single unified framework in which there would be no necessity to treat separately the bosonic and fermionic cases* and that the unified treatment would result in increased clarity and understanding. Eventually the algebraic aspects of such a unified theory came to be seen as a linear theory where all (linear) objects are systematically graded by $\mathbf{Z}_2$, just as the Hilbert space of 1-particles above was graded into bosonic and fermionic parts. In the meantime, in the early 1970s, several groups of physicists (Gol′fand-Likhtman, Volkov-Akulov, and Wess-Zumino) almost simultaneously came up with a notion of infinitesimal symmetry of such graded spaces, and viewed it as a type of symmetry not encountered hitherto—namely, a symmetry that sent bosonic states into fermionic states and vice versa. These symmetries were called *supersymmetries*, and, remarkably, they depended on parameters consisting of both usual variables and variables from a *Grassmann algebra*. The appearance of the Grassmann or exterior algebra is related to the circumstance that in quantum field theory the Fermi fields obey not commutation rules but anticommutation rules. It was soon realized (Salam and Strathdee) that a systematic theory of spaces with usual and Grassmann coordinates could be developed in great depth, and that classical field theory on these *superspaces* would lead, upon quantization, to supersymmetric quantum field theories and gauge theories (Wess, Zumino, Ferrara, Salam, and Strathdee). Then in 1976 a supersymmetric extension of Einstein's theory of gravitation (supergravity) was discovered by Ferrara, Freedman, and van Nieuwenhuizen, and a little later by Deser and Zumino. With this discovery supersymmetry became the natural context for seeking a unified field theory.[24]

The infinitesimal supersymmetries discovered by the physicists would become the *super Lie algebras* and their corresponding groups the *super Lie groups*. A systematic theory of super Lie algebras culminating in the classification of simple super Lie algebras over an algebraically closed field was carried out by V. Kac shortly after the first papers on supergroups and algebras appeared in the physics literature.[25] Of course, as long as one can work with the infinitesimal picture the theory of super Lie algebras is perfectly adequate, and it is immediately accessible because it is a linear theory and is modeled after the well-known theory of simple

*Separate but equal facilities are inherently discriminatory!

Lie algebras. However, for a fuller understanding the deeper (nonlinear) theory of supermanifolds and super Lie groups cannot be evaded. First introduced by Salam and Strathdee, the concept of supermanifolds and super Lie groups was developed by the physicists. Among mathematicians, one of the earliest pioneering efforts was that of F. A. Berezin,[26] who tried to emphasize the idea that this was a new branch of algebra and analysis. Among the important more recent works exposing the theory for mathematicians are the articles and books of B. De Witt, D. Leites, and Yu. Manin as well as the expositions of P. Deligne and J. Morgan[27] and the lectures of D. Freed.[28]

Informally speaking, a supermanifold is a manifold in which the coordinate functions are smooth functions of the usual coordinates as well as the so-called odd variables. The simplest example of this is $\mathbf{R}^p$, on which the coordinate functions form the algebra $C^\infty(\mathbf{R}^p) \otimes \mathbf{R}[\theta_1, \dots, \theta_q]$ where θ_j $(1 \le j \le q)$ are odd variables that are anticommuting, i.e., satisfy

$$\theta_j\theta_k + \theta_k\theta_j = 0\,, \quad 1 \le j, k \le q\,.$$

Such a space is denoted by $\mathbf{R}^{p|q}$, and the general supermanifold is obtained by gluing spaces that locally look like $\mathbf{R}^{p|q}$. While this definition imitates that of smooth manifolds with obvious variants in the analytic and holomorphic categories, there is a striking difference: the odd variables are *not numerical* in the sense that they all have the value 0. So they are more subtle, and a supermanifold is more like a scheme of Grothendieck on which the rings of the structure sheaf have nilpotent elements; indeed, any odd element in the structure sheaf of a supermanifold is nilpotent. So a supermanifold is a generalization of a manifold at a fundamental level. However, the techniques for studying supermanifolds did not have to be freshly created; one could simply follow the ideas of Grothendieck's theory of schemes. Supermanifolds are more general than schemes because the coordinate rings are not commutative but *supercommutative*, a mildly noncommutative variant of commutative rings. If we drop the smoothness requirement in a supermanifold, we obtain a superscheme that is the most general geometric object yet constructed. Super Lie groups, and more generally supergroup schemes, are the symmetries of these objects.

1.9. References

1. It is not often that one speaks of beauty and coherence in the physical description of the world around us. Supersymmetry, as a theory with almost no experimental confirmation, is very much like mathematics in that it relies on internal esthetics to a much larger extent than traditional physical theories, although this situation may change with the advent of supercolliders in the TeV range. The mathematical path to physical insight was the modus operandi for one of the greatest physicists of all time, P. A. M. Dirac. In his famous paper on monopoles, "Quantized singularities in the electromagnetic field" (*Proc. Roy. Soc. London A* 133: 60–72, 1931), Dirac has this to say on this issue:

   The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. . . . Non-euclidean geometry and noncommutative algebra, which were at one time considered to be purely fictions of the mind and pastimes for logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation. The theoretical worker in the future will therefore have to proceed in a more indirect way. The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and *after* each success in this direction, to try to interpret the new mathematical features in terms of physical entities.

Here is another quotation from Y. Nambu (*Broken symmetry: selected papers of Y. Nambu*, T. Eguchi and K. Nishijima, eds., World Scientific, River Edge, N.J., 1995):

The Dirac mode is to invent, so to speak, a new mathematical concept or framework first, and then try to find its relevance in the real world, with the expectation that (in a distorted paraphrasing of Dirac) a mathematically beautiful idea must have been adopted by God. Of course, the question of what constitutes a beautiful and relevant idea is where physics begins to become an art.

I think this second mode is unique to physics among the natural sciences, being most akin to the mode practiced by the mathematicians. Particle physics, in particular, has thrived on the interplay of these two modes. Among examples of this second approach, one may cite such concepts as

Magnetic monopole
Non-Abelian gauge theory
Supersymmetry

On rare occasions, these two modes can become one and the same, as in the cases of Einstein gravity and the Dirac equation.

2. Ferrara, S. Supersimmetria. *L'Italia al CERN*, 369. F. Menzinger, ed. Ufficio Pubblicazioni INFN, Frascati, 1995.

3. Listed below are books on classical physics that are close to the spirit of our discussions.

Classical mechanics:

Arnold, V. I. *Mathematical methods of classical mechanics*. Graduate Texts in Mathematics, 60. Springer, New York–Heidelberg, 1978.

Electromagnetism and gravitation:

Weyl, H. *Space, time, matter*. Dover, New York, 1952.

Quantum mechanics:

Dirac, P. A. M. *Principles of quantum mechanics*. 4th ed. University Press, Oxford, 1958.

von Neumann, J. *Mathematical foundations of quantum mechanics*. Princeton University Press, Princeton, N.J., 1955.

Weyl, H. *The theory of groups and quantum mechanics*. Dover, New York, 1950.

4. The suggestion of Y. Aharanov and Bohm is contained in:

   Aharonov, Y., and Bohm, D. Significance of electromagnetic potentials in the quantum theory. *Phys. Rev. (2)* 115: 485–491, 1959.

   For the experiment of Chambers, see:

   Chambers, R. G. Shift of an electron interference pattern by enclosed magnetic flux. *Phys. Rev. Lett.* 5: 3–5, 1960.

5. The references for foundational matters in quantum mechanics are the following:

   Beltrametti, E. G., and Cassinelli, G. *The logic of quantum mechanics*. Encyclopedia of Mathematics and Its Applications, 15. Addison-Wesley, Reading, Mass., 1981.

   Varadarajan, V. S. *Geometry of quantum theory*. 2d ed. Springer, New York, 1985.

   Wheeler, J. A., and Zurek, W. H., eds. *Quantum theory and measurement*. Princeton Series in Physics. Princeton University Press, Princeton, N.J., 1983.

6. See 3.

7. In the 1950s J. Schwinger wrote a series of papers on the foundations of quantum mechanics that were summarized in a book:

   Schwinger, J. *Quantum kinematics and dynamics*. Benjamin, New York, 1970.

   He discusses his measurement algebra in his paper given on the occasion of the centenary marking Weyl's birth:

   Schwinger, J. Hermann Weyl and quantum kinematics. *Exact sciences and their philosophical foundations*, 107–129. Lang, Frankfurt, 1988.

8. Digernes, T., Varadarajan, V. S., and Varadhan, S. R. S. Finite approximations to quantum systems. *Rev. Math. Phys.* 6(4): 621–648, 1994.

9. Schwinger, J. *Quantum mechanics*. Symbolism of atomic measurements. Springer, Berlin, 2001. Also see his article on the occasion of the Weyl centenary.[7]

10. Arens, R.; Varadarajan, V. S. On the concept of Einstein-Podolsky-Rosen states and their structure. *J. Math. Phys.* 41(2): 638–651, 2000.

11. See the preface to Weyl's book[3] and Schwinger's Weyl centenary address.[7]

12. The pioneering papers are:

    Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D. Deformation theory and quantization. II. Physical applications. *Ann. Physics* 111(1): 111–151, 1978.

Moyal, J. E. Quantum mechanics as a statistical theory. *Math. Proc. Cambridge Philos. Soc.* 45: 99–124, 1949.

The subject of quantization as deformation took off in the 1980s and led directly to the theory of quantum groups.

13. For Wigner's theorem, see my book,[5] p. 104. For more recent discussions, see:

Cassinelli, G., De Vito, E., Lahti, P. J., and Levrero, A. Symmetry groups in quantum mechanics and the theorem of Wigner on the symmetry transformations. *Rev. Math. Phys.* 9(8): 921–941, 1997.

Weinberg, S. *The quantum theory of fields*. I. Foundations, 91. Corrected reprint of the 1995 original. Cambridge University Press, Cambridge, 1996.

14. Weinberg, S. Testing quantum mechanics. *Ann. Physics* 194(2): 336–386, 1989.

15. See Weyl's book[4] for the beginnings of the theory of projective representations of Lie groups and the projective representations of finite groups going back to I. Schur. For the complete theory, see my book[5] and:

Parthasarathy, K. R. *Multipliers on locally compact groups*. Lecture Notes in Mathematics, 93. Springer, Berlin–New York, 1969.

The seminal papers are:

Bargmann, V. On unitary ray representations of continuous groups. *Ann. of Math. (2)* 59: 1–46, 1954.

Mackey, G. W. Les ensembles boréliens et les extensions des groupes. *J. Math. Pures Appl. (9)* 36: 171–178, 1957.

Mackey, G. W. Unitary representations of group extensions. I. *Acta Math.* 99: 265–311, 1958.

For more recent contributions and references, see:

Cassinelli, G., De Vito, E., Lahti, P., and Levrero, A. Symmetries of the quantum state space and group representations. *Rev. Math. Phys.* 10(7): 893–924, 1998.

Divakaran, P. P. Symmetries and quantization. Preprint. Published in *Rev. Math. Phys.* 6(2): 167–205, 1994.

16. For a general historical and mathematical discussion of Weyl's role in the discovery of gauge theory, see:

O'Raifeartaigh, L., ed. *The dawning of gauge theory*. Princeton Series in Physics. Princeton University Press, Princeton, N.J., 1997. This book contains the reprints of some of the most fundamental papers in the subject.

Varadarajan, V. S. Vector bundles and connections in physics and mathematics: some historical remarks. *A tribute to C. S. Seshadri. Perspectives in geometry and representation theory*, 502–541. Hindustan Book Agency, New Delhi, 2003.

17. Parthasarathy, K. R., Ranga Rao, R., and Varadarajan, V. S. Representations of complex semisimple Lie groups and Lie algebras. *Ann. Math.* 85: 383–429, 1967.

18. Minkowski, H. Space and time. *The principle of relativity*. A collection of original memoirs on the special and general theory of relativity. Dover, New York, 1952, 173–191. This is a translation of an address delivered at the 80th Assembly of German Natural Scientists and Physicists, Cologne, 21 September 1908. See *Gesammelte Abhandlungen von Hermann Minkowski*, Chelsea, New York, 1967, 431–444.

Minkowski's address started with these famous words:

> The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth, space by itself, and time by itself, are doomed to fade away in mere shadows, and only a kind of union of the two will preserve an independent reality.

Alexandrov, A. D. A contribution to chronogeometry. *Canad. J. Math.* 19: 1119–1128, 1967.

19. The basic reference (which also contains the Herglotz formula on p. 11) is:

Pauli, W. *Theory of relativity*. Dover, New York, 1981.

For a beautiful exposition of the basic ideas, see:

Schwinger, J. *Einstein's legacy*. Scientific American Library, New York, 1986.

20. Wigner's great 1939 paper is:

Wigner, E. P. Unitary representations of the inhomogeneous Lorentz group. *Ann. Math.* 40: 149–204, 1939.

21. See:

Effros, E. G. Transformation groups and C^*-algebras. *Ann. of Math. (2)* 81: 38–55, 1965.

For further and more recent work, see:

Becker, H., and Kechris, A. *The descriptive set theory of Polish group actions*. London Mathematical Society Lecture Note Series, 232. Cambridge University Press, Cambridge, 1996.

22. See:

Schwartz, L. *Application of distributions to the theory of elementary particles in quantum mechanics*. Gordon and Breach, New York, 1968.

23. Velo, G., and Wightman, A. S., eds. *Invariant wave equations*. Lecture Notes in Physics, 73. Springer, Berlin–New York, 1978.

24. There are two comprehensive collections of basic articles on supersymmetry that are very valuable:

Ferrara, S., ed. *Supersymmetry*, vols. 1, 2. World Scientific, Singapore, 1987.

Salam, A., and Sezgin, E. *Supergravities in diverse dimensions*. World Scientific, Singapore, 1987.

25. Kac, V. G. Lie superalgebras. *Advances in Math.* 26(1): 8–96, 1977.

26. Berezin, F. A. *Introduction to superanalysis*. Mathematical Physics and Applied Mathematics, 9. Reidel, Dordrecht, 1987.

27. The recent expositions close to the spirit of our discussions are as follows.

Kostant, B. Graded manifolds, graded Lie theory, and prequantization. *Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975)*, 177–306. Lecture Notes in Mathematics, 570. Springer, Berlin, 1977.

Leites, D. A. Introduction to the theory of supermanifolds. *Uspekhi Mat. Nauk* 35(1): 3–57, 1980; *Russian Math. Surveys* 35(1): 1–64, 1980.

Manin, Y. I. *Gauge field theory and complex geometry.* Grundlehren der Mathematischen Wissenschaften, 289. Springer, Berlin, 1988.

Deligne, P., and Morgan, J. W. Notes on supersymmetry (following Joseph Bernstein). *Quantum fields and strings: a course for mathematicians, vol. 1 (Princeton, NJ, 1996/1997)*, 41–97. American Mathematical Society, Providence, R.I., 1999.

28. Freed, D. S. *Five lectures on supersymmetry.* American Mathematical Society, Providence, R.I., 1999.

CHAPTER 2

The Concept of a Supermanifold

2.1. Geometry of Physical Space

Someone who is already familiar with the theory of differentiable manifolds or algebraic varieties can be very quickly introduced to the notion of a supermanifold and the concept of supersymmetry. Just as the manifolds and varieties are defined by first starting with local pieces on which the coordinate functions are defined, and then gluing these local pieces together, a supermanifold may be defined as a space on which locally one has coordinates $x^1, \dots, x^n, \theta^1, \dots \theta^r$ where the x^i are the usual commuting coordinates and the θ^j, the anticommuting (fermionic) coordinates, with the various sets of local chats being related by transformations of the appropriate smoothness type. Everything is then done exactly as in the classical theory. Supersymmetries are diffeomorphisms of such spaces and these form super Lie groups. One can construct a theory of differentiation and integration on such spaces and write down equations of motions of particles and fields starting from suitable Lagrangians. If one starts with a supersymmetric Lagrangian, then one obtains an action of the supersymmetric group on the solutions of the field equations thus defined. The stage on which supersymmetric quantum field theory lives is then a superspacetime, either flat or curved. However, such a treatment, in spite of being very practical and having the advantage of getting into the heart of matters very quickly, does not do full justice either to the understanding of the concepts at a deeper level or to comprehending the boldness of the generalization of conventional geometry that is involved here. In this chapter we shall take a more leisurely and foundational approach. We shall try to look more closely at the evolution of the concept of space as a geometrical object starting from Euclid and his plane (and space) and ending with the superspacetimes of the phycisists. This is, however, a very complicated story with multiple themes and replete with many twists and turns and really too complex to be discussed briefly. Nevertheless, the attempt to unravel it will provide (I hope) at least some insight into supergeometry at a fundamental level.

We begin with the evolution of geometry. Geometry is perhaps the most ancient part of mathematics. Euclid is its most celebrated expositor and his *Elements* is still the object of great admiration. Euclid's geometry is an idealized distillation of our experience of the world around us. To his successors all of Euclid's axioms except one appeared to be entirely natural. The exception was the famous axiom of parallels. Indeed, Euclid himself recognized the special nature of lines in a plane that do not meet; this is clear from the fact that he went as far as he could without the parallel axiom and started using it only when it became absolutely indispensable. One of the crucial places where it is necessary to invoke this axiom is in the proof that the sum of the angles of a triangle is equal to two right angles. One may therefore say that starting from Euclid himself the axiom of parallels was the source of a lot of discomfort and hence the object of intense scrutiny. Already Proclus in the fifth century A.D. was quite skeptical of this axiom, and so he might be regarded as one of the earliest figures who thought that an alternative system of geometry was a possibility, or at least that the axiom of parallels should be looked into more closely. One of the first people who started a systematic study of geometry where no assumptions were made about parallel lines was the Italian Jesuit priest Saccheri. Later Legendre made an intense study of the parallel axiom and at one point even thought that he had proven it to be a consequence of the remaining axioms. Eventually he settled for the weaker statement that *the sum of the angles of a triangle is always less than or equal to two right angles, and that the parallel axiom is equivalent to saying that the sum is equal to two right angles*; *and further, that if this is valid just for one triangle, then it is valid for all triangles*. In retrospect, as we shall see later, this result of Legendre would appear as the definitive formulation of the axiom of parallels that characterizes euclidean geometry, inasmuch as it describes the fact that euclidean geometry is *flat*.

Eventually this line of thought led to the discovery of noneuclidean geometry by Bolyai and Lobachevsky, although Gauss, as became clear from his unpublished manuscripts which were discovered after his death, had anticipated them. The discovery of noneuclidean geometry did not end speculations on this subject because it was not at first clear whether the new axioms were self-consistent. However, Klein and Beltrami constructed models for noneuclidean geometry *entirely within* the framework of euclidean geometry, from which it followed that noneuclidean geometry was as self-consistent as euclidean geometry. The question of the consistency of euclidean geometry was, however, not clarified properly till Hilbert came to the scene. He gave the first rigorous presentation of a complete set of axioms of euclidean geometry (using some crucial ideas of Pasch) and proved that its consistency was equivalent to the consistency of arithmetic. What happened after this—the revolution in logic—is quite well-known and is not of concern for us here.

One reason that the discovery of noneuclidean geometry took so long might have been the fact that there was universal belief that euclidean geometry was special because it described the space we live in. Stemming from this uncritical acceptance of the view that the geometry of space is euclidean was the conviction that there was no other geometry. Philosophers like Kant argued that the euclidean nature of space was a fact of nature and the weight of their authority was very powerful. From our perspective we know that the question of the geometry of space is, of course, entirely different from the question of the existence of geometries that are not euclidean. Gauss was the first person who clearly understood the difference between these two questions. In Gauss's Nachlass one can find his computations of the sums of angles of each of the triangles that occurred in his triangulation of the Hanover region, and his conclusion was that the sum was always two right angles within the limits of observational errors. Nevertheless, quite early in his scientific career Gauss became convinced of the possibility of constructing noneuclidean geometries, and in fact constructed the theory of parallels, but because of the fact that the general belief in euclidean geometry was deeply ingrained, Gauss decided not to publish his research in the theory of parallels and the construction of noneuclidean geometries for fear that there would be criticisms of such investigations by people who did not understand these things ("the outcry of the Boeotians").

Riemann took this entire circle of ideas to a completely different level. In his famous inaugural lecture of 1854 he touched on all of the aspects we have mentioned above. He pointed out to start with that a space does not have any structure except that it is a continuum in which points are specified by the values of n coordinates, n being the *dimension* of the space; on such a space one can then impose many geometrical structures. His great insight was that a geometry should be built from the infinitesimal parts. He treated in depth geometries where the distance between pairs of infinitely near points is Pythagorean, formulated the central questions about such geometries, and discovered the set of functions, the sectional curvatures, whose vanishing characterized the geometries that are euclidean, namely, those whose distance function is Pythagorean not only for infinitely near points but even for points that are a finite but small distance apart. If the space is the one we live in, he stated the principle that its geometrical structure could only be determined *empirically*. In fact, he stated explicitly that the question of the geometry of physical space does not make sense independently of physical phenomena, i.e., that space has no geometrical structure until we take into account the physical properties of matter in it, and that this structure can be determined only by measurement. Indeed, he went so far as to say that *the physical matter determined the geometrical structure of space*.

Riemann's ideas constituted a profound departure from the perceptions that had prevailed until that time. In fact, no less an authority than Newton had asserted that space by itself is an absolute entity endowed with euclidean geometric structure, and built his entire theory of motion and celestial gravitation on that premise. Riemann went completely away from this point of view. Thus, for Riemann, space derived its properties from the matter that occupied it, and that the only question that can be studied is whether the physics of the world made its geometry euclidean. It followed from this that only a mixture of geometry and physics could be tested against experience. For instance, measurements of the distance between remote points clearly depend on the assumption that a light ray travels along shortest paths. This merging of geometry and physics, which is a central and dominating theme of modern physics, may thus be traced back to Riemann's inaugural lecture.

Riemann's lecture was very concise; in fact, because it was addressed to a mostly general audience, there was only one formula in the whole paper. This circumstance, together with the fact that the paper was only published some years after his death, had the consequence that it took a long time for his successors to understand what he had discovered and to find proofs and variants for the results he had stated. The elucidation and development of the purely mathematical part of his themes was the achievement of the Italian school of differential geometers. On the other hand, his ideas and speculations on the structure of space were forgotten completely except for a "solitary echo" in the writings of Clifford.[1] This was entirely natural because most mathematicians and physicists were not concerned with philosophical speculations about the structure of space, and Riemann's ideas were unbelievably ahead of his time.

However, the whole situation changed abruptly and fantastically in the early decades of the twentieth century when Einstein discovered the theory of relativity. Einstein showed that physical phenomena already required that one should abandon the concept of space and time as objects existing independently by themselves, and that one must take the view that they are rather *phenomenological* objects, i.e., dependent on phenomena. This is just the Riemannian view except that Einstein arrived at it in a completely independent manner, and space and time were both included in the picture. It followed from Einstein's analysis that the splitting of space and time is not absolute but depends on the way an observer perceives things around oneself. In particular, only *spacetime*, the totality of physical events taking place, has an intrinsic significance, and that only phenomena can determine what its structure is. Einstein's work showed that spacetime was a differential geometric object of great subtlety, indeed, a pseudo-Riemannian manifold of signature $(+,-,-,-)$, and its geometry was noneuclidean. The central fact of Einstein's theory was that *gravitation* is just a manifestation of the *Riemannian curvature* of spacetime. Thus there was a complete fusion of geometry and physics as well as a convincing vindication of the Riemannian view.[1]

Einstein's work, which was completed by 1917, introduced curved spacetime only for discussing gravitation. The questions about the curvature of spacetime did not really have any bearing on the other great area of physics that developed in the twentieth century, namely *quantum theory*. This was because gravitational effects were not important in atomic physics due to the smallness of the masses involved, and so the merging of quantum theory and relativity could be done over *flat*, i.e., Minkowskian spacetime. However, this situation has gradually changed in recent years. The reason for this change lies in the belief that from a fundamental point of view, the world, whether in the small or in the large, is quantum mechanical, and so one should not have one model of spacetime for gravitation and another for atomic phenomena. Now gravitational phenomena become important for particles of atomic dimensions only in distances of the order of 10^{-33} cm, the so-called Planck length, and at such distances the principles of general relativity impose great obstacles to even the measurement of coordinates. Indeed, the calculations that reveal this may be thought of as the real explanations for Riemann's cryptic speculations on the geometry of space in the infinitely small. These ideas slowly led to the realization that radically new models of spacetime were perhaps needed to organize and predict fundamental quantum phenomena at extremely small distances and to unify quantum theory and gravitation. Since the 1970s a series of bold hypotheses have been advanced by physicists to the effect that spacetime at extremely small distances is a geometrical object of a type hitherto not investigated. One of these is what is called *superspace*. Together with the idea that the fundamental objects to be investigated are not point particles but extended objects like strings, the physicists have built a new theory, the theory of superstrings, that appears to offer the best chance of unification of all the fundamental forces. In the remaining sections of this chapter I shall look more closely into the first of the ideas mentioned above, that of superspace.

Superspace is just what we call a supermanifold. As I mentioned at the beginning of Chapter 1, there has been no experimental evidence that spacetime has the structure of a supermanifold. Of course, we are not speaking of direct evidence but verifications, in collision experiments, of some of the consequences of a supergeometric theory of elementary particles (for instance, the finding of the superpartners of known particles). There are reasons to expect, however, that in the next generation of collision experiments to be conducted by the new LHC (Large Hadron Collider), being built by CERN and expected to be operational by about 2005, some of these predictions will be verified. However, no matter what happens with these experiments, the idea of superspace has changed the story of the structure of space completely, and a return to the older point of view appears unlikely.

I must also mention that an even more radical generalization of space as a geometrical object has been emerging in recent years, namely, what people call *noncommutative geometry*. Unlike supergeometry, noncommutative geometry is *not localizable* and so one does not have the picture of space as being built out of its smallest pieces. People have studied the structure of physical theories on such spaces but these are even more remote from the physical world than supergeometric theories.[2]

2.2. Riemann's Inaugural Talk

On June 10, 1854, Riemann gave a talk before the Göttingen faculty that included Gauss, Dedekind, and Weber in the audience. It was the lecture that he had to give in order to regularize his position in the university. It has since become one of the most famous mathematical talks ever given.[3] The title of Riemann's talk was "*Über die Hypothesen, welche der geometrie zu Grunde liegen*" ("On the hypotheses which lie at the foundations of geometry"). The circumstances surrounding the topic of his lecture were themselves very peculiar. Following accepted convention Riemann submitted a list of three topics from which the faculty were supposed to choose the one that he would elaborate in his lecture. The topics were listed in decreasing order of preference, which was also conventional, and he expected that the faculty would select the first on his list. But Gauss, who had the decisive voice in such matters, choose the last one, that was on the foundations of geometry. So, undoubtedly intrigued by what Riemann was going to say on a topic about which he, Gauss, had spent many years thinking, and flouting all tradition, Gauss selected it as the topic of Riemann's lecture. It appears that Riemann was surprised by this turn of events and had to work intensely for a few weeks before his talk was ready. Dedekind has noted that Gauss sat in complete amazement during the lecture, and that when Dedekind, Gauss, and Weber were walking back to the department after the talk, Gauss spoke about his admiration for and astonishment at Riemann's work in terms that Dedekind said he had never observed Gauss to use in talking about the work of any mathematician, past or present[4]. If we remember that this talk contained the sketch of the entire theory of what we now call Riemannian geometry, and that this was brought to an essentially finished form in the few weeks prior to his lecture, then we would have no hesitation in regarding this work of Riemann as one of the greatest intellectual feats of all time in mathematics.

In his work on complex function theory Riemann had already discovered that it is necessary to proceed in stages: first one has to start with a space that has just a topological structure on it, and then impose complex structures on this bare framework. For example, on a torus one can have many inequivalent complex structures; this is just a restatement of the fact that there are many inequivalent fields of elliptic functions, parametrized by the quotient of the upper half-plane by the modular group. In his talk Riemann started with the concept of what we now call an n-dimensional manifold and posed the problem of studying the various geometries that can be defined on them. Riemann was thus aware that on a given manifold there are many possible metric structures, so that the problem of which structure is the one appropriate for physical space required empirical methods for its solution. Now, up to Riemann's time, both euclidean and noneuclidean geometry were defined in completely global terms. Riemann initiated the profound new idea that *geometry should be built from the infinitesimal to the global.* He showed that one should start from the form of the function that gave the distance between *infinitesimally near points*, and then determine distances between finitely separated points by computing the lengths of paths connecting these points and taking the shortest paths. As a special case one has those geometries in which the distance ds^2 (called the *metric*) between the points $(x_1, \ldots, x_n)$ and $(x_1 + dx_1, \ldots, x_n + dx_n)$ is given by the Pythagorean expression

$$ds^2 = \sum_{i,j} g_{ij}(x_1, \ldots, x_n) dx_i \, dx_j ,$$

where the g_{ij} are functions, *not necessarily constant*, on the underlying space with the property the matrix (g_{ij}) is positive definite. Euclidean geometry is characterized by the choice

$$ds^2 = dx_1^2 + \cdots + dx_n^2 .$$

Riemann also discussed briefly the case

$$ds^4 = F(x_1, \ldots, x_n, dx_1, \ldots, dx_n)$$

where F is a homogeneous polynomial of degree 4. For general, not necessarily quadratic, F the geometry that one obtains was treated by Finsler, and such geometries are nowadays called *Finslerian.*[5]

Returning to the case when ds^2 is a quadratic differential form Riemann emphasized that the structure of the metric depends on the choice of coordinates. For example, the euclidean metric takes an entirely different form in polar coordinates. It is natural to call two metrics *equivalent* if one can be obtained from the other by a change of coordinates. Riemann raised the problem of determining invariants of a metric so that two given metrics could be asserted to be equivalent if both of them have the same invariants. For a given metric Riemann introduced its *curvature*, which was a quantity depending on $n(n-1)/2$ variables, and asserted that its vanishing is the necessary and sufficient condition for the metric to be euclidean, i.e., to be equivalent to the euclidean one. The curvature at a point depended on the $n(n-1)/2$ planar directions π at that point, and given any such π, it was the *Gaussian curvature* of the infinitesimal slice of the manifold cut out by π. Obviously, for the euclidean metric, the Gaussian curvature is 0 in all planar directions at all points. Thus Riemann connected his ideas to those of Gauss but at the same generalized Gauss's work to all dimensions; moreover, he discovered the central fact in all of geometry that the euclidean geometries are precisely those that are *flat*, namely, their curvature is 0 in all planar directions at all points. The case when this curvature is a constant $\alpha \neq 0$ in all directions at all points was for him the next important case. In this case he found that for each α there was only one geometry whose ds^2 can be brought to the form

$$ds^2 = \frac{\sum dx_i^2}{[1 + \frac{\alpha}{4}\sum x_i^2]^2}$$

in suitable coordinates. The cases $\alpha >, =, < 0$ lead to *elliptic*, *euclidean*, and *hyperbolic* geometries, the hyperbolic case being the noneuclidean geometry of Bolyai and Lobachevsky. People have since discovered other models for the spaces of constant curvature. For instance, the noneuclidean plane can be modeled by the upper half-plane with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2), \quad y > 0.$$

This is often called the *Poincaré upper half-plane*. In the last part of his lecture Riemann discussed the problem of physical space, namely, the problem of determining the *actual* geometry of *physical space*. He enunciated two bold principles that went completely against the prevailing opinions:

R1: Space does not exist independently of phenomena, and its structure depends on the extent to which we can observe and predict what happens in the physical world.

R2: In its infinitely small parts space may not be accurately described even by the geometrical notions he had developed.

It is highly interesting to read the exact remarks of Riemann and see how prophetic his vision was:

> Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena....
>
> An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been

approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to ensure that this work is not hindered by too restricted concepts, and that the progress in comprehending the connection of things is not obstructed by traditional prejudices.

2.3. Einstein and the Geometry of Spacetime

It took mathematicians over 50 years to comprehend and develop the ideas of Riemann. The Italian school of geometers, notably Ricci, Bianchi, Levi-Civita, and their collaborators, discovered the tensor calculus and covariant differential calculus in terms of which Riemann's work could be most naturally understood and developed further. The curvature became a covariant tensor of rank 4, and its vanishing was equivalent to the metric being euclidean. The efforts of classical mathematicians (Saccheri, Legendre, etc.) who tried to understand the parallel axiom could now be seen as efforts to describe flatness and curvature in terms of the basic constructs of Euclid's axioms. In particular, as the deviation from two right angles of the sum of angles of a triangle is proportional to the curvature, its vanishing is the flatness characteristic of euclidean geometry.

Riemann's vision in R1 became a reality when Einstein discovered the theory of general relativity. However, it turned out that spacetime, not space, was the fundamental intrinsic object and that its structure was to be determined by physical phenomena. Thus this was an affirmation of the Riemannian point of view with the proviso that space was to be replaced by spacetime. Einstein's main discoveries were as follows.

E1: Spacetime is a pseudo-Riemannian manifold; i.e., its metric ds^2 is not euclidean but has the signature $(+, -, -, -)$ at each point.

E2: Gravitation is just the physical manifestation of the curvature of spacetime.

E3: Light travels along geodesics.

The metric of spacetime was not euclidean but has the form

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$$

at each point. This is what is nowadays called a *Lorentzian* structure. Even in the absence of matter the geometry of spacetime could not be asserted to be flat but only *Ricci flat*, i.e., that its Ricci tensor (which can be calculated from the Riemann curvature tensor) is 0. Einstein also suggested ways to put his ideas to test. One of the most famous predictions of his theory was that light rays, traveling along geodesics of the noneuclidean geometry of spacetime, would appear to be bent by the gravitational fields near a star such as the sun. Everyone knows that this was verified during an annular solar eclipse in Sobral off the coast of Brazil in 1919. Since then even more precise verifications have been made using radio astronomy.

As far as I know, however, the data are not accurate enough to decide between Einstein's theory and some alternative ones.

The second of Riemann's themes, which is hinted at in R2, lay dormant till the search for a unified field theory at the quantum level forced the physicists to reconsider the structure of spacetime at extremely small distances. One of the ideas to which their efforts led them was that the geometry of spacetime was supersymmetric with the usual coordinates supplemented by several anticommuting (fermionic) ones. This is a model that reflects the highly volatile structure of spacetime in small regions where one can pass back and forth between bosonic and fermionic particles. Modern string theory takes Riemann's vision even further and replaces the points of spacetime by strings, thereby making the geometry even more noncommutative. However, string theory is still very incomplete; no one knows the mathematical structure of a geometry that is string like at very small distances and approximates Riemannian geometry in the large.

2.4. Mathematical Evolution of the Concept of Space and Its Symmetries

Parallel to the above development of the concept of the geometry of physical space, and in counterpoint to it, was the evolution of the notion of a manifold from the mathematical side. We shall now give a very brief survey of how the concepts of a manifold or space and its symmetries evolved from the *mathematical* point of view.

Riemann Surfaces. The first truly global types of spaces to emerge were the Riemann surfaces. Riemann's work made it clear that the local complex variable z on such spaces did not have any intrinsic significance and that the really interesting questions were global. However, in Riemann's exposition, the Riemann surfaces generally appeared as a device to make multivalued functions on the complex plane single-valued. Thus they were viewed as (ramified) coverings of the (extended) complex plane. This obscured to some extent the intrinsic nature of the theory of functions on Riemann surfaces. It was Felix Klein who understood this clearly and emphasized that Riemann surfaces are independent objects and offer the correct context to study complex function theory.[6]

The first rigorous description of the concept of Riemann surface is due to Weyl. He formulated for the first time, in his famous book published in 1911,[6] the rigorous notion of a Riemann surface as a complex manifold of dimension 1 with local coordinates that are related on overlapping local domains by biholomorphic transformations. Even today, this is the way we think of not only Riemann surfaces but *all* manifolds, smooth, analytic, or complex analytic.

Weyl's work sparked the view that space is characterized by starting with a topological structure and selecting classes of local coordinates at its points. The nature of the space is then determined by the transformations in the usual affine spaces that connect the various local coordinates. If the connecting transformations are holomorphic (resp., real analytic, smooth, C^k), we obtain a holomorphic (resp., real analytic, smooth, C^k) manifold. Starting with this axiomatic view, it is natural to ask if such abstract spaces could be realized as subspaces of conventional affine or projective spaces. This leads to *imbedding theorems*. Depending on which class of spaces one is interested in, these theorems are associated with Whitney (smooth), Morrey (real analytic), Nash (Riemannian), Kodaira (Kähler), and so on.

Riemannian and Affinely Connected Manifolds. In the years following Riemann's epoch-making work the comprehension and dissemination of Riemann's ideas were carried out by Ricci, Levi-Civita, Bianchi, Weyl, and many others. In 1917 Weyl introduced a new theme.[7] He noticed that the geometry of a Riemannian manifold is controlled by the notion of parallel transport introduced by Levi-Civita, and realized that this notion could be taken as a basis for geometry without assuming that it arose from a metric. This was how the notion of a Riemannian manifold was generalized to an affinely connected manifold, i.e., a manifold *equipped with a connection*. Weyl also introduced another notion, namely, that of *conformality*, and discovered that there is a tensor, the so-called Weyl tensor, whose vanishing was equivalent to the space being conformally euclidean.

Groups of Symmetries of Space. Already in euclidean geometry one can see the appearance of transformation groups, although only implicitly. For instance, the proof of congruence of two triangles involves moving one triangle so that it falls exactly on the second triangle. This is an example of a *congruent transformation*. In the analytical model of euclidean geometry the congruent transformations are precisely the elements of the group of rigid motions of the euclidean plane, generated by the translations and rotations. In the Klein model for non-euclidean geometry the group of congruent transformations is the subgroup of the linear transformations of the projective plane that preserve a circle. It was Klein who put the group-theoretic framework in the foreground in his famous *Erlangen Programme* and established the principle that the structure of a geometry was completely determined by the group of congruent transformations belonging to it.

In the decades following Riemann's work a new theme entered this picture when Sophus Lie began the study of transformations groups that were completely general and acted on *arbitrary* manifolds, even when there was no geometrical structure on the manifolds. Roughly speaking, this was a nonlinear version of the group of affine transformations on an affine space. What was original with Lie was that the transformations depended on a *finite set of continuous parameters* and so one could, by differentiating with respect to these parameters, study their action *infinitesimally*. In modern terminology, Lie considered *Lie groups* (what else) acting on smooth manifolds. The action of the group thus gave rise to a vector space of *vector fields* on the manifold that formed an algebraic structure, namely, a *Lie algebra* that completely determined the action of the Lie group. Thus Lie did to group actions what Riemann had done for geometry, i.e., made them infinitesimal. No geometrical structure was involved and Lie's research was based on the theory of differential equations.

Originally Lie wanted to classify all actions of Lie groups on manifolds. But this turned out to be too ambitious, and he had to settle for the study of low-dimensional cases. But he was more successful with the groups themselves, which were viewed as acting on themselves by translations. His work led eventually to the basic theorems of the subject, the so-called fundamental theorems of Lie: namely, that the Lie algebra is an invariant of the group, that it determined the group in a neighborhood of the identity, and that to any Lie algebra one can associate at least a piece of a Lie group near the identity, namely, a *local Lie group*, whose associated Lie algebra is the given one. As for the classification problem the first big step was taken by Killing when he classified the *simple Lie groups*, or rather, following Lie's idea, the *simple Lie algebras*, over the *complex numbers*. However, the true elucidation of this new theme had to wait for the work of Elie Cartan.

Cartan is universally regarded as the greatest differential geometer of his generation. He took differential geometry to an entirely new level using, among other things, the revolutionary technique of "moving frames." But for our purposes it is his work on Lie groups and their associated homogeneous spaces that is of central importance. Building on the earlier but very incomplete work of Killing, Cartan obtained the rigorous classification of all simple Lie algebras over the complex numbers. He went beyond all of his predecessors by making it clear that one had to work with spaces and group actions *globally*. For instance, he established the global version of the so-called third fundamental theorem of Lie, namely, the existence of a global Lie group corresponding to a given Lie algebra. Moreover, he discovered a remarkable class of Riemannian manifolds on which the simple Lie groups over real numbers acted transitively, the so-called *Riemannian symmetric spaces*. Most of the known examples of *homogeneous spaces* were included in this scheme since they are symmetric spaces. With Cartan's work one could say that a fairly complete idea of space and its symmetries was in place from the differential geometric point of view. Cartan's work provided the foundation on which the modern development of general relativity and cosmology could be carried out.

It was during this epoch that de Rham obtained his fundamental results on the cohomology of a differentiable manifold and its relation to the theory of integration of closed exterior differential forms over submanifolds. Of course, this was already familiar in low dimensions where the theory of line and surface integrals, especially the theorems of Green and Stokes, played an important role in classical continuum physics. de Rham's work took these ideas to their proper level of generality and showed how the cohomology is completely determined by the algebra of closed exterior differential forms modulo the exact differential forms. A few years later Hodge went further and showed how, by choosing a Riemannian metric, one can describe all the cohomology by looking at the harmonic forms. Hodge's work led to the deeper understanding of the Maxwell equations and was the precursor of the modern theory of Yang-Mills equations. Hodge also pioneered the study of the topology of algebraic varieties.

Algebraic Geometry. So far we have been concerned with the evolution of the notion of space and its symmetries *from the point of view of differential geometry*. But there was, over the same period of time, a parallel development of geometry from the *algebraic* point of view. Algebraic geometry, of course, is very ancient; since it relies entirely on algebraic operations, it even predates calculus. It underwent a very intensive development in the nineteenth century when first the theory of algebraic curves, and then algebraic surfaces, were developed to a state of perfection. But it was not till the early decades of the twentieth century that the algebraic foundations were clarified and one could formulate the main questions of algebraic geometry with full rigor. This foundational development was mainly due to Zariski and Weil.

One of Riemann's fundamental theorems was that every compact Riemann surface arose as the Riemann surface of some *algebraic function*. It followed from this that there is no difference between the transcendental theory, which stressed topology and integration, and the algebraic theory, which used purely algebraic and geometric methods and worked with algebraic curves. The fact that compact Riemann surfaces and nonsingular algebraic curves were one and the same made a great impression on mathematicians and led to the search for a purely algebraic foundation for Riemann's results. The work of Dedekind and Weber started a more algebraic approach to Riemann's theory, one that was more general because it allowed the possibility to study these objects in characteristic $p > 0$. This led to a true revolution in algebraic geometry. A significant generalization of the idea of an algebraic variety occurred when Weil, as a basis for his proof of the Riemann hypothesis for algebraic curves of arbitrary genus, developed the theory of abstract algebraic varieties in any characteristic and intersection theory on them. The algebraic approach had greater scope, however, because it also automatically included singular objects; this had an influence on the analytic theory and led to the development of *analytic spaces*.

In the theory of general algebraic varieties started by Zariski and Weil and continued by Chevalley, no attempt was made to supply any geometric intuition. The effort to bring the geometric aspects of the theory of algebraic varieties more to the foreground, and to make the theory of algebraic varieties resemble the theory of differentiable manifolds more closely, was pioneered by Serre, who showed in the 1950s that the theory of algebraic varieties could be developed in a completely geometric fashion imitating the theory of complex manifolds. Serre's work revealed the geometric intuition behind the basic theorems. In particular, he showed that one can study the algebraic varieties in any characteristic by the same sheaf-theoretic methods that were introduced by him and Henri Cartan in the theory of complex manifolds, where they had been phenomenally successful.

The foundations of classical algebraic geometry developed up to this time turned out to be entirely adequate to develop the theory of groups that acted on the algebraic varieties. This was done by Chevalley in the 1950s. One of Chevalley's aims was to determine the projective varieties that admitted a *transitive* action by an *affine* algebraic group, and classify both the spaces and groups that are related in this manner. This comes down to the classification of all *simple* algebraic groups. Chevalley discovered that this was essentially the same as the Cartan-Killing classification of simple Lie algebras over **C** except that the classification of simple algebraic groups could be carried out over an algebraically closed field of arbitrary characteristic, directly working with the groups and not through their Lie algebras. This meant that his proofs were new even for the complex case of Cartan and Killing. The standard model of a projective variety with a transitive affine group of automorphisms is the Grassmannian or a flag manifold, and the corresponding group is $\mathrm{SL}(n)$. Chevalley's work went even beyond the classification. He discovered that a simple group is actually an object defined over $\mathbf{Z}$, the ring of integers; for instance, if we start with a complex simple Lie algebra $\mathfrak{g}$ and consider the group G of automorphisms of $\mathfrak{g}$, G is defined by polynomial equations with integer coefficients as a subgroup of $\mathrm{GL}(\mathfrak{g})$. So the classification yields simple groups over *any finite field*, the so-called finite groups of Lie type. It was by this method that Chevalley constructed new simple finite groups. This development led eventually to the classification of finite simple groups.

The theory of Serre was, however, not the end of the story. Dominating the landscape of algebraic geometry at that time (in the 1950s) was a set of conjectures that had been made by Weil in 1949. The conjectures related in an audacious manner the generating function of the number of points of a smooth projective variety over a finite field and its extensions with the complex cohomology of the same variety viewed as a smooth, complex projective manifold (this is only a rough description). For this purpose what was needed was a cohomology theory in *characteristic zero* of varieties defined over fields of *any characteristic*. Serre's theory furnished only a cohomology over the same field as the one over which the varieties were defined, and so was inadequate to attack the problem posed by the Weil conjectures. It was Grothendieck who developed a new and more profound view of algebraic geometry and developed a framework in which a cohomology in characteristic zero could be constructed for varieties defined over any characteristic. The conjectures of Weil were proven to be true by Deligne, who combined the Grothendieck perspective with some profound ideas of his own.

Grothendieck's work started out in an unbelievably modest way as a series of remarks on the paper of Serre that had pioneered the sheaf-theoretic ideas in algebraic geometry. Grothendieck had the audacious idea that the effectiveness of Serre's methods would be enormously enhanced if one associates to *any commutative ring with unit* a geometric object, called its *spectrum*, such that the elements of the ring could be viewed as functions on it. A conspicuous feature of Grothendieck's approach was its emphasis on generality and the consequent use of the functorial and categorical points of view. He invented the notion of a *scheme* in this process as the most general algebraic geometric object that can be constructed, and developed algebraic geometry in a setting in which *all* problems of classical geometry could be formulated and solved. He did this in a monumental series of papers called *Elements*, written in collaboration with Dieudonné, which changed the entire landscape of algebraic geometry. The Grothendieck approach initiated a view of algebraic geometry wherein the algebra and geometry were completely united. By fusing geometry and algebra he brought number theory into the picture, thereby making available for the first time a systematic geometric view of arithmetic problems. The Grothendieck perspective has played a fundamental role in all modern developments since then: in Deligne's solution of the Weil conjectures, in Faltings's solution of the Mordell conjecture, and so on.

One might therefore say that by the 1960s the long evolution of the concept of space had reached its final stage. Space was an object built by gluing local pieces, and depending on what one chooses as local models, one obtained a space that is either smooth and differential geometric or analytic or algebraic.[8]

The Physicists. However, in the 1970s, the physicists added a new chapter to this story, which had seemed to have ended with the schemes of Grothendieck and the analytic spaces. In their quest for a unified field theory of elementary particles and the fundamental forces, the physicists discovered that the Fermi-Bose symmetries that were part of quantum field theory could actually be seen classically if one worked with a suitable generalization of classical manifolds. Their ideas created spaces in which the coordinate functions depended not only on the usual coordinates but also on a certain number of *anticommuting* variables, called the *odd variables*. These odd coordinates would, on quantization, produce fermions obeying the Pauli exclusion principle, so that they may be called *fermionic coordinates*. Physicists like Salam and Strathdee, Wess and Zumino, Ferrara, and many others played a decisive role in these developments. They called these new objects *superspaces* and developed a complete theory including classical field theory on them together with their quantizations. Inspired by these developments, the mathematicians created the general theory of these geometric objects, the *supermanifolds*, that had been constructed informally by hand by the physicists. The most interesting aspect of supermanifolds is that the local coordinate rings are generated over the usual commutative rings by *Grassmann variables*, i.e., variables ξ^k such that ${\xi^k}^2 = 0$ and $\xi^k\xi^\ell = -\xi^\ell\xi^k (k \neq \ell)$. These always have zero numerical values but play a fundamental role in determining the geometry of the space. Thus the supermanifolds resemble the Grothendieck schemes in the sense that the local rings contain nilpotent elements. They are, however, more general on the one hand, since the local rings are not commutative but supercommutative, and more specialized than the schemes in the sense that they are smooth.

The mathematical physicist Berezin was a pioneer in the creation of superalgebra and supergeometry as distinct disciplines in mathematics. He emphasized superalgebraic methods and invented the notion of the *superdeterminant*, nowadays called the *Berezinian*. He made the first attempts in constructing the theory of supermanifolds and super Lie groups and emphasized that this is a new branch of geometry and analysis. Berezin's ideas were further developed by Kostant, Leites, Bernstein, and others who gave expositions of the theory of supermanifolds and their symmetries, namely, the super Lie groups. Kac classified the simple Lie superalgebras and their finite-dimensional representations. Manin, in his book, introduced the general notion of a *superscheme*. A wide-ranging perspective on supergeometry and its symmetries was given by Deligne and Morgan as a part of the volume on quantum field theory and strings.[9]

2.5. Geometry and Algebra

The idea that geometry can be described in algebraic terms is very old and goes back to Descartes. In the nineteenth century it was applied to projective geometry and led to the result that projective geometry, initially described by undefined objects called points, line, planes, and so on, and the incidence relations between them, is just the geometry of subspaces of a vector space over some division ring. However, for what we are discussing it is more appropriate to start with the work of Hilbert on algebraic geometry. Hilbert showed in his famous theorem of zeros that an affine algebraic variety, i.e., a subset of complex euclidean space $\mathbf{C}^n$ given as the set of zeros of a collection of polynomials, could be recovered as the set of homomorphisms of the algebra $A = \mathbf{C}[X_1, \ldots, X_n]/I$ where I is the ideal of polynomials that vanish on the set. In functional analysis this theme of recovering the space from the algebra of functions on it was discovered by Stone and Gel'fand in two different contexts. Stone showed that if B is a Boolean algebra, the space of all maximal filters of B can be given a canonical topology in which it becomes a totally disconnected compact Hausdorff space $X(B)$, and the Boolean algebra of subsets of $X(B)$ that are both open and closed is canonically isomorphic to B. Gel'fand showed that any compact Hausdorff space X can be recovered from the algebra $C(X)$ of complex-valued continuous functions on it as the space of homomorphisms of $C(X)$ into $\mathbf{C}$:

$$X \approx \mathrm{Hom}(C(X), \mathbf{C}) .$$

Inspired by the work of Norbert Wiener on Fourier transforms, Gel'fand introduced the concept of a commutative *Banach algebra* (with unit) and showed that if we associate to any such algebra A its *spectrum*, namely, the set

$$X(A) := \mathrm{Spec}(A) = \mathrm{Hom}(A, \mathbf{C}) ,$$

then the evaluation map

$$a \longmapsto \widehat{a} , \quad \widehat{a}(\xi) = \xi(a) , \quad a \in A, \ \xi \in X(A) ,$$

gives a representation of A as an algebra of continuous functions on $X(A)$ where $X(A)$ is equipped with the compact Hausdorff weak topology. The map

$$a \longmapsto \widehat{a} ,$$

the so-called *Gel'fand transform*; it generalizes the Fourier transform. It is an isomorphism with $C(X(A))$ if and only if A has a star structure defined by a conjugate linear involutive automorphism $a \mapsto a^*$ with the property that $\|aa^*\| = \|a\|^2$. We can thus introduce the following general heuristic principle:

Hilbert-Gel'fand Principle. The geometric structure of a space can be recovered from the commutative algebra of functions on it.

As examples of this correspondence between spaces and the algebras of functions on it, we mention the following:

- compact Hausdorff spaces $\simeq$ commutative Banach $*$-algebras,
- affine algebraic varieties over $\mathbf{C}$ $\simeq$ finitely generated algebras over $\mathbf{C}$ with no nonzero nilpotents, and
- compact Riemann surfaces $\simeq$ finitely generated fields over $\mathbf{C}$ with transcendence degree 1.

However, two important aspects of this correspondence need to be pointed out before we can use it systematically. First, the representation of the elements of the algebra as functions on the spectrum in the general case is not one-to-one.

There may be elements that are nonzero and yet go to 0 in the representation. Thus, already in both the Hilbert and Gel′fand settings, any element a such that $a^r = 0$ for some integer $r > 1$, i.e., a *nilpotent* element, necessarily goes to 0 under any homomorphism into any field or even any ring with no zero divisors, and so its representing function is 0. For instance, $\mathbf{C}[X, Y]/(X)$ is the ring $\mathbf{C}[Y]$ of (polynomial) functions on the line $X = 0$ in the XY-plane, but the map

$$\mathbf{C}[X, Y]/(X^2) \longrightarrow \mathbf{C}[X, Y]/(X) \longrightarrow \mathbf{C}[Y]$$

gives the representation of elements of $\mathbf{C}[X, Y]/(X^2)$ as functions on the line $X = 0$ in which the element X, which is nonzero but whose square is 0, goes to 0. In the Grothendieck theory this phenomenon is not ignored because it contains the mechanism to treat certain fundamental aspects (for instance, infinitesimal) of the representing space. In the example above, $\mathbf{C}[X, Y]/(X^2)$ is the ring of functions on the *double line* $X^2 = 0$ in the XY-plane. The double line is a distinctive geometric object; indeed, when we try to describe the various degenerate forms of a conic, one of the possibilities is a double line. In the Hilbert theory this idea leads to the principle that all algebras of the form $A = \mathbf{C}[X_1, \dots, X_n]/I$, where I is *any* ideal, describe geometric objects; if I is not equal to its own radical, there will be elements p such that $p \notin I$ but $p^n \in I$ for some integer $n \geq 2$, so that such p define nilpotent elements of A. Grothendieck's great insight was to realize that the full force of this correspondence between affine algebraic varieties and commutative rings can be realized only if the notions of an affine variety and functions on it are enlarged so as to make the correspondence between affine varieties and commutative rings with unit bijective, so that the following principle can be relentlessly enforced:

Grothendieck Principle. *Any* commutative ring A is essentially the ring of functions on some space X. The ring is allowed to have nilpotents whose numerical values are 0 but which play an essential role in determining the geometric structure. The functions on X may have to take their values in fields that differ from point to point.

This space, called the *spectrum* of A and denoted by $X(A) = \mathrm{Spec}(A)$, is a much more bizarre object than in the Hilbert or Gel′fand theories, and we shall not elaborate on it any further at this time. It is simply the set of all prime ideals of A, given the Zariski topology. The ring A can be localized, and so one has a sheaf of rings on $X(A)$. Thus $X(A)$ comes with a structure that allows one to consider them as objects in a category, the category of *affine schemes*, and although the objects themselves are very far from intuitive, the entire category has very nice properties. This is one of the reasons that the Grothendieck schemes work so well.[10]

The second aspect of the concept of a manifold or scheme that one has to keep in mind is that it can be *localized*. This is the idea that space should be built up from its smallest parts, and is done, as mentioned above, by investing space with a sheaf of rings on it. Thus space acquires its characteristic features from the sheaf of rings we put on it, appropriately called the *structure sheaf*. The small parts of space are then described by local models. In differential geometry the local models are $\mathbf{R}^n$ or $\mathbf{C}^n$, while in algebraic geometry they are affine schemes that are spectra of commutative rings. The general manifold is then obtained by gluing these local models. The gluing data come from the requirement that when we glue two models, we should establish a correspondence between the structure sheafs on the parts that are to be identified. The end result is then a premanifold or a *prescheme*; the notions of *smooth manifolds* or *schemes* are then obtained by adding a suitable *separation condition*. In the case of manifolds, this is just the condition that the underlying topological space is Hausdorff; for a prescheme X, this is the condition that X is closed in $X \times X$. The gluing process is indispensable because some of the most important geometrical objects are projective or compact and so cannot be described by a single set of coordinates. The geometrical objects thus defined, together with the maps between them, form a category. One of the most important properties of this category is that products exist.

Clearly, the Grothendieck scheme (or prescheme) is an object very far from the classical notion of an algebraic variety over the complex numbers, or even the notion of an algebraic variety in the sense of Serre. It is an index of the genius of Grothendieck that he saw the profound advantages of using the schemes even though at first sight they are rather unappetizing.

To conclude this brief discussion and as a simple illustration, let us consider the case of affine varieties over an algebraically closed field k and ignore the complications coming from nilpotent elements of the structure sheaf. The correspondence here is between Zariski closed subsets of affine space k^n and finitely generated algebras over k which are reduced in the sense that they have no nonzero nilpotents. In this category products exist. Because of this one can define algebraic groups G over k in the usual manner. In terms of the coordinate rings the maps of multiplication, inverse, and the unit element have to be interpreted in terms of the corresponding k-algebra. Thus the k-algebra $A = A(G)$ has a *comultiplication* that is a morphism

$$\Delta : A \longrightarrow A \otimes A\,,$$

a *coinverse*

$$\Sigma : A \longrightarrow A\,,$$

and a *counit*,

$$\Omega : A \longrightarrow k\,,$$

all of which are related by diagrams that dualize the associative law and the properties of the inverse and the unit element. The result is that A is a commutative *Hopf algebra*. Thus the category of algebraic groups over k corresponds to the category of commutative Hopf algebras. For instance, the Hopf algebra corresponding to $\mathrm{GL}(n, k)$ is

$$A = k[a_{ij}, \det^{-1}]$$

with

$$\Delta : a_{ij} \longmapsto \sum_r a_{ir} \otimes a_{rj}\,, \quad \Sigma : a_{ij} \longmapsto a^{ij}\,, \quad \Omega : a_{ij} \longmapsto \delta_{ij}\,.$$

The theory of Serre varieties provides a fully adequate framework for the theory of algebraic groups and their homogeneous spaces.

2.6. A Brief Look Ahead

To go over to the super category, one has to replace systematically all the algebras that occur on the classical theory by algebras that have a $\mathbf{Z}_2$-grading, namely, *superalgebras*. To study supervarieties one then replaces sheaves of commutative algebras by sheaves of supercommutative algebras. Here the *supercommutative algebras* are those for which any two elements either commute or anticommute depending on whether one of them is even or both of them are odd. Just as commutative rings determine geometric objects, supercommutative rings determine supergeometric objects. We give a brief run over the themes that will occupy us in the remaining chapters.

Super Linear Algebra. A *super vector space* V is nothing but a vector space over the field k that is graded by $\mathbf{Z}_2 := \mathbf{Z}/2\mathbf{Z}$, namely,

$$V = V_0 \oplus V_1 .$$

The elements of V_0 (resp., V_1) are called *even* (resp., *odd*). Morphisms between super vector spaces are linear maps that preserve the parity, where the parity function p is 1 on V_1 and 0 on V_0. A *superalgebra* is an algebra A with unit (which is necessarily even) such that the multiplication map $A \otimes A \longrightarrow A$ is a morphism, i.e., $p(ab) = p(a) + p(b)$ for all $a, b \in A$. Here and everywhere else, we shall assume that in any relation in which the parity function appears, the elements are homogeneous (that is, either even or odd), and the validity for nonhomogeneous elements is to be extended by linearity. As an example we mention the definition of supercommutative algebras: a superalgebra A is *supercommutative* if

$$ab = (-1)^{p(a)p(b)} ba , \quad a, b \in A .$$

This differs from the definition of a commutative algebra in the sign factor that appears. This is a special case of what is called the *rule of signs* in superalgebra: *whenever two elements are interchanged in a classical relation, a minus sign appears if both elements are odd.* The simplest example of a supercommutative algebra is the exterior algebra $\Lambda(U)$ of an ordinary vector space U. It is graded by $\mathbf{Z}$ (degree) but becomes a superalgebra if we introduce the coarser $\mathbf{Z}_2$-grading where an element is even or odd if its degree is even or odd. $\Lambda(U)$ is a supercommutative algebra. Linear superalgebra can be developed in almost complete analogy with linear algebra but there are a few interesting differences. Among the most important are the notions of *supertrace* and *superdeterminant* or *Berezinian.* If A is a supercommutative k-algebra and

$$R = \begin{pmatrix} L & M \\ N & P \end{pmatrix}, \quad L, P \text{ even}, \ M, N \text{ odd},$$

where the entries of the matrices are from A, then

$$\operatorname{str}(R) = \operatorname{tr}(L) - \operatorname{tr}(P) ,$$
$$\operatorname{Ber}(R) = \det(L) \det(I - MP^{-1}N) \det(P)^{-1} ,$$

where $\operatorname{Ber}(R)$ is the Berezinian of R. Unlike the classical determinant, the Berezinian is defined only when R is invertible, which is equivalent to the invertibility of L and P as matrices from the commutative k-algebra A_0, but has the important property that

$$\mathrm{Ber}(RR') = \mathrm{Ber}(R)\,\mathrm{Ber}(R'),$$

while for the supertrace we have

$$\mathrm{str}(RR') = \mathrm{str}(R'R).$$

By an *exterior algebra over a commutative k-algebra A* (k a field of characteristic 0) we mean the algebra $A[\theta_1, \dots, \theta_q]$ generated over A by elements

$$\theta_1, \dots, \theta_q$$

with the relations

$$\theta_j^2 = 0, \quad \theta_i\theta_j = -\theta_j\theta_i, , \quad i \neq j.$$

Exterior algebras are supercommutative. It must be remembered, however, that when we view an exterior algebra as a superalgebra, its $\mathbf{Z}$-grading is to be forgotten and only the coarser grading by $\mathbf{Z}_2$ into even and odd elements should be retained. In particular, they admit automorphisms that *do not preserve the original* $\mathbf{Z}$*-degree*. Thus for

$$A = k[\theta_1, \dots, \theta_r], \quad \theta_i\theta_j + \theta_j\theta_i = 0,$$

the map

$$\theta_1 \mapsto \theta_1 + \theta_1\theta_2, \theta_i \mapsto \theta_i, \quad i > 1,$$

extends to an automorphism of A that does not preserve the original $\mathbf{Z}$-grading. The existence of such automorphisms is the ingredient that invests supergeometry with its distinctive flavor.

Supermanifolds. The concept of a smooth supermanifold, say over $\mathbf{R}$, is now easy to define. A supermanifold X is just an ordinary manifold such that on sufficiently small open subsets U of it the supercoordinate ring $R(U)$ is isomorphic to a supercommutative exterior algebra of the form $C^\infty(U)[\theta_1, \dots, \theta_q]$. The integer q is independent of U and if p is the usual dimension of X, its dimension as a supermanifold is $p|q$. However, this is not the same as an exterior bundle over the ordinary manifold X; for instance, the supermanifold $\mathbf{R}^{1|2}$ has the coordinate rings $C^\infty(U)[\theta_1, \theta_2]$ but the map

$$t, \theta_1, \theta_2 \longmapsto t + \theta_1\theta_2, \theta_1, \theta_2$$

defines a superdiffeomorphism of the supermanifold but not of an exterior bundle over $\mathbf{R}$. If U is an open set in $\mathbf{R}^p$, then $U^{p|q}$ is the supermanifold whose coordinate rings are $C^\infty(U)[\theta_1, \dots, \theta_q]$. Replacing the smooth functions by real analytic or complex analytic manifolds, we have the concept of a real analytic or a complex analytic supermanifold. Unfortunately, it is not possible to define supermanifolds of class C^k for finite k because one needs the full Taylor expansion to make sense of morphisms like the one defined above. If we replace these special exterior algebras by more general supercommuting rings, we obtain the concept of a *superscheme* that generalizes the concept of a scheme.

A brief comparison between manifolds and supermanifolds is useful. The coordinate rings on manifolds are commutative, those on a supermanifold are supercommutative. However, *because the odd elements of any exterior algebra are always nilpotent*, the concept of a supermanifold is closer to that of a scheme than that of a manifold. So the techniques of studying supermanifolds are variants of those used in the study of schemes, and so more sophisticated than the corresponding ones in the theory of manifolds.

Super Lie Groups. A *super Lie group* is a group object in the category of supermanifolds. An *affine superalgebraic group* is a group object in the category of affine supervarieties. In analogy with the classical case these are the supervarieties whose coordinate algebras are *super Hopf algebras*. Here are two examples:

$\mathbf{R}^{1|1}$: The group law is given (symbolically) by

$$(t^1, \theta^1)\cdot(t^2, \theta^2) = (t^1 + t^2 + \theta^1\theta^2, \theta^1 + \theta^2)\,.$$

$\mathrm{GL}(p|q)$: Symbolically, this is the group of block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the entries are treated as coordinates, those of A and D being even and those of B and C odd. The group law is just matrix multiplication.

It may be puzzling that the group law is given so informally in the above examples. The simplest way to interpret them is to stay in the algebraic rather than the smooth category and view the formulae as defining the automorphisms of the corresponding exterior algebras. Actually, one can use the same symbolic description in all cases by utilizing the notion of *functors of points*. The idea is that any object M in the category under discussion is determined completely by the functor that associates to any object N the set $\mathrm{Hom}(N, M)$; the elements of $\mathrm{Hom}(N, M)$ are called the N-points of M. Viewed in this manner, affine supergroups are functors from the category of supercommutative rings to the category of groups, which are representable by a supercommutative Hopf algebra. Thus $\mathbf{R}^{1|1}$ corresponds to the functor that associates to any supercommuting ring R the group of all elements (t^1, θ^1) where $t^1 \in R_0$ and $\theta^1 \in R_1$, the multiplication being exactly the one given above. Similarly, the functor corresponding to $\mathrm{GL}(p|q)$ associates to any supercommuting ring R the group of all block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the entries of A and D are even elements of R and those of B and C are odd elements of R; the group law is just matrix multiplication. This group is denoted by $\mathrm{GL}(p|q)(R)$. If one wants to view these as super Lie groups in the smooth category, the functors go from the category of smooth supermanifolds to the category of groups. For instance, the functor defining the super Lie group $\mathbf{R}^{1|1}$ takes any supermanifold T to the group of all (t, θ^1, θ^2) where t, θ^1, θ^2 are global sections of $\mathcal{O}_T$ with t even and θ^i odd. Similarly, $\mathrm{GL}(p|q)$ is defined by the functor that takes T to the group $\mathrm{GL}(p|q)(R(T))$ where $R(T)$ is the supercommutative ring of global sections of T. The concept of the functor of points shows why we can manipulate the odd variables as if they are numerical coordinates. This is exactly what is done by the physicists and so the language of functor of points is precisely the one that is closest to the intuitive way of working with these objects that one finds in the physics literature.

Superspacetimes. Minkowski spacetime is the manifold $\mathbf{R}^4$ equipped with the action of the Poincaré group. To obtain *superspacetimes* one extends the abelian Lie algebra of translations by a Lie superalgebra whose odd part is what is called the *Majorana spinor module*, a module for the Lorentz group that is spinorial, real, and irreducible. This is denoted by $\mathbf{M}^{4|4}$. The super Poincaré group is the super Lie group of automorphisms of this supermanifold. Physicists call this *rigid supersymmetry* because the affine character of spacetime is preserved in this model. For *supergravity* one needs to construct local supersymmetries. Since the group involved is the group of diffeomorphisms that is infinite dimensional, this is a much deeper affair.

Once superspacetimes are introduced, one can begin the study of Lagrangian field theories on superspaces and their quantized versions. Following the classical picture this leads to supersymmetric Lagrangian field theories. They will lead to superfield equations that can be interpreted as the equations for corresponding superparticles. A superfield equation gives rise to several ordinary field equations that define a *multiplet* of particles. These developments of super field theory lead to the predictions of SUSY quantum field theory.

2.7. References

1. See Weyl's book in reference 3 of Chapter 1. The passage from Clifford's writings is quoted in more detail in the following book:

   Monastyrsky, M. *Riemann, topology, and physics*. Birkhäuser, Boston, 1987, p. 29.

   The Clifford quotation is in:

   Clifford, W. K. On the space-theory of matter. *Mathematical papers*. Chelsea, New York, 1968, p. 21.

   Clifford's deep interest in the structure of space is evident from the fact that he prepared a translation of Riemann's inaugural address. It is now part of his *Mathematical Papers*. In the paper (really just an abstract) mentioned above, here is what he says:

   > Riemann has shown that as there are different kinds of lines and surfaces, so there are different kinds of spaces of three dimensions; and that we can only find out by experience to which of these kinds the space in which we live belongs. In particular, the axioms of plane geometry are true within the limit of experiment on the surface of a sheet of paper, and yet we know that the sheet is really covered with a number of small ridges and furrows, upon which (the total curvature not being zero) these axioms are not true. Similarly, he says although the axioms of solid geometry are true within the limits of experiment

for finite portions of our space, yet we have no reason to conclude that they are true for very small portions; and if any help can be got thereby for the explanation of physical phenomena, we may have reason to conclude that they are not true for very small portions of space.

I wish here to indicate a manner in which these speculations may be applied to the investigation of physical phenomena. I hold in fact

(1) That small portions of space *are* in fact analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them.

(2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

(3) That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or ethereal.

(4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity.

I am endeavoring in a general way to explain the laws of double refraction on this hypothesis, but have not yet arrived at any results sufficiently decisive to be communicated.

2. The origins and sources for noncommutative geometry are too complicated to go into here. For some insights refer to the books of Manin and Connes:

   Connes, A. *Noncommutative geometry*. Academic, San Diego, 1994.

   Manin, Y. I. *Quantum groups and noncommutative geometry*. Université de Montréal, Centre de Recherches Mathématiques, Montreal, 1988.

   Manin, Y. I. *Topics in noncommutative geometry*. M. B. Porter Lectures. Princeton University, Princeton, N.J., 1991.

   Noncommutative geometry is closely tied up with quantum groups. See, for example:

   Chari, V., and Pressley, A. *A guide to quantum groups*. Cambridge University, Cambridge, 1994.

3. Riemann's talk is included in his collected works:

   Riemann, B. *Gesammelte mathematische Werke, wissenschaftlicher Nachlass und Nachträge* [Collected mathematical works, scientific Nachlass and addenda]. R. Narasimhan, ed. Springer, Berlin; Teubner, Leipzig, 1990.

   There are translations of Riemann's talk in English. See Clifford[1] and Spivak.[4] In addition, the collected papers contain Weyl's annotated remarks on Riemann's talk.

4. See pp. 132–134 of:

   Spivak, M. *A comprehensive introduction to differential geometry*. Vol. II. 2d ed. Publish or Perish, Wilmington, Del., 1979.

5. See Spivak,[4] p. 205.

6. See the preface (p. VII) to Weyl's book:

Weyl, H. *The concept of a Riemann surface*. Addison-Wesley, Reading, Mass.–London, 1955.

See also Klein's little monograph:

Klein, F. *On Riemann's theory of algebraic functions and their integrals*. Dover, New York, 1963.

7. Weyl's paper on pure infinitesimal geometry was the first place where the concept of a connection was freed of its metric origins and the concept of a manifold with an affine connection was introduced axiomatically. See:

   Weyl, H. Reine infinitesimalgeometrie. *Math. Z.* 2: 384–411, 1918.

8. For a wide-ranging treatment of the historical evolution of algebraic geometry, there is no better source than the book of Dieudonné:

   Dieudonné, J. *Cours de géométrie algébrique*. I. Aperçu historique sur le développement de la géométrie algébrique. Presses Universitaires de France, 1974.

9. For all the references, see notes 24 through 28 of Chapter 1.

   Berezin was a pioneer in supergeometry and superanalysis. For an account of his work and papers related to his interests, see:

   Contemporary mathematical physics. F. A. Berezin memorial volume. R. L. Dobrushin, R. A. Minlos, M. A. Shubin, and A. M. Vershik, eds. American Mathematical Society Translations, Series 2, 175. Advances in the Mathematical Sciences, 31. American Mathematical Society, Providence, R.I., 1996.

10. See Deligne's brief survey of Grothendieck's work:

    Deligne, P. Quelques idées maîtresses de l'œuvre de A. Grothendieck. *Matériaux pour l'histoire des mathématiques au XX^{e} siècle (Nice, 1996)*, 11–19. Séminaires et Congrès, 3. Société Mathématique de France, Paris, 1998.

CHAPTER 3

Super Linear Algebra

3.1. The Category of Super Vector Spaces

Super linear algebra deals with the category of super vector spaces over a field k. We shall fix k and suppose that it is of characteristic 0; in physics k is $\mathbf{R}$ or $\mathbf{C}$. The objects of this category are *super vector spaces* V over k, namely, vector spaces over k that are $\mathbf{Z}_2$-graded, i.e., have decompositions

$$V = V_0 \oplus V_1 , \quad 0, 1 \in \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z} .$$

The elements of V_0 are called *even* and those of V_1 *odd*. If d_i is the dimension of V_i, we say that V has *dimension* $d_0|d_1$. For super vector spaces V, W, the morphisms from V to W are linear maps $V \longrightarrow W$ that preserve the gradings. They form a linear space denoted by $\mathrm{Hom}(V, W)$. For any super vector space V the elements in $V_0 \cup V_1$ are called *homogeneous*, and if they are nonzero, their *parity* is defined to be 0 or 1 according as they are even or odd. The parity function is denoted by p. In any formula defining a linear or multilinear object in which the parity function appears, it is assumed that the elements involved are homogeneous (so that the formulae make sense) and that the definition is extended to nonhomogeneous elements by linearity. If we take $V = k^{p+q}$ with its standard basis e_i $(1 \le i \le p+q)$ and define e_i to be even (resp., odd) if $i \le p$ (resp., $i > p$), then V becomes a super vector space with

$$V_0 = \sum_{i=1}^{p} k e_i , \quad V_1 = \sum_{i=p+1}^{q} k e_i .$$

It is denoted by $k^{p|q}$.

The notion of direct sum for super vector spaces is the obvious one. For super vector spaces V, W, their tensor product is $V \otimes W$ whose homogeneous parts are defined by

$$(V \otimes W)_i = \sum_{j+m=i} V_j \otimes W_m$$

where i, j, m are in $\mathbf{Z}_2$ and $+$ is addition in $\mathbf{Z}_2$. Thus

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1) , \quad (V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0) .$$

For super vector spaces V, W, the so-called *internal* Hom, denoted by $\mathbf{Hom}(V, W)$, is the vector space of *all* linear maps from V to W, where the even maps are the ones preserving the grading and the odd maps are those that reverse the grading. In particular,

$$(\mathbf{Hom}(V, W))_0 = \mathrm{Hom}(V, W) .$$

If V is a super vector space, we write $\mathbf{End}(V)$ for $\mathbf{Hom}(V, V)$. The dual of a super vector space V is the super vector space V^* where $(V^*)_i$ is the space of linear functions from V to k that vanish on V_{1-i}.

The Rule of Signs and Its Consistency. The $\otimes$ in the category of vector spaces is associative and commutative in a natural sense. Thus, for ordinary, i.e., ungraded, or, what is the same, purely even, vector spaces U, V, W, we have the natural associativity isomorphism

$$(U \otimes V) \otimes W \simeq U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \longmapsto u \otimes (v \otimes w),$$

and the commutativity isomorphism

$$c_{V,W} : V \otimes W \simeq W \otimes V, \quad v \otimes w \longmapsto w \otimes v.$$

For the category of super vector spaces the associativity isomorphism remains the same, but the commutativity isomorphism is changed to

$$c_{V,W} : V \otimes W \simeq W \otimes V, \quad v \otimes w \longmapsto (-1)^{p(v)p(w)} w \otimes v.$$

This is the first example where the defining formula is given only for homogeneous elements, and it is assumed to be extended by linearity. Notice that

$$c_{V,W} c_{W,V} = \mathrm{id}.$$

This definition is the source of the *rule of signs* used by physicists, which says that whenever two terms are interchanged in a formula, a minus sign will appear if both terms are odd. The commutativity and associativity isomorphisms are compatible in the following sense. If U, V, W are super vector spaces,

$$c_{U,V\otimes W} = c_{U,W} c_{U,V}, \quad c_{V,W} c_{U,W} c_{U,V} = c_{U,V} c_{U,W} c_{V,W},$$

where we write $c_{U,V}$ for $c_{U,V} \otimes \mathrm{id}$, etc., as is easily checked. These relations can be extended to products of more than three super vector spaces. Suppose that V_i $(1 \le i \le n)$ are super vector spaces, and σ is a permutation of $\{1, \dots, n\}$. Then σ is a product $s_{i_1} \cdots s_{i_r}$ where s_j is the permutation that just interchanges j and $j+1$. Writing

$$L(s_j) = I \otimes \cdots \otimes c_{V_j,V_{j+1}} \otimes \cdots \otimes I$$

and applying these commutativity isomorphisms that successively interchange adjacent terms in $V_1 \otimes \cdots \otimes V_n$, we have an isomorphism

$$L(\sigma) = L(s_{i_1}) \cdots L(s_{i_r}) : V_1 \otimes \cdots \otimes V_n \simeq V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}.$$

This isomorphism turns out to be independent of the way σ is expressed as a composition $s_{i_1} \cdots s_{i_r}$ and is given by

$$L(\sigma) : v_1 \otimes \cdots \otimes v_n \longmapsto (-1)^{p(\sigma)} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

where

$$p(\sigma) = \{(i, j) \mid v_i, v_j \text{ odd}, \ i < j, \ \sigma(i) > \sigma(j)\}.$$

Furthermore, we can show that

$$L(\sigma\tau) = L(\sigma)L(\tau).$$

If all the V_i are the same and equal to V, we have an action of the group S_n in $V \otimes \cdots \otimes V$.

We shall now prove these results. Our convention is that the elements of S_n are mappings of the set $\{1, \ldots, n\}$ onto itself and that the product is composition of mappings. We fix a super vector space V. For $n = 1$ the group S_n is trivial. We begin by discussing the action of S_n on the n-fold tensor product of V with itself. For $n = 2$ the group S_n is $\mathbf{Z}_2$, and we send the nontrivial element to the transformation $c_{V,V}$ on $V \otimes V$ to get the action. Let us assume now that $n \geq 3$. On $V_3 := V \otimes V \otimes V$ we have operators c_{12}, c_{23} defined as follows:

$$c_{12} : v_1 \otimes v_2 \otimes v_3 \longmapsto (-1)^{p(v_1)p(v_2)} v_2 \otimes v_1 \otimes v_3 \,,$$

$$c_{23} : v_1 \otimes v_2 \otimes v_3 \longmapsto (-1)^{p(v_2)p(v_3)} v_1 \otimes v_3 \otimes v_2 \,.$$

We then find by a simple calculation that

$$c_{12}c_{23}c_{12} = c_{23}c_{12}c_{23} \,.$$

In the group S_3 the interchanges of $1, 2$ and $2, 3$ are represented by involutions s_1, s_2, respectively, and S_3 is the group generated by them with the relation

$$s_1 s_2 s_1 = s_2 s_1 s_2 \,.$$

So there is an action of S_3 on V_3 generated by the c_{ij}. This action, denoted by $\sigma \longmapsto L(\sigma)$, can be explicitly calculated for the six elements of S_3 and can be written as follows:

$$L(\sigma) : v_1 \otimes v_2 \otimes v_3 \longmapsto (-1)^{p(\sigma)} v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes v_{\sigma^{-1}(3)}$$

where

$$p(\sigma) = \sum_{(k,\ell)\in N(\sigma)} p(v_k)p(v_\ell) \,, \qquad N(\sigma) = \{(k,\ell) \mid k < \ell,\ \sigma(k) > \sigma(\ell)\} \,.$$

This description makes sense for all n and leads to the following formulation:

PROPOSITION 3.1.1 *There is a unique action L of S_n on $V_n := V \otimes \cdots \otimes V$ (n factors) such that for any $i < n$, the element s_i of S_n that sends i to $i + 1$ and vice versa and fixes all the others goes over to the map*

$$L(s_i) : v_1 \otimes \cdots \otimes v_n \longmapsto (-1)^{p(v_i)p(v_{i+1})} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n \,.$$

For arbitrary σ let $N(\sigma)$, $p(\sigma)$ be defined as above. Then

$$L(\sigma) : v_1 \otimes \cdots \otimes v_n \longmapsto (-1)^{p(\sigma)} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} \,.$$

Finally, we can write .

$$p(\sigma) = \#\{(k,\ell) \mid k < \ell,\ v_k, v_\ell \text{ both odd},\ \sigma(k) > \sigma(\ell)\} \,.$$

PROOF: The calculation above for $n = 3$ shows that for any $i < n$ we have

$$L(s_i)L(s_{i+1})L(s_i) = L(s_{i+1})L(s_i)L(s_{i+1}) \,.$$

Since S_n is generated by the s_i with the relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \,, \qquad 1 \leq i \leq n-1 \,,$$

it is immediate that there is an action of S_n on V_n that sends s_i to $L(s_i)$ for all i. If we disregard the sign factors, this is the action

$$R(\sigma) : v_1 \otimes \cdots \otimes v_n \longmapsto v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} .$$

Hence except for the sign factor we are done. We shall prove the formula for the sign factor by induction on $\ell(\sigma)$, the *length* of σ, which is by definition the cardinality $\#N(\sigma)$ of the set $N(\sigma)$.

First of all, $\ell(\sigma) = 1$ if and only if $\sigma = s_i$ for some i, and the result is then obvious. Suppose $\ell(\sigma) > 1$ and we assume the result for elements of smaller length. We can find i such that $(i, i+1) \in N(\sigma)$; we define $\tau = \sigma s_i$. It is then easily verified that $k < \ell \iff s_i k < s_i \ell$ whenever $k < \ell$ and $(k, \ell) \neq (i, i+1)$, and

$$(k, \ell) \in N(\tau) \iff (s_i k, s_i \ell) \in N(\sigma), \quad k < \ell, \ (k, \ell) \neq (i, i+1),$$

while

$$(i, i+1) \in N(\sigma), \quad (i, i+1) \notin N(\tau).$$

It follows from this that

$$\ell(\tau) = \ell(\sigma) - 1 .$$

The result is thus true for τ. Now, $\sigma = \tau s_i$ and so, because L is a well-defined action, $L(\sigma) = L(\tau)L(s_i)$. Thus

$$\begin{aligned} L(\sigma)(v_1 \otimes \cdots \otimes v_n) &= (-1)^{p(v_i)p(v_{i+1})} L(\tau)(v_{s_i 1} \otimes \cdots \otimes v_{s_i n}) \\ &= (-1)^q R(\sigma)(v_1 \otimes \cdots \otimes v_n) \end{aligned}$$

where

$$q = p(v_i)p(v_{i+1}) + \sum_{(k,\ell)\in N(\tau)} p(v_{s_i k})p(v_{s_i \ell}) = \sum_{(k',\ell')\in N(\sigma)} p(v_{k'})p(v_{\ell'}) = p(\sigma) .$$

This completes the proof. □

COROLLARY 3.1.2 *Let V_i $(i = 1, \ldots, n)$ be super vector spaces. For each $\sigma \in S_n$ let $L(\sigma)$ be the map*

$$L(\sigma) : V_1 \otimes \cdots \otimes V_n \longrightarrow V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}$$

defined by

$$L(\sigma) : v_1 \otimes \cdots \otimes v_n \longmapsto (-1)^{p(\sigma)} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} .$$

If $\sigma = s_{i_1} \cdots s_{i_r}$ then

$$L(\sigma) = L(s_{i_1}) \cdots L(s_{i_r})$$

where, for all i,

$$L(s_i) = I \otimes \cdots \otimes c_{V_i, V_{i+1}} \otimes I \cdots \otimes I .$$

In particular,

$$L(\sigma\tau) = L(\sigma)L(\tau)\mathbf{M} .$$

PROOF: Take $V = \oplus V_i$ and apply the proposition. The result is immediate. □

REMARK. The above corollary shows that the result of applying the adjacent exchanges successively at the level of tensors is independent of the way the permutation is expressed as a product of adjacent interchanges. This is the fundamental reason that the rule of signs works in super linear algebra in a consistent manner.

Superalgebras. A *superalgebra* A is a super vector space that is an associative algebra (always with unit 1 unless otherwise specified) such that multiplication is a morphism of super vector spaces from $A \otimes A$ to A. This is the same as requiring that

$$p(ab) = p(a) + p(b)\,.$$

It is easy to check that 1 is always even, that A_0 is a purely even subalgebra, and that

$$A_0 A_1 \subset A_1\,, \quad A_1^2 \subset A_0\,.$$

If V is a super vector space, $\mathbf{End}(V)$ is a superalgebra. For a superalgebra its supercenter is the span of all homogeneous elements a such that $ab = (-1)^{p(a)p(b)}ba$ for all b; it is often written as $Z(A)$. This definition is an example that illustrates the sign rule. We have

$$Z(\mathbf{End}(V)) = k \cdot 1\,.$$

It is to be mentioned that the supercenter is in general different from the center of A viewed as an ungraded algebra. Examples of this will occur in the theory of Clifford algebras that will be treated in Chapter 5. In fact, if $A = k[t]$ where t is odd and $t^2 = 1$, then A is a superalgebra that is not supercommutative since the square of the odd element t is not 0; as an ungraded algebra it is commutative, and so its center is itself while the supercenter of the superalgebra A is just k. If $V = k^{p|q}$ we write $M(p|q)$ or $M^{p|q}$ for $\mathbf{End}(V)$. Using the standard basis we have the usual matrix representations for elements of $M(p|q)$ in the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the letters denote matrices with A, B, C, D of orders, respectively, $p \times p, p \times q, q \times p, q \times q$. The even elements and odd elements are, respectively, of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

A superalgebra A is said to be *commutative* if

$$ab = (-1)^{p(a)p(b)}ba$$

for all (homogeneous) $a, b \in A$. The terminology can cause some mild confusion because $k[t]$ with t odd and $t^2 = 1$ is a superalgebra that is commutative as an algebra but not as a superalgebra. Indeed, in a commutative superalgebra, we have

$$ab + ba = 0\,, \quad a^2 = 0\,,$$

for odd a, b; in particular, odd elements are *nilpotent*. This is false for t in the above example. For this reason, and in order to avoid confusion, commutative superalgebras are often called *supercommutative*. The exterior algebra over an even vector space is an example of a supercommutative algebra. If the vector space has finite dimension, this superalgebra is isomorphic to $k[\theta_1, \ldots, \theta_q]$ where the θ_i are anticommuting, i.e., satisfy the relations $\theta_i\theta_j + \theta_j\theta_i = 0$ for all i, j. If A is supercommutative, A_0 (super)commutes with A. We can formulate supercommutativity of an algebra A by

$$\mu = \mu \circ c_{A,A}, \quad \mu : A \otimes A \longrightarrow A,$$

where μ is multiplication. Formulated in this way there is no difference from the usual definition of commutativity classically. In this definition the sign rule is hidden. In general, it is possible to hide all the signs using such devices.

A variant of the definition of supercommutativity leads to the definition of the *opposite* of a superalgebra. If A is a superalgebra, its opposite A^{opp} has the same super vector space underlying it but

$$a \cdot b = (-1)^{p(a)p(b)} ba$$

where $a \cdot b$ is the product of a and b in A^{opp}. This is the same as requiring that

$$\mu_{\mathrm{opp}} = \mu \circ c_{A,A}.$$

Thus A is supercommutative if and only if $A^{\mathrm{opp}} = A$.

Super Lie Algebras. If we remember the sign rule, it is easy to define a *Lie superalgebra* or *super Lie algebra*. It is a super vector space $\mathfrak{g}$ with a bracket $[\,\cdot\,,\cdot\,]$ that is a morphism from $\mathfrak{g} \otimes \mathfrak{g}$ to $\mathfrak{g}$ with the following properties:

(a) $[a, b] = -(-1)^{p(a)p(b)}[b, a]$ and
(b) the *(super) Jacobi identity*

$$[a, [b, c]] + (-1)^{p(a)p(b)+p(a)p(c)}[b, [c, a]] + (-1)^{p(a)p(c)+p(b)p(c)}[c, [a, b]] = 0.$$

One can hide the signs above by rewriting these relations as

(a′) $[\,\cdot\,,\cdot\,](1 + c_{\mathfrak{g},\mathfrak{g}}) = 0$ and
(b′) the (super) Jacobi identity

$$[\,\cdot, [\,\cdot\,,\cdot\,]](1 + \sigma + \sigma^2) = 0$$

where σ is the automorphism of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ corresponding to the cyclic permutation $(123) \longmapsto (312)$.

Thus, (b′) shows that the super Jacobi identity has the same form as the ordinary Jacobi identity for ordinary Lie algebras. Thus the super Lie algebra is defined in exactly the same manner in the category of super vector spaces as an ordinary Lie algebra is in the category of ordinary vector spaces. It thus appears as an entirely natural object. One might therefore say that a super Lie algebra is a *Lie object* in the category of super vector spaces.

There is a second way to comprehend the notion of a super Lie algebra that is more practical. The bracket is skew-symmetric if one of the elements is even and symmetric if both are odd. The super Jacobi identity has eight special cases depending on the parities of the three elements a, b, c. If all three are even, the definition is simply the statement that $\mathfrak{g}_0$ is a (ordinary) Lie algebra. The identities with 2 even and 1 odd say that $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module. The identities with 2 odd and 1 even say that the bracket

$$\mathfrak{g}_1 \otimes \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$$

is a *symmetric* $\mathfrak{g}_0$-map. Finally, the identities for all three odd elements reduce to

$$[a, [b, c]] + \cdots + \cdots = 0\,, \quad a, b, c \in \mathfrak{g}_1\,,$$

where $+ \cdots + \cdots$ is cyclic summation in a, b, c. It is not difficult to see that the last requirement is equivalent to

$$[a, [a, a]] = 0\,, \quad a \in \mathfrak{g}_1\,.$$

Indeed, if this condition is assumed, then replacing a by $xa + yb$ where $a, b \in \mathfrak{g}_1$ and $x, y \in k$, we find that

$$[b, [a, a]] + 2[a, [a, b]] = 0\,, \quad a, b \in \mathfrak{g}_1\,.$$

But then

$$\begin{aligned} 0 &= [a + b + c, [a + b + c, a + b + c]] \\ &= 2([a, [b, c]] + [b, [c, a]] + [c, [a, b]])\,. \end{aligned}$$

Thus a super Lie algebra is a super vector space $\mathfrak{g}$ on which a bilinear bracket $[\,\cdot\,,\cdot\,]$ is defined such that

(a) $\mathfrak{g}_0$ is an ordinary Lie algebra for $[\,\cdot\,,\cdot\,]$,
(b) $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module for the action $a \longmapsto \mathrm{ad}(a) : b \longmapsto [a, b]$ $(b \in \mathfrak{g}_1)$,
(c) $a \otimes b \longmapsto [a, b]$ is a symmetric $\mathfrak{g}_0$-module map from $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ to $\mathfrak{g}_0$, and
(d) for all $a \in \mathfrak{g}_1$, we have $[a, [a, a]] = 0$.

Except for (d) the other conditions are linear and can be understood within the framework of ordinary Lie algebras and their representations. Condition (d) is nonlinear and is the most difficult to verify in applications when Lie superalgebras are constructed by putting together an ordinary Lie algebra and a module for it satisfying (a)–(c).

If A is a superalgebra, we define

$$[a, b] = ab - (-1)^{p(a)p(b)} ba\,, \quad a, b \in A\,.$$

It is then an easy verification that $[\,\cdot\,,\cdot\,]$ converts A into a super Lie algebra. It is denoted by A_L but often we omit the suffix L. If $A = \mathbf{End}(V)$, we often write $\mathfrak{gl}(V)$ for the corresponding Lie algebra; if $V = \mathbf{R}^{p|q}$ we write $\mathfrak{gl}(p|q)$ for $\mathfrak{gl}(V)$.

Let $\mathfrak{g}$ be a super Lie algebra and for $X \in \mathfrak{g}$ let us define

$$\mathrm{ad}\, X : \mathfrak{g} \longrightarrow \mathfrak{g}\,, \quad \mathrm{ad}\, X(Y) = [X, Y]\,.$$

Then

$$\mathrm{ad} : X \longmapsto \mathrm{ad}\, X$$

is a morphism of $\mathfrak{g}$ into $\mathfrak{gl}(\mathfrak{g})$. The super Jacobi identity (b) above can be rewritten as

$$[[a, b], c] = [a, [b, c]] - (-1)^{p(a)p(b)}[b, [a, c]]\,,$$

and hence it is just the relation

$$[\mathrm{ad}\, X, \mathrm{ad}\, Y] = \mathrm{ad}[X, Y]\,, \quad X, Y \in \mathfrak{g}\,.$$

The Supertrace. Let $V = V_0 \oplus V_1$ be a finite-dimensional super vector space and let $X \in \mathbf{End}(V)$. Then we have

$$X = \begin{pmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{pmatrix}$$

where X_{ij} is the linear map of V_j to V_i such that $X_{ij}v$ is the projection on V_i of Xv for $v \in V_j$. The *supertrace* of X is now defined as

$$\mathrm{str}(X) = \mathrm{tr}(X_{00}) - \mathrm{tr}(X_{11}).$$

It is easy to verify that

$$\mathrm{str}(XY) = (-1)^{p(X)p(Y)} \,\mathrm{str}(YX), \qquad X, Y \in \mathbf{End}(V).$$

In analogy with the classical situation we write $\mathfrak{sl}(V)$ for the space of elements in $\mathfrak{gl}(V)$ with supertrace 0; if $V = \mathbf{R}^{p|q}$, then we write $\mathfrak{sl}(p|q)$ for $\mathfrak{sl}(V)$. Since the odd elements have supertrace 0, $\mathfrak{sl}(V)$ is a subsuper vector space of $\mathfrak{gl}(V)$. It is easy to verify that

$$[X, Y] \in \mathfrak{sl}(V), \qquad X, Y \in \mathfrak{gl}(V).$$

Thus $\mathfrak{sl}(V)$ is a subsuper Lie algebra of $\mathfrak{gl}(V)$. Corresponding to the classical series of Lie algebras $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, we thus have the series $\mathfrak{gl}(p|q)$, $\mathfrak{sl}(p|q)$ of super Lie algebras. In Chapter 6 we shall give Kac's classification of simple super Lie algebras over an algebraically closed field of which the $\mathfrak{sl}(p|q)$ are particular examples.

3.2. The Super Poincaré Algebra of Gol′fand-Likhtman and Volkov-Akulov

Although we have given a natural and simple definition of super Lie algebras, historically they emerged first in the works of physicists. Gol′fand and Likhtman constructed the super Poincaré algebra in 1971[1] as did Volkov and Akulov in 1973,[1] and Wess and Zumino constructed the superconformal algebra in 1974.[2] These were ad hoc constructions, and although it was realized that these were new algebraic structures, their systematic theory was not developed till 1975, when Kac introduced Lie superalgebras in full generality and classified the simple ones over **C** and **R**.[3] We shall discuss these two examples in some detail because they contain much of the intuition behind the construction of superspacetimes and their symmetries. We first take up the super Poincaré algebra of Gol′fand-Likhtman and Volkov-Akulov.

Let $\mathfrak{g}$ be a super Lie algebra. This means that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module with appropriate properties. The basic assumptions in all constructions of superspacetimes are the following:

(a) $\mathfrak{g}$ is a *real* Lie superalgebra.
(b) $\mathfrak{g}_1$ is a very special type of $\mathfrak{g}_0$-module, namely, it is spinorial.
(c) The spacetime momenta should be captured among the commutators $[A, B]$ where $A, B \in \mathfrak{g}_1$; i.e., $[\mathfrak{g}_1, \mathfrak{g}_1]$ should contain the translation subspace of $\mathfrak{g}_0$.

Condition (b) means that either $\mathfrak{g}_0$ or some quotient of it is an orthogonal Lie algebra, $\mathfrak{g}_0$ acts on $\mathfrak{g}_1$ through this quotient, and the module $\mathfrak{g}_1$ is spinorial; i.e., its complexification is a direct sum of spin modules of this orthogonal Lie algebra. This restriction of $\mathfrak{g}_1$ has its source in the fact that in quantum field theory the objects obeying the anticommutation rules were the spinor fields, and this property was then taken over in the definition of spinor fields at the classical level.

In the example of Gol'fand-Likhtman and Volkov-Akulov, $\mathfrak{g}_0$ is the Poincaré-Lie algebra, i.e.,

$$\mathfrak{g}_0 = \mathfrak{t} \oplus \mathfrak{l}$$

where $\mathfrak{t} \simeq \mathbf{R}^4$ is the abelian Lie algebra of spacetime translations, $\mathfrak{l}$ is the Lorentz Lie algebra $\mathfrak{so}(1,3)$, namely, the Lie algebra of $\mathrm{SO}(1,3)^0$, and the sum is semidirect with respect to the action of $\mathfrak{l}$ on $\mathfrak{t}$; in particular, $\mathfrak{t}$ is an abelian ideal. $\mathfrak{g}_0$ is thus the Lie algebra of the Poincaré group and hence $\mathfrak{g}$ is to be thought of as a *super Poincaré algebra*. We shall also assume that $\mathfrak{g}$ is minimal in the sense that there is no subsuper Lie algebra that strictly includes $\mathfrak{g}_0$ and is strictly included in $\mathfrak{g}$.

The Poincaré group P acts on $\mathfrak{g}_1$ and one can analyze its restriction to the translation subgroup in a manner analogous to what was done in Chapter 1 for unitary representations except now the representation is finite dimensional and not unitary. Since $\mathfrak{t}$ is abelian, by Lie's theorem on solvable actions we can find eigenvectors in the complexification $(\mathfrak{g}_1)_{\mathbf{C}}$ of $\mathfrak{g}_1$ for the action of $\mathfrak{t}$. So there is a linear function λ on $\mathfrak{t}$ such that

$$V_\lambda := \{v \in (\mathfrak{g}_1)_{\mathbf{C}} \mid [X, v] = \lambda(X)v,\ \ X \in \mathfrak{t}\} \neq 0\,.$$

If L is the Lorentz group $\mathrm{SO}(1,3)^0$ and we write $X \longmapsto X^h$ for the action of $h \in L$ on $\mathfrak{t}$ as well as $\mathfrak{g}_1$, we have

$$\mathrm{ad}(X^h) = h\,\mathrm{ad}(X)h^{-1}\,.$$

This shows that h takes V_λ to V_{λ^h} where $\lambda^h(X) = \lambda(X^{h^{-1}})$ for $X \in \mathfrak{t}$. But $\mathfrak{g}_{1,\mu}$ can be nonzero only for a finite set of linear functions μ on $\mathfrak{t}$, and hence $\lambda = 0$. But then $\mathfrak{g}_{1,0}$ is stable under L so that $\mathfrak{g}_0 \oplus \mathfrak{g}_{1,0}$ is a super Lie algebra. By minimality it must be all of $\mathfrak{g}$. Hence in a minimal $\mathfrak{g}$ the action of $\mathfrak{t}$ on $\mathfrak{g}_1$ is 0. This means that $\mathfrak{g}_0$ acts on $\mathfrak{g}_1$ through $\mathfrak{l}$ so that it makes sense to say that $\mathfrak{g}_1$ is spinorial. Furthermore, if $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is any Lie superalgebra and $\mathfrak{h}$ is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$, then $\mathfrak{g}_0 \oplus \mathfrak{h}$ is a subsuper Lie algebra of $\mathfrak{g}$. Hence if $\mathfrak{g}$ is a minimal extension of $\mathfrak{g}_0$, then $\mathfrak{g}_1$ must be *irreducible*. Since we are working over $\mathbf{R}$, we must remember that $\mathfrak{g}_1$ may not be irreducible after extension of scalars to $\mathbf{C}$.

The irreducible representations of the group $\mathrm{SL}(2, \mathbf{C})$, viewed as a real Lie group, are precisely the representations $\mathbf{k} \otimes \overline{\mathbf{m}}$ where for any integer $r \geq 1$, we write $\mathbf{r}$ for the irreducible *holomorphic* representation of dimension r, and $\overline{\mathbf{m}}$ for the complex conjugate representation of the representation $\mathbf{m}$. Recall that $\mathbf{1}$ is the trivial representation in dimension 1 and $\mathbf{2}$ is the defining representation in $\mathbf{C}^2$. Of these $\mathbf{2}$ and $\overline{\mathbf{2}}$ are the spin modules. To get a real irreducible spinorial module we take notice that $\mathbf{2} \oplus \overline{\mathbf{2}}$ has a real form. Indeed, with $\mathbf{C}^2$ as the space of $\mathbf{2}$, the representation $\mathbf{2} \oplus \overline{\mathbf{2}}$ can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} g \cdot u \\ \overline{g} \cdot \overline{v} \end{pmatrix}, \quad g \in \mathrm{SL}(2, \mathbf{C}),\ u, v \in \mathbf{C}^2,$$

where a bar over a letter denotes complex conjugation. This action commutes with the conjugation

$$\sigma : \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} \overline{v} \\ \overline{u} \end{pmatrix}.$$

We define $\mathfrak{m}$ to be the real form of $\mathbf{2} \oplus \overline{\mathbf{2}}$ defined by σ. We have

$$\mathfrak{m}_{\mathbf{C}} = \mathbf{2} \oplus \overline{\mathbf{2}}, \quad \mathfrak{m} = (\mathfrak{m}_{\mathbf{C}})^{\sigma}.$$

Since $\mathbf{2}$ and $\overline{\mathbf{2}}$ are inequivalent, the above is the *only* possible decomposition of $\mathfrak{m}_{\mathbf{C}}$ into irreducible pieces, and so there is no proper submodule of $\mathfrak{m}$ stable under the conjugation σ. Thus $\mathfrak{m}$ is irreducible under $\mathfrak{g}_0$. This is the so-called *Majorana spinor*. Any real spinorial representation of $\mathrm{SL}(2, \mathbf{C})$ is a direct sum of copies of $\mathfrak{m}$, and so minimality forces $\mathfrak{g}_1$ to be $\mathfrak{m}$. Our aim is to show that there is a structure of a super Lie algebra on

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m}$$

satisfying (3) above. The irreducibility of $\mathfrak{m}$ ensures the minimality of $\mathfrak{g}$ as an extension of $\mathfrak{g}_0$.

To make $\mathfrak{g}$ into a super Lie algebra, we must find a symmetric $\mathfrak{g}_0$-map

$$[\cdot, \cdot] : \mathfrak{m} \otimes \mathfrak{m} \longrightarrow \mathfrak{g}_0$$

such that

$$[a, [a, a]] = 0, \quad a \in \mathfrak{m}.$$

Now

$$\mathfrak{m}_{\mathbf{C}} \otimes \mathfrak{m}_{\mathbf{C}} = (\mathbf{2} \otimes \mathbf{2}) \oplus (\overline{\mathbf{2}} \otimes \overline{\mathbf{2}}) \oplus (\mathbf{2} \otimes \overline{\mathbf{2}}) \oplus (\overline{\mathbf{2}} \otimes \mathbf{2}).$$

We claim that there is a projectively unique symmetric $\mathfrak{l}$-map

$$L : \mathfrak{m}_{\mathbf{C}} \otimes \mathfrak{m}_{\mathbf{C}} \longrightarrow \mathfrak{t}_{\mathbf{C}}$$

where the right side is the complexification of the 4-dimensional representation of $\mathrm{SO}(1, 3)^0$ viewed as a representation of $\mathrm{SL}(2, \mathbf{C})$. To see this, we first note that $\mathbf{2} \otimes \overline{\mathbf{2}}$ descends to a representation of $\mathrm{SO}(1, 3)^0$ because -1 acts as -1 on both factors and so acts as 1 on their tensor product. Moreover, it is the only irreducible representation of dimension 4 of $\mathrm{SO}(1, 3)^0$, and we write this as $\mathbf{4}_{\mathrm{v}}$, the vector representation in dimension 4; of course, $\mathbf{4}_{\mathrm{v}} \simeq \mathfrak{t}$. Furthermore, using the map $F : u \otimes v \longmapsto v \otimes u$ we have

$$\mathbf{2} \otimes \overline{\mathbf{2}} \simeq \overline{\mathbf{2}} \otimes \mathbf{2} \simeq \mathbf{4}_{\mathrm{v}}.$$

Thus $W = (\mathbf{2} \otimes \overline{\mathbf{2}}) \oplus (\overline{\mathbf{2}} \otimes \mathbf{2}) \simeq \mathbf{4}_{\mathrm{v}} \oplus \mathbf{4}_{\mathrm{v}}$. On the other hand, W is stable under F and so splits as the direct sum of subspaces symmetric and skew-symmetric with respect to F, these being also submodules. Hence each of them is isomorphic to $\mathbf{4}_{\mathrm{v}}$. Now $\mathbf{2} \otimes \mathbf{2}$ is $\mathbf{1} \oplus \mathbf{3}$ where $\mathbf{1}$ occurs in the skew-symmetric part and $\mathbf{3}$ occurs in the symmetric part, and a similar result is true for the complex conjugate modules. Hence

$$(\mathfrak{m}_{\mathbf{C}} \otimes \mathfrak{m}_{\mathbf{C}})^{\mathrm{sym}} \simeq \mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{4}_{\mathrm{v}},$$

showing that there is a projectively unique symmetric $\mathfrak{l}$-map L from $\mathfrak{m}_{\mathbf{C}} \otimes \mathfrak{m}_{\mathbf{C}}$ to $\mathfrak{t}_{\mathbf{C}}$.

We put

$$[a, b] = L(a \otimes b), \quad a, b \in \mathfrak{m}_{\mathbf{C}}.$$

Since L goes into the translation part $\mathfrak{t}_{\mathbf{C}}$ that acts as 0 on $\mathfrak{m}_{\mathbf{C}}$, we have automatically

$$[c, [c, c]] = 0, \quad c \in \mathfrak{m}_{\mathbf{C}}.$$

We have thus obtained a Lie superalgebra

$$\mathfrak{s} = (\mathfrak{g}_0)_{\mathbf{C}} \oplus \mathfrak{m}_{\mathbf{C}}.$$

This is the complexified super Poincaré algebra of Gol'fand-Likhtman and Volkov-Akulov.

To obtain the appropriate real form of $\mathfrak{s}$ is now easy. Let us denote by φ the conjugation of $\mathfrak{t}_{\mathbf{C}}$ that defines $\mathfrak{t}$. Then, on the one-dimensional space of symmetric $\mathfrak{l}$-maps from $\mathfrak{m}_{\mathbf{C}} \otimes \mathfrak{m}_{\mathbf{C}}$ to $\mathfrak{t}_{\mathbf{C}}$, we have the conjugation

$$M \longmapsto \varphi \circ M \circ (\sigma \otimes \sigma),$$

and so there is an element N fixed by this conjugation. If

$$[a, b] = N(a \otimes b), \quad a, b \in \mathfrak{m},$$

then N maps $\mathfrak{m} \otimes \mathfrak{m}$ into $\mathfrak{t}$ and so, as before,

$$[[c, c], c] = 0, \quad c \in \mathfrak{m}.$$

Thus we have a Lie superalgebra structure on

$$\mathfrak{g} := \mathfrak{g}_0 \oplus \mathfrak{m}.$$

This is the super Poincaré algebra constructed by Gol'fand-Likhtman and Volkov-Akulov.

Over $\mathbf{C}$, the map $N : \mathfrak{m}_{\mathbf{C}} \times \mathfrak{m}_{\mathbf{C}} \longrightarrow \mathfrak{t}_{\mathbf{C}}$ is unique up to a multiplicative scalar. If we consider the two super commutators $[Z_1, Z_1'] = N(Z_1, Z_1')$ and $[Z_1, Z_1'] = cN(Z_1, Z_1')$ between odd elements, the map $Z_0 + Z_1 \longmapsto Z_0 + \gamma Z_1$ is an isomorphism of the two corresponding super Lie algebras if $\gamma^2 = c^{-1}$. Thus, over $\mathbf{C}$, the super Poincaré algebra is unique up to isomorphism. Over $\mathbf{R}$ the same argument works but gives the result that we have two models up to isomorphism, corresponding to $c = \pm 1$. We may say that these two, which differ only in the sign of the superbrackets of odd elements, are *isomers*.

It is to be noted that in constructing this example we have made the following assumptions about the structure of $\mathfrak{g}$:

(a) $\mathfrak{g}_1$ is spinorial,
(b) $[\cdot, \cdot]$ is not identically zero on $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ and maps it into $\mathfrak{t}$, and
(c) $\mathfrak{g}$ is minimal under conditions (a) and (b).

Indeed, $\mathfrak{g}$ is *esssentially uniquely determined by these assumptions*. However, there are other examples if some of these assumptions are dropped. Since $\mathfrak{t}$ is irreducible as a module for $\mathfrak{l}$, it follows that the map $\mathfrak{g}_1 \otimes \mathfrak{g}_1 \longrightarrow \mathfrak{t}$ is surjective and so the spacetime momenta P_μ (which form a basis of $\mathfrak{t}$) can be expressed as anticommutators of the spinorial odd elements (also called spinorial charges). This leads to the positivity of the energy operator in SUSY theories. This aspect of $\mathfrak{g}$ has prompted a heuristic understanding of $\mathfrak{g}$ as the *square root of the Poincaré algebra*.

We can make all of the above discussion very explicit. Let $\mathbf{C}^2$ be the space of column vectors on which SL(2, $\mathbf{C}$) acts naturally from the left. We take

$$\mathfrak{m}_{\mathbf{C}} = \mathbf{C}^2 \oplus \mathbf{C}^2 = \{(u, v) \mid u, v \in \mathbf{C}^2\}.$$

We allow SL(2, $\mathbf{C}$) to act on $\mathfrak{m}_{\mathbf{C}}$ by

$$g \cdot (u, v) = (gu, \overline{g}v).$$

Here $v \longmapsto \overline{v}$ is the standard conjugation on $\mathbf{C}^2$. The conjugation σ that determines the real form $\mathfrak{m}$ is given by

$$\sigma(u, v) = (\overline{v}, \overline{u})$$

so that

$$\mathfrak{m} = \{(u, \overline{u}) \mid u \in \mathbf{C}^2\}.$$

The bracket for odd elements is then defined by

$$[(u, v), (u', v')] = (1/2)(uv'^{\mathrm{T}} + u'v^{\mathrm{T}})$$

where a^{T} is the transpose of a. The space $\mathfrak{t}_{\mathbf{C}}$ is the space of 2×2 complex matrices equipped with the conjugation $x \longmapsto \overline{x}^{\mathrm{T}}$ so that $\mathfrak{t}$ is the space of 2×2 Hermitian matrices. We have

$$[(u, \overline{u}), (v, \overline{v})] = (1/2)(u\overline{v}^{\mathrm{T}} + v\overline{u}^{\mathrm{T}}) \in \mathfrak{t}.$$

We could have chosen the negative sign in the bracket above but the choice above has the following positivity property: let C^+ be the forward light cone of all elements x in $\mathfrak{t}$ that are nonzero and nonnegative definite, and let C^{++} be its interior; then

$$[X, X] \in C^+, \quad 0 \neq X \in \mathfrak{m}.$$

If $\langle s, t\rangle$ is the Lorentz invariant symmetric bilinear form on $\mathfrak{t} \times \mathfrak{t}$ whose quadratic form is $x \longmapsto \det(x)$, the inclusion $x \in C^+$ is equivalent to $\langle x, y\rangle > 0$ for all $y \in C^{++}$. Thus the above positivity is equivalent to

$$\langle y, [X, X]\rangle > 0, \quad y \in C^{++}.$$

Note that if e_1, e_2 are the standard basis vectors in $\mathbf{C}^2$, then for $X = (e_1, e_1) \in \mathfrak{m}$ we have

$$[X, X] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \sim v_0 = (1, 0, 0, 0).$$

In any unitary representation π, the operator $H = -i(d/dt)_{t=0}\pi(\exp tv_0)$ is the energy operator, while the above commutation rule implies that $H = \pi(X)^\dagger \pi(X)$ so that the operator corresponding to energy is positive. We do not go into more detail here.

3.3. Conformal Spacetime

The second example we wish to discuss is the superconformal algebra of Wess and Zumino. To this end we begin with some preliminary discussion of conformality.

The relevance of conformality to the physics of radiation goes back to Weyl. Conformal maps are defined as maps of one Riemannian or pseudo-Riemannian manifold into another that take one metric to a multiple of the other, the multiplying constant being a strictly positive function that is allowed to vary from point to point. The simplest example is the dilation $x \mapsto cx$ on the space $\mathbf{R}^{p,q}$, which is euclidean space $\mathbf{R}^{p+q}$ equipped with a metric of signature (p, q), where $c > 0$ is a constant. Less trivial examples are complex analytic maps f from a domain D in the complex plane to another domain D', df never being 0 on D. Such maps are classically known as conformal maps, which is why the maps in the more general context are also called conformal. Weyl noticed that the Maxwell equations are invariant under all conformal transformations; we have seen this in our discussion of the Maxwell equations. The idea that for radiation problems the symmetry group should be the conformal group on Minkowski spacetime is also natural because the conformal group is the group whose action preserves the forward light cone structure of spacetime. In euclidean space $\mathbf{R}^n$ (with the usual positive definite metric), the so-called *inversions* are familiar from classical geometry; these are maps $P \mapsto P'$ with the property that P' is on the same ray as OP (O is the origin) and satisfies

$$OP \cdot OP' = 1\,;$$

this determines the map as

$$x \longmapsto x' = \frac{x}{\|x\|^2}\,.$$

It is trivial to check that

$$ds'^2 = \frac{1}{r^4}\,ds^2$$

so that this map is conformal; it is undefined at O, but by going over to the one-point compactification S^n of $\mathbf{R}^n$ via stereographic projection and defining ∞ to be the image of O, we get a conformal map of S^n. This is typical of conformal maps in the sense that they are globally defined only after a suitable compactification. The compactification of Minkowski spacetime and the determination of the conformal extension of the Poincaré group go back to the nineteenth century and the work of Felix Klein. It is tied up with some of the most beautiful parts of classical projective geometry. It was resurrected in modern times by the work of Penrose.

In two dimensions the conformal groups are infinite dimensional because we have more or less arbitrary holomorphic maps that act conformally. However, this is not true in higher dimensions; for $\mathbf{R}^{p,q}$ with $p+q \geq 3$, the vector fields which are conformal in the sense that the corresponding one parameter (local) groups of diffeomorphisms are conformal, already form a *finite-dimensional* Lie algebra, which is in fact isomorphic to $\mathfrak{so}(p+1, q+1)$. Thus $\mathrm{SO}(p+1, q+1)$ acts conformally on a compactification of $\mathbf{R}^{p,q}$ and contains the inhomogeneous group $\mathrm{ISO}(p, q)$ as the subgroup that leaves invariant $\mathbf{R}^{p,q}$. In particular, $\mathrm{SO}(1, n+1)$ is the conformal extension of $\mathrm{ISO}(n)$ acting on S^n viewed as the one-point compactification of $\mathbf{R}^n$. We shall discuss these examples a little later. For the moment we shall concern ourselves with the case of dimension 4.

The Variety of Lines in Projective Space: The Klein Quadric. We now treat the case of dimension 4 in greater detail. We start with a complex vector space T of dimension 4 and the corresponding projective space $\mathbf{P} \simeq \mathbf{CP}^3$ of lines (= one-dimensional linear subspaces) in T. We denote by $\mathbf{G}$ the Grassmannian of all 2-dimensional subspaces of T, which can be thought of as the set of all lines in $\mathbf{P}$. The group $\mathrm{GL}(T) \simeq \mathrm{GL}(4, \mathbf{C})$ acts transitively on $\mathbf{G}$, and so we can identify $\mathbf{G}$ with the homogeneous space $\mathrm{GL}(4, \mathbf{C})/P_0$, where P_0 is the subgroup of elements leaving invariant the plane π_0 spanned by e_1, e_2, $(e_n)_{1 \le n \le 4}$ being a basis of T.

Thus P_0 consists of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A, B, D are 2×2 matrices, so that $\mathbf{G}$ becomes a complex manifold of dimension $16 - 12 = 4$. The group $\mathrm{SL}(T) \simeq \mathrm{SL}(4, \mathbf{C})$ already acts transitively on $\mathbf{G}$. We omit the reference to $\mathbf{C}$ hereafter and write $\mathrm{GL}(T)$, $\mathrm{SL}(T)$, etc., for the above groups. Associated to T we also have its second exterior power $E = \Lambda^2(T)$. The action of $\mathrm{GL}(T)$ on T lifts to a natural action on E: for $g \in \mathrm{GL}(T)$, $g(u \wedge v) = gu \wedge gv$. It is well-known that this action gives an irreducible representation of $\mathrm{SL}(T)$ on E.

We shall now exhibit an $\mathrm{SL}(T)$-equivariant imbedding of $\mathbf{G}$ in the projective space $\mathbf{P}(E)$ of E. If π is a plane in T and a, b is a basis for it, we have the element $a \wedge b \in E$; if we change the basis to another $(a', b') = (a, b)u$ where u is an invertible 2×2 matrix, then $a' \wedge b' = \det(u) a \wedge b$, and so the image $[a \wedge b]$ of $a \wedge b$ in the projective space $\mathbf{P}(E)$ of E is uniquely determined. This gives the *Plücker map* $P\ell$:

$$P\ell : \pi \longmapsto [a \wedge b], \quad a, b \text{ a basis of } \pi.$$

The Plücker map is an *imbedding*. To see this, recall first that if a, b are two linearly independent vectors in T, then, for any vector c the condition $c \wedge a \wedge b = 0$ is necessary and sufficient for c to lie in the plane spanned by a, b; this is obvious if we take $a = e_1, b = e_2$ where $(e_i)_{1 \le i \le 4}$ is a basis for T. So, if $a \wedge b = a' \wedge b'$ where a', b' are also linearly independent, then $c \wedge a' \wedge b' = 0$ when $c = a, b$, and hence a, b lie on the plane spanned by a', b'; thus the planes spanned by a, b and a', b' are the same. Finally, it is obvious that $P\ell$ is equivariant under $\mathrm{GL}(T)$.

If we choose a basis (e_i) for T and define $e_{ij} = e_i \wedge e_j$, then $(e_{ij})_{i<j}$ is a basis for E, and one can compute for any plane π of T the homogeneous coordinates of $P\ell(\pi)$. Let π be a plane in T with a basis (a, b) where

$$a = \sum_i a_i e_i, \quad b = \sum_i b_i e_i.$$

Let $y_{ij} = -y_{ji}$ be the minor defined by rows i, j in the 4×2 matrix

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}$$

so that

$$a \wedge b = \sum_{i<j} y_{ij} e_i \wedge e_j \,.$$

The $(y_{ij})_{i<j}$ are by definition the (homogeneous) *Plücker coordinates* of π. These, of course, depend on the choice of a basis for T.

The image of **G** under the Plücker map can now be determined completely. In fact, if $p = a \wedge b$, then $p \wedge p = 0$; conversely, if $p \in E$, say $p = (y_{12}, y_{23}, y_{31}, y_{14}, y_{24}, y_{34})$, and $p \wedge p = 0$, we claim that there is a $\pi \in \mathbf{G}$ such that $[p]$ is the image of π under $P\ell$. The condition $p \wedge p = 0$ becomes

$$\text{(K)} \qquad y_{12}y_{34} + y_{23}y_{14} + y_{31}y_{24} = 0 \,.$$

To prove our claim, we may assume, by permuting the ordering of the basis vectors e_i of T if necessary, that $y_{12} \neq 0$, and hence that $y_{12} = 1$. Then

$$y_{34} = -y_{31}y_{24} - y_{23}y_{14}$$

so that we can take

$$p = a \wedge b \,, \quad a = e_1 - y_{23}e_3 - y_{24}e_4 \,, \quad b = e_2 - y_{31}e_3 + y_{14}e_4 \,,$$

which proves the claim.

Actually, the quadratic function defined by the left side of (**K**) depends only on the choice of a volume element on T, i.e., a basis for $\Lambda^4(T)$. Let $0 \neq \mu \in \Lambda^4(T)$. Then

$$p \wedge p = Q_\mu(p)\mu \,.$$

If $\mu = e_1 \wedge e_2 \wedge e_3 \wedge e_4$, then $Q_\mu(p)$ is given by the left side of equation (**K**). The equation

$$Q_\mu(p) = 0 \iff p \wedge p = 0 \,,$$

which is equation (**K**) above, in the Plücker coordinates with respect to the basis (e_{ij}), defines a quadric in the projective space $\mathbf{P}(E)$. It is called the *Klein quadric* and is denoted by **K**. Klein discovered it and used it extensively in the study of the geometry of lines in projective space. The Plücker map is then a bijection of **G** with **K**. The variety **K** is nonsingular because the gradient of the function Q never vanishes at any point of **K**.

By the definition of Q_μ we have, for any $y \in E$,

$$y \wedge y = Q_\mu(y)\mu \,,$$

and so it follows at once that

$$Q_\mu(g \cdot y) = Q_\mu(y) \,, \quad g \in \mathrm{SL}(T) \,.$$

Thus the action of $\mathrm{SL}(T)$ in E maps $\mathrm{SL}(T)$ into the complex orthogonal group $\mathrm{O}(E) \simeq \mathrm{O}(6)$; it is actually into $\mathrm{SO}(E) \simeq \mathrm{SO}(6)$ because the image has to be connected. It is easy to check that the kernel of this map is ± 1. In fact, if $g(u\wedge v) = u \wedge v$ for all u, v, then g leaves all 2-planes stable and hence all lines, and so is a scalar c with $c^4 = 1$; then $u \wedge v = c^2 u \wedge v$ so that $c^2 = 1$. Since both $\mathrm{SL}(T)$ and $\mathrm{SO}(E)$ have dimension 15, we then have the exact sequence

$$1 \longrightarrow (\pm 1) \longrightarrow \mathrm{SL}(T) \longrightarrow \mathrm{SO}(E) \longrightarrow 1\,.$$

We may therefore view $\mathrm{SL}(T)$ as the spin group of $\mathrm{SO}(E)$. Let $\mathbf{4}$ be the defining 4-dimensional representation of $\mathrm{SL}(T)$ (in T) and $\mathbf{4}^*$ its dual representation (in T^*). Then $\mathbf{4}$ and $\mathbf{4}^*$ are the two spin representations of $\mathrm{SO}(E)$. This follows from the fact (see Chapter 5, Lemma 5.6.1) that all other nontrivial representations of $\mathrm{SL}(4)$ have dimension greater than 4.

Let $(e_i)_{1\le i\le 4}$ be a basis for T. Let π_0 be the plane spanned by e_1, e_2 and π_∞ the plane spanned by e_3, e_4. We say that a plane π is *finite* if its intersection with π_∞ is 0. This is equivalent to saying that the projection $T \longrightarrow \pi_0$ corresponding to the direct sum $T = \pi_0 \oplus \pi_\infty$ is an isomorphism of π with π_0. In this case we have a uniquely determined basis

$$a = e_1 + \alpha e_3 + \gamma e_4\,, \qquad b = e_2 + \beta e_3 + \delta e_4\,,$$

for π, and conversely, any π with such a basis is finite. It is also the same as saying that $y_{12} \neq 0$, as we have seen above. Indeed, if $y_{12} \neq 0$ and (a_i), (b_i) are the coordinate vectors of a basis for π, the 12-minor of the matrix with columns as these two vectors is nonzero and so by right multiplying by the inverse of the 12-submatrix we have a new basis for π of the above form. Let $\mathbf{K}^\times$ be the set of all finite planes. Because it is defined by the condition $y_{12} \neq 0$, we see that $\mathbf{K}^\times$ is an open subset of $\mathbf{K}$ that is easily seen to be dense. Thus, the assignment

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \pi(A)$$

where $\pi(A)$ is the plane spanned by a, b above gives a parametrization of the open dense set of finite planes in the Klein quadric. Since $\mathbf{K}$ also has dimension 4, we see that the Plücker map allows us to view the Klein quadric as the *compactification* of *complex spacetime* with coordinates $\alpha, \beta, \gamma, \delta$, identified with the space $M_2 = M_2(\mathbf{C})$ of complex 2×2 matrices A. If $g \in \mathrm{GL}(T)$ and π is a finite plane parametrized by the matrix A, then for

$$g = \begin{pmatrix} L & M \\ N & R \end{pmatrix}$$

the plane $\pi' = g{\cdot}\pi$ has the basis

$$\begin{pmatrix} L + MA \\ N + RA \end{pmatrix}$$

so that π' is parametrized by

$$(N + RA)(L + MA)^{-1}$$

provided it is also finite; the condition for π' to be finite is that $L + MA$ be invertible. This g acts on $\mathbf{K}$ generically as the *fractional linear map*

$$g : A \longmapsto (N + RA)(L + MA)^{-1} .$$

The situation is reminiscent of the action of SL(2) on the Riemann sphere by fractional linear transformations, except that in the present context the complement of the set of finite planes is not a single point but a variety that is actually a cone, the cone at infinity. It consists of the planes that have nonzero intersection with π_∞. In the interpretation in $\mathbf{CP}^3$ these are lines that meet the line π_∞.

We shall now show that the subgroup P of $\mathrm{SL}(T)$ that leaves $\mathbf{K}^\times$ invariant is precisely the subgroup P_∞ that fixes π_∞. The representation of π_∞ is the matrix

$$\begin{pmatrix} 0 \\ I \end{pmatrix}$$

and so, since

$$g\begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} M \\ R \end{pmatrix}, \quad g = \begin{pmatrix} L & M \\ N & R \end{pmatrix},$$

g fixes π_∞ if and only if $M = 0$.

The condition that g leaves $\mathbf{K}^\times$ invariant is that $L + MA$ be invertible for all A. Taking $A = 0$ we find that L is invertible, and so, setting $X = L^{-1}M$, we find that $I + XA$ should be invertible for all A. If M, and hence X and X^*, were not 0, then $XX^* \neq 0$ and so, being nonnegative, would have an eigenvalue $\gamma > 0$; then $I - \gamma^{-1}XX^*$ would not be invertible.

We have thus proven that $P = P_\infty$ and is the subgroup of all g of the form

$$\begin{pmatrix} L & 0 \\ NL & R \end{pmatrix} =: (N, L, R) , \quad L, R \text{ invertible.}$$

The action of P on $\mathbf{K}^\times$ is given by

$$A \longmapsto N + RAL^{-1} .$$

Using the correspondence $g \longmapsto (N, L, R)$ we may therefore identify P with the semidirect product

$$P = M_2'H \simeq M_2 \times' H ,$$

where

$$H = \mathrm{SL}(2 \times 2) := \left\{ \begin{pmatrix} L & 0 \\ 0 & R \end{pmatrix} \middle| \, L, R \in GL(2), \ \det(L)\det(R) = 1 \right\},$$

and

$$M_2' = \left\{ \begin{pmatrix} I & 0 \\ N & I \end{pmatrix} \right\} \simeq M_2 = M_2(\mathbf{C}) ,$$

with H and M_2' acting on M_2, respectively, by

$$A \longmapsto RAL^{-1} , \quad A \longmapsto N + A .$$

The group $\mathrm{SL}(2) \times \mathrm{SL}(2)$ is a subgroup of H and as such its action is just the action $A \longmapsto g_2Ag_1^{-1}$. H itself is the product of this subgroup and the group of dilations consisting of elements (c, c^{-1}) which act by $A \longmapsto c^{-2}A$. We have thus imbedded the complex spacetime inside its compactification $\mathbf{K}$ and the complex Poincaré group (plus the dilations) inside $\mathrm{SL}(T)$ as P in such a way that the Poincaré action goes over to the action by its image in P.

We shall now show that the action of $\mathrm{SL}(T)$ on $\mathbf{K}$ is conformal. To this end we should first define a conformal metric on $\mathbf{K}$. A conformal metric on a complex or real manifold X is an assignment that associates to each point x of X a set of nonsingular quadratic forms on the tangent space at x, any two of which are nonzero scalar multiples of each other, such that on a neighborhood of each point we can choose a holomorphic (resp., smooth, real analytic) metric whose quadratic forms belong to this assignment; we then say that the metric defines the conformal structure on that neighborhood. The invariance of a conformal metric under an automorphism α of X has an obvious definition, namely, that if α takes x to y, the set of metrics at x goes over to the set at y under $d\alpha$; if this is the case, we say that α is *conformal*. We shall now show that on the tangent space at each point π of $\mathbf{K}$ there is a set F_π of metrics uniquely defined by the requirement that they are changed into multiples of themselves under the action of the stabilizer of π and further that any two members of F_π are proportional. Moreover, we shall show that on a suitable neighborhood of any π, we can choose a metric whose multiples define this structure. This will show that $\pi \mapsto F_\pi$ is the unique conformal metric on $\mathbf{K}$ invariant for the action of $\mathrm{SL}(T)$. To verify the existence of F_π we can, in view of the transitivity of the action of $\mathrm{SL}(T)$, take $\pi = \pi_0$. Then the stabilizer P_{π_0} consists of the matrices

$$\begin{pmatrix} L & M \\ 0 & R \end{pmatrix}, \qquad L, R \text{ invertible.}$$

Now $\pi_0 \in \mathbf{K}^\times \simeq M_2$, where the identification is

$$A \longmapsto \begin{pmatrix} I \\ A \end{pmatrix}$$

with the action

$$A \longmapsto RA(L + MA)^{-1}.$$

We identify the tangent space at π_0 with M_2; the tangent action of the element of the stabilizer above is then

$$A \longmapsto \left(\frac{d}{dt}\right)_{t=0} tRA(L + tMA)^{-1} = RAL^{-1}.$$

The map

$$q : A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \det(A) = \alpha\delta - \beta\gamma$$

is a nondegenerate quadratic form on M_2 that changes into cq where $c = \det(R)\det(L)^{-1}$ under the above tangent action. Moreover, as the subgroup of the stabilizer defined by $M = 0$, $R, L \in \mathrm{SL}(2)$ has no characters, any quadratic form that is changed into a multiple of itself by elements of this subgroup will have to be invariant under it, and as the action of this subgroup is already irreducible, such a form has to be a multiple of q. We may then take F_π to be the set of nonzero multiples of q. It is easy to construct a *holomorphic* metric on $\mathbf{K}^\times$ that defines the conformal structure. The *flat* metric

$$\mu = d\alpha\, d\delta - d\beta\, d\gamma$$

on $\mathbf{K}^\times$ is invariant under the translations, and, as the translations are already transitive on $\mathbf{K}^\times$, *μ has to define* the conformal structure on $\mathbf{K}^\times$. The form of the metric on $\mathbf{K}^\times$ is not the usual flat one, but if we write

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix},$$

then

$$d\alpha\, d\delta - d\beta\, d\gamma = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2,$$

the right side of which is the usual form of the metric.

We now turn to what happens over the reals, where the story gets more interesting. Any conjugation of T defines a real form of T and hence defines a real form of E. The corresponding real form of $\mathbf{K}$ is simply the Klein quadric of the real form of T. For our purposes we need a conjugation of E that does not arise in this manner. We have already seen that real Minkowski space can be identified with the space of 2×2 *Hermitian* matrices in such a way that SL(2) acts through $A \mapsto gAg^*$. So it is appropriate to start with the conjugation

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix} = A^*$$

on $\mathbf{K}^\times$. Since the Plücker coordinates of the corresponding plane are

$$(1, -\alpha, -\beta, \delta, -\gamma, \alpha\delta - \beta\gamma),$$

it is immediate that the conjugation θ on E defined by

$$\theta : (y_{12}, y_{23}, y_{31}, y_{14}, y_{24}, y_{34}) \longmapsto (\overline{y_{12}}, \overline{y_{23}}, \overline{y_{24}}, \overline{y_{14}}, \overline{y_{31}}, \overline{y_{34}})$$

determines a conjugation on $\mathbf{P}(E)$ preserving $\mathbf{K}$ that extends the conjugation defined above on $\mathbf{K}^\times$. This is clearly unique and we shall write θ again for it. Let

$$E_{\mathbf{R}} = \{e \in E \mid e^\theta = e\}.$$

Then $E_{\mathbf{R}}$ is a real form of E; $y \in E_{\mathbf{R}}$ if and only if $y_{12}, y_{23}, y_{34}, y_{14}$ are real and $y_{31} = \overline{y_{24}}$. The restriction $Q_{\mathbf{R}}$ of Q to $E_{\mathbf{R}}$ is then real and is given by

$$Q_{\mathbf{R}}(y) = y_{12}y_{34} + y_{23}y_{14} + y_{31}\overline{y_{31}}, \qquad y \in E_{\mathbf{R}},$$

which is real and has signature $(4, 2)$. Let $\mathbf{K}_{\mathbf{R}}$ be the fixed point set of θ on $\mathbf{K}$. Then $\mathbf{K}_{\mathbf{R}}$ is the image of the set of zeros of Q on $E_{\mathbf{R}}$. In fact, let $u \in E$ be such that its image lies in $\mathbf{K}_{\mathbf{R}}$; then $u^\theta = cu$ for some $c \neq 0$; since θ is involutive, we must have $|c| = 1$, and so we can write $c = \overline{d}/d$ for some d with $|d| = 1$. Then for $v = d^{-1}u$ we have $v^\theta = v$. Thus

$$\mathbf{K}_{\mathbf{R}} = \{[y] \mid y \in E_{\mathbf{R}},\ Q_{\mathbf{R}}(y) = 0\}.$$

We also note at this time that $\mathrm{SO}(E_{\mathbf{R}})^0$ is transitive on $K_{\mathbf{R}}$. In fact, in suitable real coordinates (u, v) with $u \in \mathbf{R}^4$, $v \in \mathbf{R}^2$, the equation to $K_{\mathbf{R}}$ is $u{\cdot}u - v{\cdot}v = 0$; given a nonzero point (u, v) on this cone, we must have both u and v nonzero, and so without changing the corresponding point in projective space, we may assume that $u \cdot u = v \cdot v = 1$. Then we can use $\mathrm{SO}(4, \mathbf{R}) \times \mathrm{SO}(2, \mathbf{R})$ to move (u, v) to $((1,0,0,0),(1,0))$. If $\mathbf{K}_{\mathbf{R}}^{\times} = \mathbf{K}_{\mathbf{R}} \cap \mathbf{K}^{\times}$, then $\mathbf{K}_{\mathbf{R}}^{\times}$ is open in $\mathbf{K}_{\mathbf{R}}$, and it is not difficult to check that it is dense in $\mathbf{K}_{\mathbf{R}}$; indeed, in suitable coordinates $y = (a_i)$, we have $\mathbf{Q}_{\mathbf{R}}(y) = a_1^2 - a_2^2 + a_3^2 - a_4^2 + a_5^2 + a_6^2$, and it is a question of checking that the points with $a_1 - a_2 \neq 0$ are dense in $\mathbf{K}_{\mathbf{R}}$.

Now θ induces an involution $Q' \longmapsto Q'^{\theta}$ on the space of quadratic forms on E: $Q'^{\theta}(u) = Q(u^{\theta})^{\mathrm{conj}}$. Since Q and Q^{θ} coincide on $E_{\mathbf{R}}$ they must be equal, i.e., $Q = Q^{\theta}$. Hence $g^{\theta} := \theta g \theta$ lies in $\mathrm{SO}(Q)$ if and only if $g \in \mathrm{SO}(Q)$. So we have a conjugation $g \mapsto g^{\theta}$ on $\mathrm{SO}(E)$. It is easy to check that the subgroup of fixed points for this involution is $\mathrm{SO}(E_{\mathbf{R}})$, the subgroup of $\mathrm{SO}(E)$ that leaves $E_{\mathbf{R}}$ invariant. Since $\mathrm{SL}(T)$ is simply connected, θ lifts to a unique conjugation of $\mathrm{SL}(T)$, which we shall also denote by θ. Let

$$G = \mathrm{SL}(T)^{\theta}.$$

We wish to show the following:

(a) G is connected and is the full preimage of $\mathrm{SO}(E_{\mathbf{R}})^0$ in $\mathrm{SL}(T)$ under the (spin) map $\mathrm{SL}(T) \longrightarrow \mathrm{SO}(E)$.

(b) There is a Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(2,2)$ on T such that G is the subgroup of $\mathrm{SL}(T)$ preserving it, so that $G \simeq \mathrm{SU}(2,2)$.

(c) A plane in T defines a point of $\mathbf{K}_{\mathbf{R}}$ if and only if it is a null plane with respect to $\langle \cdot, \cdot \rangle$ and that G acts transitively on the set of null planes.

The differential of the spin map is the identity on the Lie algebra and so, whether G is connected or not, the image of G^0 under the spin map is all of $\mathrm{SO}(E_{\mathbf{R}})^0$. We shall first prove that G^0 is the full preimage of $\mathrm{SO}(E_{\mathbf{R}})^0$, and for this it is enough to show that $-1 \in G^0$. Consider, for $z \in \mathbf{C}$ with $|z| = 1$,

$$\delta(z) = \mathrm{diag}(z, \bar{z}, z, \bar{z}).$$

Its action on E is by the matrix

$$\gamma(z) = \mathrm{diag}(1, 1, z^2, 1, \bar{z}^2, 1).$$

Then $\gamma(z)$ leaves $E_{\mathbf{R}}$ invariant and so lies in $\mathrm{SO}(E_{\mathbf{R}})^0$ for all z. If h is the map $\mathrm{SL}(T) \longrightarrow \mathrm{SO}(E)$ and $h(g)^{\theta} = h(g)$, then $g^{\theta} = \pm g$. Hence $\delta(z)^{\theta} = \pm\delta(z)$ for all z. By continuity we must have the $+$ sign for all z and so $\delta(z) \in G$ for all z, hence $\delta(z) \in G^0$ for all z. But $\delta(1) = 1$, $\delta(-1) = -1$, proving that $-1 \in G^0$.

Now it is known that any real form of $\mathfrak{sl}(4)$ is conjugate to one of $\mathfrak{sl}(4, \mathbf{R})$, $\mathfrak{su}(p,q)$ $(0 \le p \le q,\ p+q = 4)$, and hence any conjugation of $\mathfrak{sl}(4)$ is conjugate to either $X \longmapsto X^{\mathrm{conj}}$ or to $X \longmapsto -FX^*F$ where F is the diagonal matrix with p entries equal to 1 and q entries equal to -1. The corresponding conjugations of $\mathrm{SL}(4)$ are $g \longmapsto g^{\mathrm{conj}}$ and $g \longmapsto Fg^{*-1}F$, respectively. The fixed-point groups of conjugations of $\mathrm{SL}(4)$ are thus conjugate to $\mathrm{SL}(4, \mathbf{R})$ and $\mathrm{SU}(p,q)$. But these are all connected.[4] So G is connected. Furthermore, if K is a maximal compact subgroup of G, then G goes *onto* a maximal compact subgroup of $\mathrm{SO}(4,2)$ with kernel $\{\pm\}$ and so, because the dimension of the maximal compacts of $\mathrm{SO}(4,2)$, which are all conjugate to $\mathrm{SO}(4) \times \mathrm{SO}(2)$, is 7, the dimension of the maximal compacts of G is also 7. But the maximal compacts of SL(4, **R**), SU(4), SU(1, 3), SU(2, 2) are, respectively, SO(4), SU(4), $(\mathrm{U}(1) \times \mathrm{U}(3))_1$, $(\mathrm{U}(2) \times \mathrm{U}(2))_1$ where the suffix 1 means that the determinant has to be 1, and these are of dimension 6, 15, 9, 7, respectively. Hence $G \simeq \mathrm{SU}(2,2)$. However, a calculation is needed to determine the Hermitian form left invariant by G and to verify that the planes that are fixed by θ are precisely the null planes for this Hermitian form. It is interesting to notice that the images of the real forms SL(4, **R**), SU(4), SU(1, 3), SU(2, 2) are, respectively, SO(3, 3), SO(6), SO*(6), SO(4, 2). In particular, the real form SO(5, 1) is *not* obtained this way.

Let $\mathfrak{g}$ be the Lie algebra of G. If Z is an endomorphism of T, its action $\rho(Z)$ on E is given by $e_i \wedge e_j \mapsto Ze_i \wedge e_j + e_i \wedge Ze_j$. If $Z = (z_{ij})_{1\le i,j\le 4}$, the matrix of $\rho(Z)$ in the basis $e_{12}, e_{23}, e_{14}, e_{34}, e_{31}, e_{24}$ is

$$\begin{pmatrix} z_{11}+z_{22} & -z_{13} & z_{24} & 0 & -z_{23} & -z_{14} \\ -z_{31} & z_{22}+z_{33} & 0 & -z_{24} & -z_{21} & z_{34} \\ z_{42} & 0 & z_{11}+z_{44} & z_{13} & -z_{43} & z_{12} \\ 0 & -z_{42} & z_{31} & z_{33}+z_{44} & z_{41} & z_{32} \\ -z_{32} & -z_{12} & -z_{34} & z_{14} & z_{33}+z_{11} & 0 \\ -z_{41} & z_{43} & z_{21} & z_{23} & 0 & z_{22}+z_{44} \end{pmatrix}.$$

The condition that $Z \in \mathfrak{g}$ is that the action of this matrix commutes with θ. If Θ is the matrix of θ, this is the condition

$$\rho(Z)\Theta = \Theta\overline{\rho(Z)}.$$

Now

$$\Theta = \begin{pmatrix} I_4 & 0 \\ 0 & J_2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and so, writing

$$\rho(Z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

we get

$$A = \overline{A}, \quad DJ_2 = J_2\overline{D}, \quad BJ_2 = \overline{B}, \quad J_2\overline{C} = C.$$

By using that $\sum z_{jj} = 0$, these reduce to the conditions

$$z_{11}+z_{22},\ z_{22}+z_{33} \in \mathbf{R},$$
$$z_{13}, z_{24}, z_{31}, z_{42} \in \mathbf{R},$$
$$z_{22}+z_{44} \in (-1)^{1/2}\mathbf{R},$$

and

$$z_{14} = \overline{z_{23}}, \quad z_{34} = -\overline{z_{21}}, \quad z_{12} = -\overline{z_{43}}, \quad z_{32} = \overline{z_{41}}.$$

It is not difficult to check that these are equivalent to saying that Z must be of the form

$$\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}, \quad B, C \text{ Hermitian, } A \text{ arbitrary,}$$

where * denotes adjoints and all letters denote 2×2 matrices. If

$$F = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix},$$

then this amounts to

$$FZ^* + ZF = 0.$$

In other words, the Lie algebra $\mathfrak{g}$ of G is the same as the fixed-point set of the conjugation

$$Z \longmapsto -FZ^*F,$$

and so

$$Z^\theta = -FZ^*F, \quad Z \in \mathfrak{sl}(T).$$

This means that

$$g^\theta = Fg^{*-1}F, \quad g \in \mathrm{SL}(T).$$

Let us write $(\cdot, \cdot)$ for the usual positive definite Hermitian form on $\mathbf{C}^4$, and let

$$F(u, v) = (Fu, v), \quad u, v \in \mathbf{C}^4.$$

Then F is a Hermitian form and G is the subgroup of $\mathrm{SL}(T)$ that leaves it invariant. This Hermitian form has signature (2, 2); indeed, if $T^\pm$ are the 2-dimensional eigenspaces of F for the eigenvalues ± 1, then $T = T^+ \oplus T^-$ and for $u = u^+ + u^-$ with $u^\pm \in T^\pm$, we have

$$F(u, u) = \|u^+\|^2 - \|u^-\|^2$$

where $\|\cdot\|$ is the usual norm in $\mathbf{C}^4$. This finishes the proof that $G \simeq \mathrm{SU}(2, 2)$. We write $G = \mathrm{SU}(F)$.

The plane π_0 is certainly a null plane for F. Because G is transitive on $\mathbf{K_R}$ (because $\mathrm{SO}(E_\mathbf{R})$ is transitive), it follows that all the planes that are fixed by θ are null planes for F. There are no other null planes. To see this, let π be a null plane and $\{f_1, f_2\}$ be an orthonormal basis for π. Since $F(u, v) = 0$ for $u, v \in \pi$, the image $F\pi$ of π under the linear transformation defined by F is orthogonal to π. Since F is unitary, we see that if $f_3 = -iFf_1$, $f_4 = -iFf_2$, $\{f_3, f_4\}$ is an orthonormal basis for $F\pi$. Then $(f_j)_{1 \le j \le 4}$ is an orthonormal basis for T and $(Fe_j, e_k) = (Ff_j, f_k)$ for all j, k. Hence the map g that takes e_j to f_j for all j is unitary and preserves F. Choose a scalar c with $|c| = 1$ such that $h = cg$ has determinant 1; then $h \in G$ and $h\pi_0 = \pi$. Thus $\pi \in \mathbf{K_R}$. All our claims are therefore proven.

Recall that $\mathbf{K_R}$ is the fixed-point set of the Klein quadric $\mathbf{K}$ with respect to the conjugation θ. Then real Minkowski space (corresponding to Hermitian A in M_2) is imbedded as a dense open set $\mathbf{K}_\mathbf{R}^\times$ of $\mathbf{K_R}$. If

$$g = \begin{pmatrix} L & M \\ NL & R \end{pmatrix} \in \mathrm{SU}(F)$$

is to leave $\mathbf{K}_{\mathbf{R}}^{\times}$ invariant, the condition is that $L+MA$ be invertible for all Hermitian A. As before, taking $A = 0$ we see that L must be invertible. Hence $I + XA$ is invertible for all Hermitian A where $X = L^{-1}M$. If we write

$$g = \begin{pmatrix} L & M \\ NL & R \end{pmatrix},$$

then the condition that $g \in \mathrm{SU}(F)$, which is $g^*Fg = F$, is equivalent to the conditions that N and R^*M are Hermitian and $R^* - M^*N = L^{-1}$; in particular, $X = L^{-1}M = R^*M - M^*NM$ is Hermitian. As before, if M and hence $X \neq 0$, then X^2 is nonzero and nonnegative and so has an eigenvalue $\gamma > 0$; then $I + XA$ is not invertible for $A = -\gamma^{-1}X$. Thus the subgroup $P_{\mathbf{R}}$ leaving $\mathbf{K}_{\mathbf{R}}^{\times}$ invariant is the set of all matrices of the form

$$\begin{pmatrix} L & 0 \\ NL & L^{*-1} \end{pmatrix}, \quad N \text{ Hermitian}, \ \det(L) \in \mathbf{R},$$

with the action on $\mathbf{K}_{\mathbf{R}}^{\times}$ given by

$$A \longmapsto N + L^{*-1}AL^{-1}.$$

The group $P_{\mathbf{R}}$ is $\mathbf{R}^{\times} P_{\mathbf{R}}^{1}$ where $P_{\mathbf{R}}^{1}$ is the group of all $g_{N,L}$ with N Hermitian and $L \in \mathrm{SL}(2)$. So it is the Poincaré group plus the dilations. The conformal metric on $\mathbf{K}$ becomes $(dx^0)^2-(dx^1)^2-(dx^2)^2-(dx^3)^2$ on $\mathbf{K}_{\mathbf{R}}^{\times}$, which is real and of signature $(1,3)$. Since $\det(A)$ goes over to $\det(A)\det(L)^{-2}$, it follows that we have a real conformal structure that is invariant under G. The group G thus acts conformally and transitively on $\mathbf{K}_{\mathbf{R}}$, and the subgroup that leaves real Minkowski space invariant is the Poincaré group plus the dilations. Notice that ± 1 act trivially so that we have an imbedding of the inhomogeneous Lorentz group inside $\mathrm{SO}(4,2)^0$. The Lie algebra of $\mathrm{SU}(2,2)$ is isomorphic to $\mathfrak{so}(2,4)$, which is thus viewed as the conformal extension of the Poincaré-Lie algebra.

Conformality in Higher Dimensions. The above considerations can be generalized considerably. In fact, it is true that $\mathbf{R}^{m,n}$, the affine Minkowski space of signature (m,n), can be imbedded as a dense open subset of a compact manifold that has a conformal structure and on which the group $\mathrm{SO}(m+1, n+1)$ acts transitively and conformally, and further that the Poincaré group (= the inhomogeneous group $\mathrm{ISO}(m,n)^0 = \mathbf{R}^{m,n} \times' \mathrm{SO}(m,n)^0$) can be imbedded inside $\mathrm{SO}(m+1, n+1)$ in such a fashion that the action of $\mathrm{ISO}(m,n)^0$ goes over to the action of $\mathrm{SO}(m+1,n+1)^0$. For $m = 1, n = 3$, we obtain the imbedding of the usual Poincaré group inside $\mathrm{SO}(2,4)^0$ treated above. Throughout we assume that $0 \leq m \leq n$ and $n \geq 1$.

We start with the imbedding of $\mathrm{ISO}(m,n)^0$ in $\mathrm{SO}(m+1,n+1)^0$. Let V be a real vector space of dimension $m+n+2$ with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ of signature $(m+1, n+1)$ given on it. Let Ω be the "light cone" of V, namely, the variety of nonzero elements p of V such that $(p,p) = 0$;

$$\Omega = \{p \in V \mid p \neq 0, \ (p,p) = 0\}.$$

We write $H = \mathrm{SO}(V)^0$. In a linear basis for V in which the quadratic form of V becomes

$$Q = x_0^2 + \cdots + x_m^2 - y_0^2 - \cdots - y_n^2 ,$$

the equation defining Ω is a homogeneous quadratic polynomial and so defines a quadric cone in the projective space $\mathbf{P}(V) \simeq \mathbf{RP}^{m+n+1}$ of V, stable under the action of $\mathrm{SO}(V)$. We write $[\Omega]$ for this cone and in general $[p]$ for the image in projective space of any nonzero $p \in V$. Since the gradient of Q is never zero at any point of Ω, it follows that Ω and $[\Omega]$ are both smooth manifolds and that the map $\Omega \longrightarrow [\Omega]$ is submersive everywhere. Clearly $\dim([\Omega]) = m + n$. The action of H on Ω gives rise to an action of H on $[\Omega]$. Let $p \in \Omega$ and let V_p (resp., $[\Omega]_p$) be the tangent space to Ω (resp., $[\Omega]$) at p (resp., $[p]$). V_p is the orthogonal complement of p in V. Finally, let H_p be the stabilizer of p in H. In what follows we shall fix p and choose $q \in \Omega$ such that $(q, p) = 1$. This is always possible, and we write W_p for the orthogonal complement of the span of the linear span of q, p. It is easy to see that $V = \mathbf{R}p \oplus \mathbf{R}q \oplus W_p$, $V_p = \mathbf{R}p \oplus W_p$, and that W_p has signature (m, n).

Notice first that the tangent map $V_p \longrightarrow [\Omega]_p$ has kernel $\mathbf{R}p$ and so $(\cdot\,,\cdot)$ induces a quadratic form on $[\Omega]_p$. It is immediate from the above decomposition of V that $W_p \simeq [\Omega]_p$ and so $[\Omega]_p$ has signature (m, n) with respect to this quadratic form. If p' is any other point of Ω above $[p]$, then $p' = \lambda p$ ($\lambda \neq 0$), and the quadratic form induced on $[\Omega]_p$ gets multiplied by λ^2 if we use p' in place of p. Moreover, if we change p to $h{\cdot}p$ for some $h \in H$, the quadratic forms at $h{\cdot}[p]$ are the ones induced from the quadratic forms at $[p]$ by the tangent map of the action of h. It follows that we have a *conformal structure* on $[\Omega]$ and that the action of H on $[\Omega]$ is conformal.

We shall first verify that H^0 acts transitively on $[\Omega]$. We use coordinates and write the equation of Ω as

$$x^2 = y^2 , \quad x^2 = x_0^2 + \cdots + x_m^2 , \; y^2 = y_0^2 + \cdots + y_n^2 .$$

Clearly, $\mathbf{x} := (x_0, \ldots, x_m), \mathbf{y} := (y_0, \ldots, y_n)$ are both nonzero for any point of Ω. So without changing the image in projective space, we may assume that $x^2 = y^2 = 1$. Then we can use the actions of $\mathrm{SO}(m+1, \mathbf{R})$ and $\mathrm{SO}(n+1, \mathbf{R})$ to assume that $\mathbf{x} = (1, 0, \ldots, 0), \mathbf{y} = (1, 0, \ldots, 0)$; in case $m = 0$, we have to take $\mathbf{x}$ as $(\pm 1, 0, \ldots, 0)$. So the transitivity is proven when $m > 0$. If $m = 0$ we change $\mathbf{y}$ to $(\pm 1, 0, \ldots, 0)$ so that in all cases any point of $[\Omega]$ can be moved to the image of $(1, 0 \ldots, 0, 1, 0, \ldots, 0)$. This proves transitivity and hence also the connectedness of $[\Omega]$.

We shall now show that $H_p^0 \simeq \mathrm{ISO}(m, n)^0$ giving us an imbedding of the latter in $\mathrm{SO}(m+1, n+1)^0$. We proceed as in Chapter 1, Section 5. Let $h \in H_p$. Then h leaves V_p stable, and so we have a flag $\mathbf{R}p \subset V_p \subset V$ stable under h. We claim that $h{\cdot}q - q \in V_p$. Certainly $h{\cdot}q = bq + v$ for some $v \in V_p$. But then $1 = (q, p) = (h{\cdot}q, p) = b$ showing that $b = 1$. It is then immediate that $h{\cdot}r - r \in V_p$ for any r. We write $t(h)$ for the image of $h{\cdot}q - q$ in $W_p' := V_p/\mathbf{R}p \simeq W_p$. On the other hand, h induces an action on $V_p/\mathbf{R}p$ that preserves the induced quadratic form there and so we have a map $H_p^0 \longrightarrow \mathrm{SO}(W_p')$ that must go into $\mathrm{SO}(W_p')^0$. So we have the image $r(h) \in \mathrm{SO}(W_p')^0$ of h. We thus have a map

$$J : h \longmapsto (t(h), r(h)) \in \mathrm{ISO}(W_p')^0, \quad h \in H_p^0 .$$

We claim that J is an isomorphism. In what follows we shall often identify W_p and W_p'. First of all, J is a homomorphism. Indeed, let $h, h' \in H_p^0$. Then $h' \cdot q = q + t(h') + c(h')p$ where $t(h') \in W_p$ so that $hh' \cdot q - q \equiv t(h) + r(h) \cdot t(h') \bmod \mathbf{R}p$, showing that $J(hh') = J(h)J(h')$. It is obvious that J is a morphism of Lie groups. Now the dimension of the Lie algebra of H_p, which is the dimension subspace $\{L \in \mathrm{Lie}(H) \mid Lp = 0\}$, is easily computed to be $\dim \mathrm{ISO}(m, n)$; for this one can go to the complexes and work in the complex orthogonal groups and take p to be the point $(1, i, 0, \ldots, 0)$, and then it is a trivial computation. Hence if we prove that J is injective, we can conclude that it is an isomorphism.

Suppose, then, that $J(h)$ is the identity. This means that $h \cdot q = q + a(h)p$, and further, that for any $v \in V_p$, $h \cdot v = v + b(h)p$. Taking the scalar product of the first relation with $h \cdot q$, we find that $0 = (q, q) = (h \cdot q, h \cdot q) = 2a(h)$, giving $h \cdot q = q$. Taking the scalar product of the second relation with $h \cdot q$, we find $(v, q) = (h \cdot v, h \cdot q) = (v + b(h)p, q) = (v, q) + b(h)$, giving $b(h) = 0$. So $h \cdot v = v$, proving that $h = 1$. We thus have

$$\mathrm{ISO}(m, n)^0 \simeq H_p^0 \hookrightarrow \mathrm{SO}(m + 1, n + 1)^0 .$$

For $h \in H_p^0$ we also write $t(h)$ for the element of $\mathrm{ISO}(W_p')$ that is the translation by $t(h)$.

The tangent space V_p of Ω at p intersects Ω in a cone; we write C_p for it and $C_{[p]}$ for its image in $[\Omega]$. Clearly, H_p fixes $C_{[p]}$. Let $A_{[p]} = [\Omega] \setminus C_{[p]}$. Then $A_{[p]}$ is an open dense subset of $[\Omega]$, stable under H_p^0; the density is an easy verification. We wish to show that there is an isomorphism of $A_{[p]}$ with W_p' in such a manner that the action of H_p^0 goes over to the action of $\mathrm{ISO}(W_p')$.

Let T and M be the preimages under J in H_p^0 of the translation and linear subgroups of $\mathrm{ISO}(W_p')$. Now $[q] \in A_{[p]}$, and we shall first prove that for any $[r] \in A_{[p]}$ there is a unique $h \in T$ such that the translation $t(h)$ takes $[q]$ to $[r]$. Since $[r] \in A_{[p]}$, we have $(r, p) \neq 0$, and so we may assume that $(r, p) = 1$. Hence $t' = r - q \in V_p$ and hence defines an element $t \in W_p'$. There is then a unique $h \in T$ such that $t(h) = t$; i.e., $J(h)$ is translation by t. We claim that h takes $[q]$ to $[r]$. By definition of h we have that $h \cdot q - q$ has t as its image in W_p'. Then $r - q$ and $h \cdot q - q$ have the same image in W_p', and so $h \cdot q - r \in V_p$ and has image 0 in W_p'. So $h \cdot q = r + bp$. But then $0 = (q, q) = (h \cdot q, h \cdot q) = (r, r) + 2b(r, p) = 2b$, showing that $b = 0$. In other words, the translation group T acts simply transitively on $A_{[p]}$. We thus have a bijection $h \cdot [q] \longrightarrow t(h)$ from $A_{[p]}$ to W_p'. It is an easy check that the action of H_p^0 on $A_{[p]}$ goes over to the action of $\mathrm{ISO}(W_p')$.

The metric on W_p' induced from V has signature (m, n) as we saw earlier. We can regard it as a flat metric on W_p' invariant under $\mathrm{ISO}(W_p')$; if we transport it to $A_{[p]}$, it becomes invariant under the action of H_p^0. Clearly it belongs to the conformal structure on $[\Omega]$. So all of our assertions are proven.

3.4. The Superconformal Algebra of Wess and Zumino

In 1974 Wess and Zumino[2] constructed a real super Lie algebra whose even part *contains* the *conformal extension* $\mathfrak{so}(4,2) \simeq \mathfrak{su}(2,2)$ of the Poincaré-Lie algebra considered above. The (complexified) Wess-Zumino algebra was the first example constructed of a *simple* super Lie algebra.

A word of explanation is in order here about the word *contains* above. Ideally, one would like to require that the superextension have the property that its even part is *exactly* $\mathfrak{so}(4,2)$. This turns out to be impossible, and for a superextension of minimal dimension (over $\mathbf{C}$), the even part of the superextension becomes $\mathfrak{so}(4,2) \oplus \mathbf{R}$, where the action of the elements of $\mathbf{R}$ on the odd part of the superextension generates a *rotation group* ("compact R-symmetry").

Let us operate first over $\mathbf{C}$. The problem is then the construction of super Lie algebras whose even parts are isomorphic to $\mathfrak{sl}(4)$, at least up to a central direct factor. We have already come across the series $\mathfrak{sl}(p|q)$ of super Lie algebras. The even part of $\mathfrak{g} = \mathfrak{sl}(p|q)$ consists of complex matrices

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

where X, Y are $p \times p$, $q \times q$ and

$$\mathrm{tr}(X) = \mathrm{tr}(Y)\,.$$

Thus the even part is isomorphic to $\mathfrak{sl}(p) \oplus \mathfrak{sl}(q) \oplus \mathbf{C}$. In particular, the even part of $\mathfrak{sl}(4,1)$ is $\mathfrak{sl}(4) \oplus \mathbf{C}$. The elements of the odd part of $\mathfrak{sl}(4|1)$ are matrices of the form

$$\begin{pmatrix} 0 & a \\ b^{\mathsf{T}} & 0 \end{pmatrix}, \quad a, b \text{ column vectors in } \in \mathbf{C}^4\,,$$

so that the odd part is the module $\mathbf{4} \oplus \mathbf{4}^*$. Now $[\mathfrak{g}_1, \mathfrak{g}_1]$ is stable under the adjoint action of $\mathfrak{sl}(4)$ and has nonzero intersection with both $\mathfrak{sl}(4)$ and the one-dimensional center of $\mathfrak{g}_0$. It is then immediate that $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_0$. At this time it is noteworthy that the even part is not precisely $\mathfrak{sl}(4)$ but has a one-dimensional central component with basis element R given by

$$R = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{3} \end{pmatrix}.$$

We note the following formulae: if we write

$$(X, x) = \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix}, \quad \mathrm{tr}(X) = x\,, \quad (a, b) = \begin{pmatrix} 0 & a \\ b^{\mathsf{T}} & 0 \end{pmatrix},$$

then

$$[(X, x), (a, b)] = ((X - x)a, -(X - x)^{\mathsf{T}} b)\,,$$
$$[(a, b), (a', b')] = (ab'^{\mathsf{T}} + a'b^{\mathsf{T}}, b^{\mathsf{T}} a' + b'^{\mathsf{T}} a)\,.$$

In particular, the projections $X(a, b)$, $R(a, b)$ of $[(a, b), (a, b)]$ on $\mathfrak{sl}(4)$, $\mathbf{C}R$, respectively, are given by

$$X(a, b) = 2ab^{\mathsf{T}} - \left(\frac{1}{2}\right)(b^{\mathsf{T}}a)I\,, \quad R(a, b) = \left(\frac{3}{2}\right)(b^{\mathsf{T}}a)R\,;$$

note that $X(a, b)$ has trace 0 as it should have. R acts nontrivially on the odd part; indeed, ad R is -1 on **4** and $+1$ on **4***, as is seen easily from the above formulae.

We shall first show that there does not exist a super Lie algebra $\mathfrak{h}$ with the properties: (1) $\mathfrak{h}_0 = \mathfrak{sl}(4)$ and $\mathfrak{h}_1$ spinorial, and (2) $[\mathfrak{h}_1, \mathfrak{h}_1]$ has a nonzero intersection with $\mathfrak{h}_0$ (then we must have that this commutator is all of $\mathfrak{h}_0$). The spinorial condition means that $\mathfrak{h}_1$ should be the sum of copies of **4** and **4***. It is not possible that $\mathfrak{h}_1$ contains only copies of **4**. To see this, note that $\mathbf{4} \otimes \mathbf{4}$ cannot contain the trivial representation as **4** and **4*** are not equivalent, and so, because its dimension is 16, it cannot contain the adjoint representation either, which has dimension 15. We thus see that both **4** and **4*** must occur in the odd part. So, for a particular choice of subspaces of type **4** and **4***, the space $\mathfrak{h}_1 = \mathfrak{sl}(4) \oplus \mathbf{4} \oplus \mathbf{4}^*$ is a super Lie algebra with the same properties as $\mathfrak{h}$. Since the even part is exactly $\mathfrak{sl}(4)$, we must have an $\mathfrak{sl}(4)$-map from $\mathbf{4} \otimes \mathbf{4}^*$ into $\mathfrak{sl}(4)$ satisfying the cubic condition for super Lie algebras.

We claim that this is impossible. To see this, notice that such a map is projectively unique, and so has to be a multiple of the map obtained from the map $[\,\cdot\,,\cdot\,]$ of the super Lie algebra $\mathfrak{sl}(4|1)$ by following it with the projection on the $\mathfrak{sl}(4)$ factor. From the formula above, we find that

$$[(a, b), X(a, b)] = \left(-\left(\frac{3}{2}\right)(b^{\mathsf{T}}a)a, +\left(\frac{3}{2}\right)(b^{\mathsf{T}}a)b\right),$$

which is obviously not identically zero. So there is no super Lie algebra with properties (1) and (2). The dimension of any super Lie algebra with properties (1) and (2) above with the modification in (1) that the even part *contains* $\mathfrak{sl}(4)$ must then be at least 24; if it is to be 24, then the even part has to be the direct sum of $\mathfrak{sl}(4)$ and a one-dimensional central factor. We shall now show that up to isomorphism, $\mathfrak{g}$ is the only super Lie algebra in dimension 24 of the type we want. Let $[\,\cdot\,,\cdot\,]'$ be another super bracket structure on $\mathfrak{g}$ such that $[\,\cdot\,,\cdot\,]'$ coincides with $[\,\cdot\,,\cdot\,]$ on $\mathfrak{sl}(4) \times \mathfrak{g}_1$. Then $\mathrm{ad}'(R)$ will act like a scalar $-\alpha$ on the **4** part and a scalar β on the **4*** part. Moreover, if Z_{00}, Z_{01} denote projections of an element $Z \in \mathfrak{g}_0$ into $\mathfrak{sl}(4)$ and $\mathbf{C}R$, respectively, then there must be nonzero constants γ, δ such that

$$[Y_1, Y_2]' = \gamma[Y_1, Y_2]_{00} + \delta[Y_1, Y_2]_{01}\,, \quad Y_1, Y_2 \in \mathfrak{g}_1\,.$$

Thus

$$[(a, b), (a, b)]' = \gamma X(a, b) + \delta R(a, b)\,.$$

The cubic condition then gives the relations

$$\alpha = \beta\,, \quad \gamma = \alpha\delta\,.$$

Thus there are nonzero α, δ such that

$$[R, Y]' = \alpha[R, Y],$$
$$[Y_1, Y_2]' = \alpha\delta[Y_1, Y_2]_{00} + \delta[Y_1, Y_2]_{01}.$$

If τ is the linear automorphism of $\mathfrak{g}$ such that

$$\tau(Z) = Z(Z \in \mathfrak{sl}(4)), \quad \tau(R) = \alpha R, \quad \tau(Y) = (\alpha\delta)^{1/2} Y, \quad Y \in \mathfrak{g}_1,$$

then

$$\tau([X_1, X_2]') = [\tau(X_1), \tau(X_2)], \quad X_1, X_2 \in \mathfrak{g}.$$

We thus have the following:

THEOREM 3.4.1 *There is no super Lie algebra whose odd part is spinorial and whose even part is* $\mathfrak{sl}(4)$ *and is spanned by the commutators of odd elements. Moreover,* $\mathfrak{sl}(4|1)$ *is the unique (up to isomorphism) super Lie algebra of minimum dimension with spinorial odd part such that* $\mathfrak{sl}(4)$ *is contained in the even part and is spanned by the commutators of odd elements.*

The Real Form. We now examine the real forms of $\mathfrak{g}$. We are only interested in those real forms whose even parts have their semisimple components isomorphic to $\mathfrak{su}(2, 2)$. We shall show that up to an automorphism of $\mathfrak{g}$ there are only two such and that the central factors of their even parts act on the odd part with respective eigenvalues $\mp i$, $\pm i$ on the **4** and **4*** components. The two have the same underlying super vector space, and the only difference in their bracket structures is that the commutator of two odd elements in one is the negative of the corresponding commutator in the other. They are, however, not isomorphic over **R**. One may call them *isomers* as we did in the case of the super Poincaré algebra.

The Unitary Super Lie Algebras. We begin with a description of the general unitary series of super Lie algebras. Let V be a complex super vector space. A *super Hermitian form* is a morphism

$$f : V \otimes_{\mathbf{R}} V \longrightarrow \mathbf{C}$$

of super vector spaces over **R** that is linear in the first variable and conjugate-linear in the second variable such that

$$f \circ c_{V,V} = f^{\text{conj}}.$$

This means that the complex-valued map $(u, v) \longmapsto f(u, v)$ is linear in u, conjugate-linear in v and has the symmetry property

$$f(v, u) = (-1)^{p(u)p(v)} \overline{f(u, v)}$$

and the consistency property

$$f(u, v) = 0, \quad u, v \text{ are of opposite parity.}$$

Thus f (resp., if) is an ordinary Hermitian form on $V_0 \times V_0$ (resp., $V_1 \times V_1$). Suppose that f is a nondegenerate super Hermitian form; i.e., its restrictions to the even and odd parts are nondegenerate. We define the super Lie algebra $\mathfrak{su}(V; f)$ to be the super vector space spanned by the set of all homogeneous $Z \in \mathfrak{sl}(V)$ such that

$$f(Zu, v) = -(-1)^{p(Z)p(u)} f(u, Zv).$$

It is not difficult to check that the above formula defines a real super Lie algebra. Let $V = \mathbf{C}^{p|q}$ with $V_0 = \mathbf{C}^p$, $V_1 = \mathbf{C}^q$, and let $f_\pm$ be given by

$$f_\pm((u_0, u_1), (v_0, v_1)) = (Fu_0, v_0) \pm i(Gu_1, v_1)$$

with

$$F = \begin{pmatrix} I_r & 0 \\ 0 & -I_{p-r} \end{pmatrix}, \quad G = \begin{pmatrix} I_s & 0 \\ 0 & -I_{q-s} \end{pmatrix}.$$

Here I_t is the unit $t \times t$ matrix. We denote the corresponding super Lie algebra by $\mathfrak{su}(r, p-r|s, q-s)_\pm$. To see that this is a real form of $\mathfrak{sl}(p|q)$, we shall construct a conjugation of $\mathfrak{sl}(p|q)$ whose fixed points form $\mathfrak{su}(r, p-r|s, q-s)_\pm$. Let

$$\sigma_\pm : \begin{pmatrix} X & A \\ B^\top & Y \end{pmatrix} \longmapsto \begin{pmatrix} -F\overline{X}^\top F & \pm iF\overline{B}G \\ \pm i(F\overline{A}G)^\top & -G\overline{Y}^\top G \end{pmatrix}.$$

It is a simple calculation to verify that $\sigma_\pm$ are conjugate-linear and preserve the superbracket on $\mathfrak{sl}(p|q)$. Hence they are conjugations of the super Lie algebra structure, and their fixed points constitute real super Lie algebras. Thus $\mathfrak{su}(r, p-r|s, q-s)_\pm$ are super Lie algebras that are real forms of $\mathfrak{sl}(p|q)$ defined by

$$\mathfrak{su}(r, p-r|s, q-s)_\pm = \left\{ \begin{pmatrix} X & A \\ B^\top & Y \end{pmatrix} \middle| X = -F\overline{X}^\top X,\ Y = -G\overline{Y}^\top G,\ A = \pm iF\overline{B}G \right\}.$$

The commutation rules involving the odd elements are given by (with obvious notation)

$$[(X, Y), (A, B)] = (XA - AY, BY^\top - X^\top B)$$
$$[(A_1, B_1), (A_2, B_2)] = \left(A_1B_2^\top + A_2B_1^\top, B_1^\top A_2 + B_2^\top A_1\right).$$

Notice that we can take B to be a completely arbitrary *complex* matrix of order $q \times p$, and then A is determined by the equation $A = \pm iF\overline{B}G$. We thus have

$$\mathfrak{su}(r, p-r|s, q-s)_\pm = \mathfrak{su}(r, p-r) \oplus \mathfrak{su}(s, q-s) \oplus M(p, q)$$

with the commutation rules involving the odd elements given by

$$[(X, Y), (0, B)] = (0, BY^\top - X^\top B)$$
$$[(0, B_1), (0, B_2)] = \pm i\left(F\overline{B}_1GB_2^\top + F\overline{B}_2GB_1^\top, B_1^\top F\overline{B}_2G + B_2^\top F\overline{B}1G\right).$$

It is clear that these two super Lie algebras are isomers.

We return to the case of $\mathfrak{g} = \mathfrak{sl}(4|1)$. In this case let

$$\mathfrak{g}_{\mathbf{R},\pm} = \mathfrak{su}(2, 2|1, 0)_\pm.$$

This is the precise definition of the super Lie algebras discovered by Wess and Zumino. They are the real forms defined by the conjugations

$$\sigma_\pm : \begin{pmatrix} X & a \\ b^{\mathsf{T}} & x \end{pmatrix} \longmapsto \begin{pmatrix} -F\overline{X}^{\mathsf{T}}F & \pm iF\overline{b} \\ \pm i(F\overline{a})^{\mathsf{T}} & -\overline{x} \end{pmatrix}, \quad F = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

We shall now show that up to an automorphism of $\mathfrak{g}$ these are the only real forms whose even parts have their simple components $\simeq \mathfrak{su}(2,2)$.

In the first place, suppose $\mathfrak{h}$ is such a real form. Write V for the super vector space with $V_0 = \mathbf{C}^4$, $V_1 = \mathbf{C}$, with the standard unitary structure. The restriction of $\mathfrak{h}_0$ to V_0 is the Lie algebra of elements X such that $X = -H\overline{X}^{\mathsf{T}}H$ where H is a Hermitian matrix of signature $(2,2)$. By a unitary isomorphism of V_0, we can change H to F where

$$F = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

Let τ be the conjugation of $\mathfrak{sl}(4|1)$ that defines $\mathfrak{h}$. Then τ and $\sigma_\pm$ coincide on $\mathfrak{su}(2,2)$.

Now, on $\mathfrak{g}_1$, we have $\mathrm{ad}(X) = \lambda\, \mathrm{ad}(X)\lambda^{-1}$ for $\lambda = \tau, \sigma_\pm$ and $X \in \mathfrak{su}(2,2)$. So $\tau = \rho\sigma_\pm$, where ρ is a linear automorphism of $\mathfrak{g}_1$ that commutes with the action of $\mathfrak{su}(2,2)$ on $\mathfrak{g}_1$. Thus ρ must be of the form

$$(a,b) \longmapsto (k_1'a, k_2'b)$$

for nonzero scalars k_1', k_2'. In other words, τ is given on $\mathfrak{g}_1$ by

$$(a,b) \longmapsto (k_1F\overline{b}, k_2F\overline{a}), \quad k_1\overline{k_2} = 1;$$

the condition on the k's is a consequence of the fact that τ is an involution. The condition

$$[(a,b)^\tau, (a',b')^\tau] = [(a,b),(a',b')]^\tau$$

plus the fact that the commutators span $\mathfrak{g}_0$ shows that on $\mathfrak{g}_0$ one must have

$$\tau : \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \longmapsto \begin{pmatrix} k_1k_2F\overline{X}^{\mathsf{T}}F & 0 \\ 0 & k_1k_2\overline{x} \end{pmatrix}.$$

Taking $x = 0$ we find that $k_1k_2 = -1$ since $\tau = \sigma$ on $\mathfrak{su}(2,2)$. Thus $k_1 = ir$, $k_2 = ir^{-1}$ where r is a nonzero real number, and

$$\tau : \begin{pmatrix} X & a \\ b^{\mathsf{T}} & x \end{pmatrix} \longmapsto \begin{pmatrix} -F\overline{X}^{\mathsf{T}}F & irF\overline{b} \\ ir^{-1}(F\overline{a})^{\mathsf{T}} & -\overline{x} \end{pmatrix}, \quad 0 \neq r \in \mathbf{R}.$$

Let θ be the linear automorphism of $V = \mathbf{C}^{4|1}$ that is $|r|^{1/2}I_4$ on V_0 and $|r|^{1/2}I_1$ on V_1. We also write θ for the corresponding automorphism of $\mathfrak{g}$. It is a simple calculation that

$$\tau = \theta\sigma_{\mathrm{sgn}(r)}\theta^{-1}.$$

Thus all real forms of $\mathfrak{g}$ of the type we are interested in are conjugate to $\mathfrak{g}_{\mathbf{R},\pm}$ by an automorphism of $\mathfrak{g}$ coming from an automorphism of $\mathbf{C}^{4|1}$.

THEOREM 3.4.2 *Any real form of* $\mathfrak{sl}(4|1)$ *whose even part has the simple component* $\simeq \mathfrak{su}(2,2)$ *is conjugate to one of* $\mathfrak{su}(2,2|1,0)_{\pm}$.

3.5. Modules over a Supercommutative Superalgebra

In the theory of manifolds, what happens at one point is entirely linear algebra, mostly of the tangent space and the space of tensors and spinors at that point. However, if one wants an algebraic framework for what happens on even a small open set, one needs the theory of modules over the smooth functions on that open set. For example, the spaces of vector fields and exterior differential forms on an open set are modules over the algebra of smooth functions on that open set. The situation is the same in the theory of supermanifolds as well. We shall therefore discuss some basic aspects of the theory of modules over supercommutative algebras.

Let A be a supercommutative superalgebra over the field k (of characteristic 0 as always). Modules are vector spaces over k on which A acts from the left; the action

$$a \otimes m \longmapsto a \cdot m\,, \quad a \in A,\ m \in M\,,$$

is assumed to be a morphism of super vector spaces, so that

$$p(a \cdot m) = p(a) + p(m)\,.$$

In particular, M_0 and M_1 are stable under A_0 and are interchanged under A_1. We often write am instead of $a \cdot m$. As in the classical theory, left modules may be viewed as right modules and vice versa, but in the super case this involves sign factors; thus M is viewed as a right module for A under the action

$$m \cdot a = (-1)^{p(a)p(m)} a \cdot m\,, \quad a \in A,\ m \in M\,.$$

A morphism $M \longrightarrow N$ of A-modules is an even k-linear map T such that $T(am) = aT(m)$. For modules M, N one has $M \otimes N$ defined in the usual manner by dividing $M \otimes_k N$ by the k-linear subspace spanned by the relations

$$ma \otimes n = m \otimes an\,, \quad a \in A\,.$$

The internal Hom $\mathbf{Hom}(M,N)$ is defined to be the space of k-linear maps T from M to N such that $T(am) = (-1)^{p(T)p(a)} aT(m)$. It is easily checked that $\mathrm{Hom}(M,N)$ is the space of k-linear maps T from M to N such that $T(ma) = T(m)a$. Thus

$$T \in (\mathbf{Hom}(M,N))_0 \iff T(am) = aT(m)\,,$$

$$T \in (\mathbf{Hom}(M,N))_1 \iff T(am) = (-1)^{p(a)} aT(m)\,.$$

$\mathbf{Hom}(M,N)$ is again an A-module if we define

$$(aT)(m) = aT(m)\,.$$

If we take $N = A$, we obtain the module dual to M, namely, M',

$$M' = \mathbf{Hom}(M,A)\,.$$

In all of these definitions, it is noteworthy how the rule of signs is used in carrying over to the super case the familiar concepts of the commutative theory. If $M = N$ we write $\mathbf{End}(M)$ for $\mathbf{Hom}(M,M)$ and $\mathrm{End}(M)$ for $\mathrm{Hom}(M,M)$.

A *free module* is an A-module that has a free *homogeneous* basis. If $(e_i)_{1\le i\le p+q}$ is a basis with e_i even or odd according as $i \le p$ or $p+1 \le i \le p+q$, we denote it by $A^{p|q}$, and define its *rank* as $p|q$. Thus

$$A^{p|q} = Ae_1 \oplus \cdots \oplus Ae_{p+q}\,, \quad e_i \text{ even or odd as } i \le \text{ or } > p.$$

To see that p, q are uniquely determined, we use the argument of "taking all the odd variables to 0." More precisely, let

$$J = \text{the ideal in } A \text{ generated by } A_1.$$

Then

$$J = A_1 \oplus A_1^2\,, \quad A_1^2 \subset A_0\,, \quad \frac{A}{J} \simeq \frac{A_0}{A_1^2}\,.$$

All elements of J are nilpotent and so $1 \notin J$; i.e., J is a proper ideal. Now A/J is a commutative ring with unit and so we can find a field F and a homomorphism of A/J into F. Then $A^{p|q} \otimes_F F$ is $F^{p|q}$, a super vector space of dimension $p|q$. Hence p and q are uniquely determined.

Morphisms between different $A^{p|q}$ can as usual be described through matrices, but a little more care than in the commutative case is necessary. We write elements of $A^{p|q}$ as $m = \sum_i e_i x^i$ so that

$$m \longleftrightarrow \begin{pmatrix} x^1 \\ \vdots \\ x^{p+q} \end{pmatrix}.$$

This means that m is even (resp., odd) if and only if the x^i are even (resp., odd) for $i \le p$ and odd (resp., even) for $i > p$, while for m to be odd, the conditions are reversed. If

$$T : A^{p|q} \longrightarrow A^{r|s}\,, \quad T \in \mathbf{Hom}(M, N)\,,$$

then

$$Te_j = \sum_j e_i t^i_j$$

so that T may be identified with the matrix (t^i_j); composition then corresponds to matrix multiplication. The matrix for T is then of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and T is even or odd according as the matrix is of the form

$$\begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix}$$

where "even" and "odd" refer to matrices whose elements are all even or all odd, respectively. If $A = k$, there are no odd elements of A, and so we recover the description given earlier. In the general case $\mathrm{Hom}(A^{p|q}, A^{p|q})$ is a superalgebra, and the associated super Lie algebra is denoted by $\mathfrak{gl}_A(p|q)$. Because there are in general nonzero odd elements in A, the definition of the supertrace has to be slightly modified. We put

$$\mathrm{str}(T) = \mathrm{tr}(A) - (-1)^{p(T)}\,\mathrm{tr}(D)\,, \quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

It can be checked that

$$\mathrm{str}(TU) = (-1)^{p(T)p(U)}\,\mathrm{str}(UT)\,.$$

Let M be a free module over A of rank $p|q$ with basis e_i where the e_i are even for $i \le p$ and odd for $i > p$. Let $M' = \mathbf{Hom}(M, A)$. Let $e'_i \in M'$ be defined by

$$e'_i(e_j) = \delta^i_j\,.$$

Then

$$p(e'_i) = 0\,, \quad 1 \le i \le p\,, \qquad p(e'_i) = 1\,, \quad p+1 \le i \le p+q\,,$$

and (e'_i) is a free homogeneous basis for M' so that $M' \simeq A^{p|q}$ also. For $m' \in M', m \in M$ we write

$$m'(m) = \langle m', m\rangle\,.$$

If $T \in \mathbf{Hom}(M, N)$ we define $T' \in \mathbf{Hom}(N', M')$ by

$$\langle T'n', m\rangle = (-1)^{p(T)p(n')}\langle n', Tm\rangle\,.$$

If

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then

$$T' = \begin{pmatrix} A^{\mathsf T} & C^{\mathsf T} \\ -B^{\mathsf T} & D^{\mathsf T} \end{pmatrix}, \quad T \text{ even},$$

$$T' = \begin{pmatrix} A^{\mathsf T} & -C^{\mathsf T} \\ B^{\mathsf T} & D^{\mathsf T} \end{pmatrix}, \quad T \text{ odd},$$

as can be easily checked. Unlike the commutative case, $T \longmapsto T'$ is not of period 2 but of period 4. We have

$$p(S') = p(S)\,, \quad (ST)' = (-1)^{p(S)p(T)}T'S'\,.$$

Derivations of Superalgebras. Let A be a superalgebra that need not be associative. A (homogeneous) *derivation* of A is a k-linear map $D(A \longrightarrow A)$ such that

$$D(ab) = (Da)b + (-1)^{p(D)p(a)}a(Db)\,.$$

Notice the use of the sign rule. If D is even, this reduces to the usual definition, but for odd D this gives the definition of the *odd derivations*. Let $\mathfrak{D} :=: \mathrm{Der}(A)$ be the super vector space of derivations. Then $\mathfrak{D}$ becomes a super Lie algebra if we define

$$[D_1, D_2] = D_1D_2 - (-1)^{p(D_1)p(D_2)}D_2D_1\,.$$

If we define aD for $a \in A$ as $x \longmapsto aD(x)$, D being an element of $\mathfrak{D}$, then $\mathfrak{D}$ becomes a A-module.

3.6. The Berezinian (Superdeterminant)

One of the most striking discoveries in super linear algebra is the notion of *superdeterminant*. It is due to Berezin, who was a pioneer in supergeometry and superanalysis, and who stressed the fact that this subject is a vast generalization of classical geometry and analysis. After his untimely and unfortunate death, the superdeterminant became known as the *Berezinian*. Unlike the classical determinant, the Berezinian *is defined only for invertible linear transformations*; this is already an indication that it is more subtle than its counterpart in the classical theory. It plays the role of the classical Jacobian in problems where we have to integrate over supermanifolds and have to change coordinates. At this time we shall be concerned only with the linear algebraic aspects of the Berezinian.

In the simplest case when $A = k$ and $T \in \mathbf{End}(M)$ is even, the matrix of T is

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

If the relation $\det(e^X) = e^{\operatorname{tr}(X)}$ in the commutative situation is to persist in the supercommutative situation where the supertrace replaces the trace, one *has to make the definition*

$$\operatorname{Ber}(T) = \det(A)\det(D)^{-1},$$

since the supertrace of the matrix $X = \left(\begin{smallmatrix} U & 0 \\ 0 & V \end{smallmatrix}\right)$ is $\operatorname{tr}(U) - \operatorname{tr}(V)$. Thus already we must have D invertible. In the general case when we are dealing with modules over a general supercommutative superalgebra A, we first observe the following lemma:

LEMMA 3.6.1 *If*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{End}(M)$$

is even, then T is invertible if and only if A and D are invertible matrices over the commutative ring A_0; i.e., $\det(A)$ *and* $\det(D)$ *are units of A_0.*

PROOF: As in a previous situation, we do this by going to the case when the odd variables are made 0. Let $J = A_1 + A_1^2$ be the ideal in A generated by A_1, and let $\overline{A} = A/J$. For any matrix L over A, let $\overline{L}$ be the matrix over $\overline{A}$ obtained by applying the map $A \longrightarrow \overline{A}$ to each entry of L.

We claim first that L is invertible if and only if $\overline{L}$ is invertible over $\overline{A}$. We consider only right inverses because the argument for left inverses is the same (recall that if both right and left inverses exist, they are equal). If L is invertible it is obvious that $\overline{L}$ is invertible. Indeed, if $LM = 1$, then $\overline{L}\,\overline{M} = 1$. Suppose that $\overline{L}$ is invertible. This means that we can find a matrix M over A such that $LM = I + X$ where X is a matrix over A such that all its entries are in J. It is enough to prove that $I + X$ is invertible, and for this it is sufficient to show that X is nilpotent, i.e., $X^r = 0$ for some integer $r \geq 1$. There are odd elements $o_1, \dots, o_N$ such that any entry of X is of the form $\sum_i a_i o_i$ for suitable $a_i \in A$. If $r = N + 1$, it is clear that any product $o_{i_1} \cdots o_{i_r} = 0$ because two of the o_i's have to be identical. Hence $X^r = 0$. This proves our claim.

This said, we return to the proof of the lemma. Since T is even, A, D have even entries and B, C have odd entries. Hence

$$\overline{T} = \begin{pmatrix} \overline{A} & 0 \\ 0 & \overline{D} \end{pmatrix}$$

so that T is invertible if and only if $\overline{A}$ and $\overline{D}$ are invertible, which in turn happens if and only if A and D are invertible. The lemma is proven. □

For any T as above, we have the easily verified decomposition

$$(*) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}.$$

Since we want the Berezinian to be multiplicative, this shows that we have no alternative except to define

$$\mathrm{Ber}(T) = \det(A - BD^{-1}C)\det(D)^{-1}, \quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, D \text{ even.}$$

We take this as the definition of $\mathrm{Ber}(T)$. With this definition we have

$$\mathrm{Ber}(T) = 1, \quad T = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}, \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}, \quad B, C \text{ odd.}$$

The roles of A and D appear to be different in the definition of $\mathrm{Ber}(X)$. This is, however, only an apparent puzzle. If we use the decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix},$$

then, assuming that Ber is multiplicative, we obtain

$$\mathrm{Ber}(X) = \det(D - CA^{-1}B)^{-1}\det(A), \quad X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

That the two definitions are the same will follow after we have shown that Ber is multiplicative and has the obvious definition on the even (block) diagonal elements. Notice that all the matrices whose determinants are taken have even entries, and so the determinants make sense. In particular, $\mathrm{Ber}(T)$ is an element of A_0.

Let $\mathrm{GL}_A(p|q)$ denote the group of all invertible even elements of $\mathrm{End}(\mathbf{R}^{p|q})$. We then have the basic theorem.

THEOREM 3.6.2 *Let*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an even element of $\mathbf{End}(\mathbf{R}^{p|q})$. *Then*

(i) *T is invertible if and only if A and D are invertible, and*

(ii) $\mathrm{Ber}(X)$ *is an element of* A_0.

If $X, Y \in \mathrm{GL}_A(p|q)$, *then*

$$\mathrm{Ber}(XY) = \mathrm{Ber}(X)\,\mathrm{Ber}(Y), \quad X, Y \in \mathrm{GL}_A(p|q).$$

In particular, $\mathrm{Ber}(X)$ *is a unit of* A_0.

PROOF: The first statement has already been established. We now prove (ii). Let $G = \mathrm{GL}_A(p|q)$ and let G^+, G^0, G^- be the subgroups of G consisting of elements of the respective form g^+, g^0, g^- where

$$g^+ = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \quad g^0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad g^- = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}.$$

From $(*)$ we see that any element $g \in G$ can be expressed as a triple product $g = g^+g^0g^-$. We then have $\mathrm{Ber}(g^{\pm}) = 1$ and $\mathrm{Ber}(g) = \mathrm{Ber}(g^0) = \det(A)\det(D)^{-1}$. The triple product decompositions of g^+g, g^0g, gg^0, gg^- are easy to obtain in terms of the one for g, and so it is easily established that $\mathrm{Ber}(XY) = \mathrm{Ber}(X)\,\mathrm{Ber}(Y)$ for all Y if $X \in G^+, G^0$, and for all X if $Y \in G^-, G^0$. The key step is now to prove that

$$(*) \qquad \mathrm{Ber}(XY) = \mathrm{Ber}(X)\,\mathrm{Ber}(Y)$$

for all X if $Y \in G^+$. It is clearly enough to assume that $X \in G^-$. Thus we assume that

$$X = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}.$$

Now

$$B \longmapsto \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$$

maps the additive group of A_1 homomorphically into G^+, and so we may assume in proving $(*)$ that B is elementary; i.e., all but one entry of B is 0, and that one is an odd element β. Thus we have

$$X = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & E \\ 0 & 1 \end{pmatrix}, \quad E \text{ elementary}.$$

Then, remembering that $1 + CE$ is invertible,

$$XY = \begin{pmatrix} 1 & E \\ C & 1 + CE \end{pmatrix}, \quad \mathrm{Ber}(XY) = \det(1 - E(1+CE)^{-1}C)\det(1+CE)^{-1},$$

so that we have to prove that

$$\det(1 - E(1+CE)^{-1}C)\det(1+CE)^{-1} = 1.$$

Since E has a single nonzero entry β, which is odd, all entries of any matrix of the form EX, XE are divisible by β. Hence the product of any two elements of any two of these matrices is 0. This means, in the first place, that $(CE)^2 = 0$, and so

$$(1 + CE)^{-1} = 1 - CE$$

and hence

$$1 - E(1+CE)^{-1}C = 1 - E(1 - CE)C = 1 - EC.$$

If L is any matrix of even elements such that the product of any two entries of L is 0, then a direct computation shows that

$$\det(1 + L) = 1 + \mathrm{tr}(L).$$

Hence

$$\det(1 - E(1 + CE)^{-1}C) = \det((1 - EC)) = 1 - \mathrm{tr}(EC)\,.$$

Moreover,

$$\det((1 + CE)^{-1}) = (\det(1 + CE))^{-1} = (1 + \mathrm{tr}(CE))^{-1}\,.$$

Hence

$$\det(1 - E(1 + CE)^{-1}C)\det(1 + CE)^{-1} = (1 - \mathrm{tr}(EC))(1 + \mathrm{tr}(CE))^{-1}\,.$$

But, as C, E have only odd entries, $\mathrm{tr}(CE) = -\mathrm{tr}(EC)$ so that

$$\det(1 - E(1 + CE)^{-1}C)\det(1 + CE)^{-1} = (1 + \mathrm{tr}(CE))(1 + \mathrm{tr}(CE))^{-1} = 1\,,$$

as we wanted to prove.

The proof of the multiplicativity of Ber can now be completed easily. Let G' be the set of all $Y \in G$ such that $\mathrm{Ber}(XY) = \mathrm{Ber}(X)\,\mathrm{Ber}(Y)$ for all $X \in G$. We have seen earlier that G' is a subgroup containing G^-, G^0, and we have seen just now that it contains G^+ as well. Hence $G' = G$, finishing the proof of the theorem. □

COROLLARY 3.6.3 *If*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then

$$\mathrm{Ber}(T) = \det(A)\det(D - CA^{-1}B)^{-1}\,.$$

Let M be a free module of rank $p|q$ over A. Then $M \simeq A^{p|q}$, and any invertible $\mathbf{End}(M)$ can be represented by a matrix $X^{\sim}$. If we choose another basis for M, the matrix for X changes to $X'^{\sim} = CX^{\sim}C^{-1}$ where C is some invertible *even* matrix. Hence $\mathrm{Ber}(X^{\sim}) = \mathrm{Ber}(X'^{\sim})$. If we define $\mathrm{Ber}(X)$ as $\mathrm{Ber}(X^{\sim})$, then $\mathrm{Ber}(X)$ is well-defined and gives a homomorphism

$$\mathrm{Ber} : \mathrm{Aut}(M) \longrightarrow A_0^{\times} = \mathrm{GL}_A(1|0)$$

where $A_0^{\times}$ is the group of units of A_0. The following properties are now easy to establish:

(a) $\mathrm{Ber}(X^{-1}) = \mathrm{Ber}(X)^{-1}$.
(b) $\mathrm{Ber}(X') = \mathrm{Ber}(X)$.
(c) $\mathrm{Ber}(X \oplus Y) = \mathrm{Ber}(X)\,\mathrm{Ber}(Y)$.

3.7. The Categorical Point of View

The category of vector spaces and the category of super vector spaces, as well as the categories of modules over commutative and supercommutative rings, are examples of categories where there is a notion of tensor product that is functorial in each variable. Such categories first appeared implicitly in the work of Tannaka, who proved a duality theorem for compact nonabelian groups that generalized the Pontryagin duality for abelian groups.[5] The equivalence classes of irreducible representations of a nonabelian compact group do not form any reasonable algebraic structure, but if for any compact group G one considers the *category* $\mathrm{Rep}(G)$ of all finite-dimensional unitary G-modules, we obtain a category in which there is an algebraic operation, namely, that of $\otimes$, the tensor product of two representations. If $g \in G$, then for each unitary G-module V, we have the element $V(g)$ that gives the action of g on V; $V(g)$ is an element of the unitary group $\mathcal{U}(V)$ of V, and the assignment

$$V \longmapsto V(g)$$

is a functor compatible with tensor products and duals. The celebrated *Tannaka duality theorem* may then be formulated as the statement that *G can be identified with the group of all such functors*. The first systematic study of abstract categories with a tensor product was that of Saavedra,[6] in which many ideas of Grothendieck played a critical role. Subsequently tensor categories have been the object of study by Deligne and Milne,[7] Deligne,[8] and Doplicher and Roberts.[9] In this section we shall give a brief discussion of how the point of view of tensor categories illuminates the theory of super vector spaces and supermodules. For more information see the above references.

The basic structure from the categorical point of view is that of an abstract category $\mathcal{C}$ with a binary operation $\otimes$,

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}, \quad X, Y \longmapsto X \otimes Y,$$

where $X \otimes Y$ is the "tensor product" of X and Y. We shall not go into the precise details about the axioms but confine ourselves to some remarks. The basic axiom is that the operation $\otimes$ satisfies the following:

Associativity constraint: This means that there is a functorial isomorphism

$$(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$$

satisfying what is called the pentagon axiom involving four objects.

Commutativity constraint: There is a functorial isomorphism

$$X \otimes Y \simeq Y \otimes X$$

satisfying the so-called hexagon axiom involving three objects.

Unit constraint: There is a unit object 1 with an isomorphism $1 \simeq 1 \otimes 1$ such that $X \longmapsto 1 \otimes X$ is an equivalence of categories of $\mathcal{C}$ with itself. In particular, we have unique functorial isomorphisms

$$X \simeq 1 \otimes X, \quad X \simeq X \otimes 1.$$

To this one adds the general assumption that $\mathcal{C}$ is an abelian category. For any object X we write $\mathrm{End}(X)$ for the ring $\mathrm{Hom}(X, X)$. The category is said to be k-linear, k a field, if $k \subset \mathrm{End}(1)$. Then all $\mathrm{Hom}(X, Y)$ become vector spaces over k and $\mathrm{End}(X)$ become k-algebras.

In the category of vector spaces or modules over a commutative ring with unit, the unit object is the ring itself, and the commutativity isomorphism is just the map

$$u \otimes v \longrightarrow v \otimes u.$$

In the super categories it is the map

$$u \otimes v \longrightarrow (-1)^{p(u)p(v)} v \otimes u.$$

In the general case one can use the associativity and commutativity constraints to define the tensor products of arbitrary finite families of objects in a natural manner and actions of the permutation group on tensor powers of a single object. We have done this in detail in the category of super vector spaces already.

In order to do anything serious, one has to assume that the category $\mathcal{C}$ admits the so-called *internal* Hom, written **Hom**. Before we do this, we shall briefly describe a general method by which objects are defined in a category. Suppose $\mathcal{T}$ is any category. For any object A the assignment

$$T \longmapsto \mathrm{Hom}(T, A)$$

is then a contravariant functor from $\mathcal{T}$ to the category of sets. If A, B are objects in $\mathcal{T}$ and $f(A \longrightarrow B)$ is an isomorphism, it is immediate that for any object T, there is a functorial bijection

$$\mathrm{Hom}(T, A) \simeq \mathrm{Hom}(T, B)\,, \quad x \leftrightarrow fx\,.$$

Conversely, suppose that A, B are two objects in $\mathcal{T}$ with the property that there is a functorial bijection

$$\mathrm{Hom}(T, A) \simeq \mathrm{Hom}(T, B)\,.$$

Then A and B are isomorphic; this is a consequence of the so-called *Yoneda lemma.* Indeed, taking $T = A$, let f be the element of $\mathrm{Hom}(A, B)$ that corresponds under the above bijection to id_A; similarly, taking $T = B$ let g be the element of $\mathrm{Hom}(B, A)$ that corresponds to id_B. It is then an easy exercise to show that $fg = \mathrm{id}_B, gf = \mathrm{id}_A$, proving that A and B are uniquely isomorphic given this data. However, if we have a contravariant functor F from $\mathcal{T}$ to the category of sets, it is not always true that there is an object A in the category such that we have a functorial identification

$$F(T) \simeq \mathrm{Hom}(T, A)\,.$$

By Yoneda's lemma, we know that A, if it exists, is determined up to a unique isomorphism. Given F, if A exists, we shall say that F is *representable* and is *represented by* A.

If A and B are any two objects, then for any map $A \longrightarrow B$ we have an associated map

$$\mathrm{Hom}(T, A) \longrightarrow \mathrm{Hom}(T, B)$$

that is functorial in T. Yoneda's lemma is the statement that the mapping of sets

$$\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(\mathrm{Hom}(\,\cdot\,, A), \mathrm{Hom}(\,\cdot\,, B))$$

is a bijection. The proof is essentially the same as in the preceding discussion.

This said, let us return to the category $\mathcal{C}$. We now assume that for each pair of objects X, Y the functor

$$T \longmapsto \mathrm{Hom}(T \otimes X, Y)$$

is representable. This means that there is an object $\mathbf{Hom}(X, Y)$ with the property that

$$\mathrm{Hom}(T, \mathbf{Hom}(X, Y)) = \mathrm{Hom}(T \otimes X, Y)$$

for all objects T. This object is unique up to unique isomorphism. This assumption leads to a number of consequences. Using $X \simeq 1 \otimes X$ we have

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(1, \mathbf{Hom}(X, Y)).$$

In the vector space or module categories Hom is the same as **Hom**. However, in the super categories, Hom is the space of even maps, while **Hom** is the space of all maps. If we take T to be $\mathbf{Hom}(X, Y)$ itself, we find that corresponding to the identity map of $\mathbf{Hom}(X, Y)$ into itself there is a map

$$\mathrm{ev}_{X,Y} : \mathbf{Hom}(X, Y) \otimes X \longrightarrow Y.$$

This is the so-called *evaluation map*, so named because in the category of modules it is the map that takes $L \otimes v$ to $L(v)$. It has the property that for any $t \in \mathrm{Hom}(T \otimes X, Y)$, the corresponding element $s \in \mathrm{Hom}(T, \mathbf{Hom}(X, Y))$ is related to t by

$$\mathrm{ev}_{X,Y} \circ (s \otimes \mathrm{id}) = t.$$

Finally, we can define the dual of any object by

$$X^* = \mathbf{Hom}(X, 1), \quad \mathrm{Hom}(T, X^*) = \mathrm{Hom}(T \otimes X, 1).$$

We have the evaluation map

$$\mathrm{ev}_X := \mathrm{ev}_{X,1} : X^* \otimes X \longrightarrow 1.$$

Using the commutativity isomorphism, we then have the map

$$X \otimes X^* \longrightarrow 1,$$

which gives a map

$$X \longrightarrow X^{**}.$$

An object is called *reflexive* if

$$X = X^{**}.$$

Already, in the category of vector spaces, only finite-dimensional spaces are reflexive. More generally, free modules of finite rank are reflexive. In the category of modules over a supercommutative k-algebra A, the free modules $A^{p|q}$ are easily seen to be reflexive. However, in the category of modules of finite rank, not all objects are reflexive.

Consider now an object X such that

$$X^* \otimes Y \longrightarrow \mathbf{Hom}(X, Y)$$

is an isomorphism for all objects Y. This is the case, for example, in the category of modules over a supercommutative A when $X = A^{p|q}$, as we shall see presently. In particular,

$$X^* \otimes X \longrightarrow \mathbf{End}(X)$$

is an isomorphism. We thus have the composite map

$$\mathrm{Tr} : \mathbf{End}(X) \longrightarrow X^* \otimes X \longrightarrow 1$$

where the last map is the evaluation map described above. This is the trace map, defined on all of $\mathbf{End}(X)$.

Let us see how Tr reduces to the supertrace in the category of modules over a supercommutative k-algebra A with unit. We take

$$X = A^{p|q}.$$

In this case we can explicitly write the isomorphism

$$\mathbf{Hom}(A^{p|q}, Y) \simeq (A^{p|q})^* \otimes Y.$$

Let (e_i) be a homogeneous basis for $A^{p|q}$, and let $p(i) = p(e_i)$. Let (ξ^j) be the dual basis for $(A^{p|q})^*$ so that $\xi^j(e_i) = \delta_{ij}$. The map

$$(A^{p|q})^* \otimes Y \simeq \mathbf{Hom}(A^{p|q}, Y)$$

is then given by

$$\xi \otimes y \longmapsto t_{\xi\otimes y}, \quad t_{\xi\otimes y}(x) = (-1)^{p(x)p(y)}\xi(x)y.$$

A simple calculation shows that any homogeneous $f \in \mathbf{Hom}(A^{p|q}, Y)$ can be expressed as

$$f = \sum_j (-1)^{p(j)+p(f)p(j)}\xi^j \otimes f(e_j).$$

If we take $Y = A^{p|q}$, we get

$$\delta(f) = \sum_j (-1)^{p(j)(1+p(f))}\xi^j(f(e_j)).$$

Suppose now f is represented by the matrix (M^i_j) so that

$$f(e_j) = \sum_i e_i M^i_j.$$

Then

$$\delta(f) = \sum_{ij} (-1)^{p(j)(1+p(f))}\xi^j(e_i M^i_j)$$

so that

$$\mathrm{Tr}(f) = \sum_{ij} (-1)^{p(j)(1+p(f))}\delta^j_i M^i_j = \sum_{a \text{ even}} M^a_a - (-1)^{p(f)} \sum_{b \text{ odd}} M^b_b.$$

We have thus recovered our ad hoc definition. This categorical definition shows that the supertrace is independent of the basis used to compute it.

Even Rules. In the early days of the discovery of supersymmetry, the physicists used the method of introduction of auxiliary odd variables as a guide to make correct definitions. As an illustration, let us suppose we want to define the correct symmetry law for the superbracket. If X, Y are odd elements, we introduce auxiliary odd variables ξ, η that supercommute. Since ξX and ηY are both even, we have

$$[\xi X, \eta Y] = -[\eta Y, \xi X].$$

But, using the sign rule, we get

$$[\xi X, \eta Y] = -\xi\eta[X, Y], \quad [\eta Y, \xi X] = -\eta\xi[Y, X],$$

so that, as $\xi\eta = -\eta\xi$, we have

$$[X, Y] = [Y, X].$$

A similar argument can be given for the definition of the super Jacobi identity. These examples can be generalized into a far-reaching principle from the categorical point of view.

The Even Rules Principle. For any vector space V over k and any supercommutative k-algebra B, we write

$$V(B) = (V \otimes B)_0 = \text{the even part of } V \otimes B.$$

Clearly, $B \longmapsto V(B)$ is functorial in B. If

$$f : V_1 \times \cdots \times V_N \longrightarrow V$$

is multilinear, then, for any B, we have a natural extension

$$f_B : V_1(B) \times \cdots \times V_N(B) \longrightarrow V(B),$$

which is B_0-multilinear and functorial in B. The definition of f_B is simply

$$f_B(b_1 v_1, \ldots, b_N v_N) = (-1)^{m(m-1)/2} b_1 \cdots b_N f(v_1, \ldots, v_N),$$

where the $b_i \in B$, $v_i \in V_i$ are homogeneous and m is the number of b_i (or v_i) that are odd. The system (f_B) is functorial in B. The *principle of even rules* states that any functorial system (f_B) of B_0-multilinear maps

$$f_B : V_1(B) \times \cdots \times V_N(B) \longrightarrow V(B)$$

arises from a unique k-multilinear map

$$f : V_1 \times \cdots \times V_N \longrightarrow V.$$

The proof is quite simple; see Deligne and Morgan.[10] The proof just formalizes the examples discussed above. It is even enough to restrict the B's to the exterior algebras. These are just the auxiliary odd variables used heuristically.

The categorical view is, of course, hardly needed while making calculations in specific problems. However, it is essential for an understanding of super linear algebra at a fundamental level. One can go far with this point of view. As we have seen earlier, one can introduce Lie objects in a tensor category and one can even prove the Poincaré-Birkhoff-Witt theorem in the categorical context. For this and other aspects, see Deligne and Morgan.[10]

Everything discussed so far is based on the assumption that k has characteristic 0. In positive characteristic the main results on the tensor categories require interesting modifications.[11]

3.8. References

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CHAPTER 4

Elementary Theory of Supermanifolds

4.1. The Category of Ringed Spaces

The unifying concept that allows us to view differentiable, analytic, or holomorphic manifolds, and also algebraic varieties, is that of a *ringed space*. This is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ (written as $\mathcal{O}$ when there is no doubt as to what X is) is a sheaf of commutative rings (with units) on X. For instance, let X be a Hausdorff second countable space carrying a smooth structure, and let $C^\infty(U \longrightarrow C^\infty(U))$ be the sheaf of rings where, for each open set $U \subset X$, $C^\infty(U)$ is the $\mathbf{R}$-algebra of all smooth functions on U. Then (X, C^∞) is a ringed space that is locally isomorphic to the ringed space associated to a ball in $\mathbf{R}^n$ with its smooth structure.

To formulate this notion more generally, let us start with a topological space X. For each open $U \subset X$, let $R(U)$ be an $\mathbf{R}$-algebra of real *functions* such that the assignment

$$U \longmapsto R(U)$$

is a *sheaf* of algebras of functions. This means that the following conditions are satisfied:

(a) Each $R(U)$ contains the constants, and if $V \subset U$, then the restriction map takes $R(U)$ into $R(V)$.

(b) If U is a union of open sets U_i and $f_i \in R(U_i)$, and if the (f_i) are compatible, i.e., given i, j, f_i, and f_j have the same restriction to $U_i \cap U_j$, then the function f such that f_i is the restriction of f to U_i belongs to $R(U)$.

We call (X, R) a *ringed space of functions*. If (X, R) and (Y, S) are two such spaces, a *morphism* between (X, R) and (Y, S) is a continuous map $\psi(X \longrightarrow Y)$ such that the pullback map ψ^* takes $S(V)$ into $R(\psi^{-1}(V))$ for each open set $V \subset Y$; here

$$(\psi^*(g))(x) = g(\psi(x)), \quad g \in S(V).$$

We have thus obtained the *category* of ringed spaces of functions. If (X, R) is a ringed space of functions and $Y \subset X$ is an open set, the space (Y, R_Y) is also a ringed space of functions if $R_Y = R|Y$ is the *restriction of R to Y*, i.e., for any open set $V \subset Y$, $R_Y(U) = R(U)$. We refer to (Y, R_Y) as the *open subspace* of (X, R) defined by Y; the identity map from Y to X is then a morphism.

In order to define specific types of ringed spaces of functions, we choose local models and define the corresponding types of ringed spaces of functions as those locally isomorphic to the local models. For example, to define a smooth manifold we start with the ringed spaces $(\mathbf{R}^n, C_n^\infty)$ where

$$C_n^\infty : U \longmapsto C_n^\infty(U),$$

$C_n^\infty(U)$ being the $\mathbf{R}$-algebra of smooth functions on U. Then a differentiable or a smooth manifold can be defined as a ringed space (X, R) of functions such that for each point $x \in X$ there is an open neighborhood U of x and a homeomorphism of h of U with an open set $U^\sim \subset \mathbf{R}^n$ such that h is an isomorphism of (U, R_U) with the ringed space of functions $(U^\sim, C_n^\infty|U^\sim)$, i.e., if $V \subset U$ is open, the algebra $R(V)$ is precisely the algebra of all functions $g \circ h$ where g is a smooth function on $h(V)$. To define an analytic or a complex analytic manifold, the procedure is similar; we simply replace $(\mathbf{R}^n, C_n^\infty)$ by $(\mathbf{R}^n, A_n)$ or $(\mathbf{C}^n, H_n)$ where A_n (resp., H_n) is the sheaf of algebras of analytic (resp., complex analytic) functions. It is usual to add additional conditions of separation and globality on X, for instance, that X be Hausdorff and second countable.

In algebraic geometry, Serre pioneered an approach to algebraic varieties by defining them as ringed spaces of functions locally isomorphic to the ringed spaces coming from affine algebraic sets over an algebraically closed field. See Dieudonné[1] for the theory of these varieties, which he calls *Serre varieties*. It is possible to go far in the Serre framework; for instance, it is possible to give quite a practical and adequate treatment of the theory of affine algebraic groups.

However, as we have mentioned before, Grothendieck realized that ultimately the Serre framework is inadequate and that one has to replace the coordinate rings of affine algebraic sets with *completely arbitrary commutative rings with unit*, i.e., in the structure sheaf the rings of functions are replaced by arbitrary commutative rings with unit. This led to the more general definition of a ringed space leading to the Grothendieck schemes. It turns out that this more general notion of a ringed space is essential for super geometry.

DEFINITION A *sheaf of rings on a topological space* X is an assignment

$$U \longmapsto R(U)$$

where $R(U)$ is a commutative ring with unit, with the following properties:

(i) If $V \subset U$ there is a homomorphism from $R(U)$ to $R(V)$, called *restriction to* V, denoted by r_{VU}; for three open sets $W \subset V \subset U$, we have $r_{WV} r_{VU} = r_{WU}$.

(ii) If U is the union of open sets U_i and $f_i \in R(U_i)$ are given, then for the existence of $f \in R(U)$ that restricts on U_i to f_i for each i, it is necessary and sufficient that f_i and f_j have the same restrictions on $U_i \cap U_j$; moreover, f, when it exists, is unique.

A *ringed space* is a pair $(X, \mathcal{O})$ where X is a topological space and $\mathcal{O}$ is a sheaf of rings on X. $\mathcal{O}$ is called the *structure sheaf* of the ringed space. For any open set U, the elements of $\mathcal{O}(U)$ are called *sections over* U. If it is necessary to call attention to X, we write $\mathcal{O}_X$ for $\mathcal{O}$.

If $x \in X$ and U, V are open sets containing x, we say that two elements $a \in \mathcal{O}(U)$, $b \in \mathcal{O}(V)$ are *equivalent* if there is an open set W with $x \in W \subset U \cap V$ such that a and b have the same restrictions to W. The equivalence classes are as usual called *germs* of sections of $\mathcal{O}$ and form a ring $\mathcal{O}_x$ called the *stalk* of the sheaf at x. The notion of a *space* is then obtained if we make the following definition.

DEFINITION A ringed space is called a *space* if the stalks are all local rings.

Here we recall that a commutative ring with unit is called *local* if it has a unique maximal ideal. The unique maximal ideal of $\mathcal{O}_x$ is denoted by $\mathfrak{m}_x$. The elements of $\mathcal{O}_x \setminus \mathfrak{m}_x$ are precisely the invertible elements of $\mathcal{O}_x$.

The notion of an open subspace of a ringed space is obtained as before; one just restricts the sheaf to the open set in question. In defining morphisms between ringed spaces one has to be careful because the rings of the sheaf are abstractly attached to the open sets and there is no automatic pullback as in the case when the rings were rings of functions. But the solution to this problem is simple. One also gives the pullbacks in defining morphisms. Thus a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a continuous map ψ from X to Y together with a sheaf map of $\mathcal{O}_Y$ to $\mathcal{O}_X$ above ψ, i.e., a collection of homomorphisms

$$\psi_V^* : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(\psi^{-1}(V)), \quad V \text{ open} \subset Y,$$

that commute with restrictions. The notion of an isomorphism of ringed spaces follows at once. We have thus obtained the *category* of ringed spaces. If the objects are spaces, we require that the pullback, which induces a map $\mathcal{O}_{Y,\psi(x)} \longrightarrow \mathcal{O}_{X,x}$, is local; i.e., it takes the maximal ideal $\mathfrak{m}_{\psi(x)}$ of $\mathcal{O}_{Y,\psi(x)}$ into the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$.

In the case when the rings $\mathcal{O}(U)$ are actually rings of *functions* with values in a field k, the pullbacks defined earlier are in general the only ones possible. To see this, assume that X and Y are ringed spaces of functions and that the stalks are local rings. For $x \in X$, the elements of $\mathcal{O}_{X,x}$ vanishing at x form an ideal I_x and so are contained in $\mathfrak{m}_x$. Since I_x has codimension 1, being the kernel of the evaluation map $f \longmapsto f(x)$, we must have $I_x = \mathfrak{m}_x$. Then if an element has a nonzero value at a point, its restriction to some open set V containing x is invertible in $\mathcal{O}_X(V)$. Now suppose that we have an arbitrary pullback ψ^* defined as above. Fix $x \in X$ and let $\psi(x) = y$. If $\psi^*(g)(x) \neq g(\psi(x))$ for some $g \in S(V)$, we may, by adding a constant to g, assume that $\psi^*(g)(x) = 0$, $g(\psi(x)) \neq 0$. So g is invertible on some V; hence $\psi^*(g)$ is invertible in an open neighborhood of x, contradicting the assumption that $\psi^*(g)(x) = 0$. This also shows that in this case the locality condition is automatically satisfied.

Using very general results from commutative algebra, one can represent any commutative ring with unit as a ring of "functions" on some space, even though the field in which these functions take their values will in general vary from point to point. Indeed, the space is the set of *prime ideals* of the ring, and at any prime ideal we have the field of quotients of the integral domain, which is the ring modulo the prime ideal; the value of an element of the ring at this prime ideal is its image in this field. But, as we explained in Chapter 2, this representation need not be faithful; there will be elements that go to the zero function. For instance, this is the case for nilpotent elements. This fact makes the discussion of schemes more subtle.

To get supergeometric objects, we know that we have to replace everywhere the commutative rings by supercommutative rings. Thus a *super ringed space* is a topological space X with a sheaf of supercommuting rings with units, called the *structure sheaf.* The restriction homomorphisms of the sheaf must be morphisms in the super category and so must preserve the gradings. The definition of morphisms of super ringed spaces is exactly the same as for ringed spaces, with the only change being that the pullback maps (ψ_V^*) must be morphisms in the category of supercommutative rings, i.e., preserve the gradings. We thus obtain the category of super ringed spaces. For any two objects X, Y in this category, $\mathrm{Hom}(X, Y)$ denotes as usual the set of morphisms $X \longrightarrow Y$. A *superspace* is a super ringed space such that the stalks are local supercommutative rings. A supermanifold is a special type of superspace.

Here we must note that a supercommutative ring is called *local* if it has a unique maximal homogeneous ideal. Since the odd elements are nilpotent, they are in any homogeneous maximal ideal, and so this amounts to saying that the even part is a commutative local ring. More precisely, we have the following:

LEMMA 4.1.1 *Let A be a supercommutative ring. Then A is local if and only if the even part A_0 of A is local in the usual sense. In this case, the maximal ideal* $\mathfrak{m}_0$ *of A_0 and the maximal homogeneous ideal* $\mathfrak{m}$ *of A are related by*

$$\mathfrak{m}_0 = A_0 \cap \mathfrak{m}\,, \quad \mathfrak{m} = \mathfrak{m}_0 \oplus A_1\,.$$

Moreover, $\mathfrak{m} \supset J$ *where J is the ideal generated by A_1. Finally $k = A/\mathfrak{m}$ is a field.*

PROOF: We have already seen in Section 3.6 that $J = A_1^2 \oplus A_1$ consists of nilpotent elements and that A_1^2 is an ideal of A_0. If $\mathfrak{a}_0$ is any ideal of A_0, $1 \notin \mathfrak{a}_0 + A_1^2$, for if $1 = a_0 + b$, $a_0 \in A_0$, $b \in A_1^2$, then $a_0 = 1 - b$ is invertible since b is nilpotent, and so $1 \in \mathfrak{a}_0$, a contradiction. Hence $\mathfrak{a}_0 + A_1^2$ is an ideal. This said, suppose first that A is local and $\mathfrak{m}$ is its maximal homogeneous ideal. Then $J \subset \mathfrak{m}$, and if $\mathfrak{a}_0$ is any ideal of A_0, then

$$\mathfrak{a}_0 \subset \mathfrak{a}_0 \oplus J = \left(\mathfrak{a}_0 + A_1^2\right) \oplus A_1 \subset \mathfrak{m}$$

and so $\mathfrak{a}_0 \subset \mathfrak{m} \cap A_0$. Thus A_0 is local and $\mathfrak{m} \cap A_0$ is its maximal ideal. Conversely, let A_0 be local and $\mathfrak{m}_0$ its maximal ideal. Then $\mathfrak{m}_0 \supset A_1^2$ and so $\mathfrak{m} = \mathfrak{m}_0 \oplus A_1$ is a homogeneous ideal of A. If $\mathfrak{m}'$ is any homogeneous ideal of A, then

$$\mathfrak{m}' = (\mathfrak{m}' \cap A_0) \oplus (\mathfrak{m}' \cap A_1) \subset \mathfrak{m}_0 \oplus A_1 = \mathfrak{m}\,.$$

So A is local and $\mathfrak{m}$ is its maximal homogeneous ideal. Since $A/\mathfrak{m} = A_0/\mathfrak{m}_0$, $k = A/\mathfrak{m}$ is a field. □

4.2. Supermanifolds

To introduce supermanifolds we follow the example of classical manifolds and introduce first the *local models.* A *superdomain* $U^{p|q}$ is the super ringed space

$(U^p, \mathcal{C}^{\infty p|q})$ where U^p is an open set in $\mathbf{R}^p$ and $\mathcal{C}^{\infty p|q}$ is the sheaf of supercommuting rings defined by

$$\mathcal{C}^{\infty p|q} : V \longmapsto C^\infty(V)[\theta^1, \theta^2, \ldots, \theta^q], \quad V \subset U \text{ open},$$

where the θ^j are anticommuting variables (indeterminates) satisfying the relations

$$\theta^{i^2} = 0, \quad \theta^i\theta^j = -\theta^j\theta^i, \quad i \neq j \iff \theta^i\theta^j = -\theta^j\theta^i, \quad 1 \leq i, j \leq q.$$

Thus each element of $\mathcal{C}^{\infty p|q}(V)$ can be written as

$$\sum_{I \subset \{1,\ldots,q\}} f_I \theta^I$$

where the $f_I \in C^\infty(V)$ and θ^I is given by

$$\theta^I = \theta^{i_1} \cdots \theta^{i_r}, \quad I = \{i_1, \ldots, i_r\},\ i_1 < \cdots < i_r.$$

The dimension of this superdomain is defined to be $p|q$. We omit the reference to the sheaf and call $U^{p|q}$ itself the superdomain. In particular, we have the superdomains $\mathbf{R}^{p|q}$. A *supermanifold* of dimension $p|q$ is a super ringed space that is locally isomorphic to $\mathbf{R}^{p|q}$. Morphisms between supermanifolds are morphisms between the corresponding super ringed spaces. We add the condition that the underlying topological space of a supermanifold should be Hausdorff and second countable. The superdomains $\mathbf{R}^{p|q}$ and $U^{p|q}$ are special examples of supermanifolds of dimension $p|q$. An open submanifold of a supermanifold is defined in the obvious way. The $U^{p|q}$ are open submanifolds of the $\mathbf{R}^{p|q}$.

The definition of supermanifold given is in the smooth category. To yield definitions of real analytic and complex analytic supermanifolds, we simply change the local models. Thus a real analytic supermanifold is a super ringed space locally isomorphic to $\mathbf{R}^{p|q}_{\text{an}}$ that is the super ringed space with

$$U \longmapsto \mathcal{A}^{p|q}(U) = \mathcal{A}(U)[\theta^1, \ldots, \theta^q]$$

as its structure sheaf where $\mathcal{A}(U)$ is the algebra of all real analytic functions on U. For a complex analytic supermanifold, we take as local models the spaces $\mathbf{C}^{p|q}$ whose structure sheaves are given by

$$\mathbf{C}^{p|q}(U) = H(U)[\theta^1, \ldots, \theta^q],$$

where $H(U)$ is the algebra of holomorphic functions on U. Actually one can even define, as Manin does,[2] more general geometric objects, like superanalytic spaces, and even superschemes.

Examples.

$\mathbf{R}^{p|q}$: We have already seen $\mathbf{R}^{p|q}$. The coordinates x_i of $\mathbf{R}^p$ are called the *even coordinates* and the θ^j are called the *odd coordinates*.

GL(1|1): Although we shall study this and other super Lie groups in more detail later, it is useful to look at them at the very beginning. Let G be the open subset of $\mathbf{R}^2$ with $x_1 > 0$, $x_2 > 0$. Then GL(1|1) is the open submanifold of the supermanifold $\mathbf{R}^{2|2}$ defined by G. This is an example of a super Lie group, and for making this aspect very transparent, it is convenient to write the coordinates as a matrix:

$$\begin{pmatrix} x^1 & \theta^1 \\ \theta^2 & x^2 \end{pmatrix}.$$

We shall take up the Lie aspects of this example a little later.

GL(p|q): We start with $\mathbf{R}^{p^2+q^2|2pq}$ whose coordinates are written as a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where

$$A = (a_{ij}), \quad D = (d_{\alpha\beta}),$$

are the even coordinates and

$$B = (b_{i\beta}), \quad C = (c_{\alpha j}),$$

are the odd coordinates. If G is the subset where $\det(A)\det(D) \neq 0$, then G is open and the supermanifold $\mathrm{GL}(p|q)$ is the open submanifold of $\mathbf{R}^{p^2+q^2|2pq}$ defined by G. Here again the multiplicative aspects will be taken up later.

Exterior Bundles of Vector Bundles on a Classical Manifold and Their Relation to Supermanifolds. Let M be a classical manifold and let V be a vector bundle on M. Then we have the *exterior bundle* E of V, which is also a vector bundle on M. If V_x is the fiber of V at $x \in M$, then the fiber of E at x is $\Lambda(V_x)$, the exterior algebra of V_x. Let $\mathcal{O}$ be the sheaf of sections of E. Then locally on M the sheaf is isomorphic to $U^{p|q}$ where $p = \dim(M)$ and $q = \mathrm{rank}(V)$, the rank of V being defined as the dimension of the fibers of V. Indeed, if V is the trivial bundle on M with sections θ_i, then the sections of E are of the form $\sum_I f_I \theta_I$ where $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_r}$ so that the sections over M of E can be identified with elements of $C^\infty(N)[\theta_1, \ldots, \theta_q]$. Thus $(M, \mathcal{O})$ is a supermanifold. Let us write $E^\flat$ for this supermanifold. Clearly, every supermanifold is locally isomorphic to a supermanifold of the form $E^\flat$; indeed, this is almost the definition of a supermanifold. The extent to which supermanifolds are globally not of the form $E^\flat$ is thus a cohomological question. One can prove (not surprisingly) that any *differentiable* supermanifold is isomorphic to some $E^\flat$, and that this result is no longer true in the analytic category (see Manin's discussion[2]). However, even in the differentiable category, we cannot simply replace supermanifolds by the supermanifolds of the form $E^\flat$. The point is that the isomorphism $M \simeq E^\flat$ is not canonical; indeed, as we shall elaborate later on, supermanifolds have *many more* morphisms than the exterior bundles because of the possibility, essential in the applications to physics, that the even and odd coordinates can be mixed under transformations. In other words, between two supermanifolds $E_1^\flat$, $E_2^\flat$ there are more morphisms in general than the morphisms that one obtains by requiring that they preserve the bundle structure.

The Imbedded Classical Manifold of a Supermanifold. If X is a supermanifold, its underlying topological space is often denoted by $|M|$. We shall now show that there is a natural smooth structure on $|M|$ that converts it into a smooth manifold. This gives the intuitive picture of M as essentially this classical manifold surrounded by a cloud of odd stuff. We shall make this more precise through our discussion below.

Let us first observe that if R is a commutative ring, then in the exterior algebra $E = R[\xi_1, \dots, \xi_r]$, an element

$$s = s_0 + \sum_j s_j \xi_j + \sum_{j<m} s_{jm} \xi_j \xi_m + \cdots ,$$

where the coefficients s_0, s_j, etc., are in R, is invertible in E if and only if s_0 is invertible in R. The map $s \mapsto s_0$ is clearly a homomorphism into R, and so if s is invertible, then s_0 is invertible in R. To prove that s is invertible if s_0 is, it is clear that by replacing s with $s_0^{-1}s$ we may assume that $s_0 = 1$; then $s = 1 - n$ where n is in the ideal generated by the ξ_j and so is nilpotent, so that s is invertible with inverse $1 + \sum_{m \geq 1} n^m$. Taking $R = C^\infty(V)$ where V is an open neighborhood of the origin 0 in $\mathbf{R}^p$, we see that for any section s of E, we can characterize $s_0(0)$ as the unique real number λ such that $s - \lambda$ is not invertible on any neighborhood of 0. We can now transfer this to any point x of a supermanifold M. Then to any section of $\mathcal{O}_M$ on an open set containing x we can associate its *value* at x as the unique real number $s^\sim(x)$ such that $s - s^\sim(x)$ is not invertible in any neighborhood of x. The map

$$s \longmapsto s^\sim(x)$$

is a homomorphism of $\mathcal{O}(U)$ into $\mathbf{R}$. Allowing x to vary in U, we see that

$$s \longmapsto s^\sim$$

is a homomorphism of $\mathcal{O}(U)$ onto an algebra $\mathcal{O}'(U)$ of real functions on U. It is clear that the assignment

$$U \longmapsto \mathcal{O}'(U)$$

is a *presheaf* on M. In the case when $(U, \mathcal{O}_U)$ is actually $(V, \mathcal{O}_V)$ where V is an open set in $\mathbf{R}^p$, we see that $\mathcal{O}'_V = C^\infty_V$ and so is actually a sheaf. In other words, for any point of M there is an open neighborhood U of it such that the restriction of $\mathcal{O}'$ to U is a sheaf and defines the structure of a smooth manifold on U. So, if we define $\mathcal{O}^\sim$ to be the sheaf of algebras of functions generated by $\mathcal{O}'$, then $\mathcal{O}^\sim$ defines the structure of a smooth manifold on M. We write $M^\sim$ for this smooth manifold. It is also called the *reduced manifold* and is also written as M_{red}. It is clear that this construction goes through in the real and complex analytic categories also. For $M = U^{p|q}$ we have $M^\sim = U$.

One can also describe the sheaf in another way. If we write

$$\mathcal{J}(U) = \{s \mid s^\sim = 0 \text{ on } U\},$$

then it is clear that $\mathcal{J}$ is a subsheaf of $\mathcal{O}$. We then have the exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^\sim \longrightarrow 0,$$

showing that $\mathcal{O}^\sim$ is the quotient sheaf $\mathcal{O}/\mathcal{J}$. The construction above exhibits $\mathcal{O}^\sim$ explicitly as a sheaf of algebras of *functions* on M.

From our definition of morphisms and the sheaf map $\mathcal{O} \longrightarrow \mathcal{O}^\sim$, it is now clear that the identity map $M^\sim \longrightarrow M$ is a morphism of the classical manifold $M^\sim$ into the supermanifold M. Since the pullback is surjective, this is an imbedding and so justifies the intuitive picture that M is essentially the classical manifold $M^\sim$ surrounded by a cloud of odd stuff. Actually, we can go further.

We introduce the sheafs $\mathcal{J}^r$ for $r = 1, 2, \ldots$, and define

$$M^r = (M, \mathcal{O}/\mathcal{J}^r)$$

so that $M^1 = M^\sim$. Then one can think of M^r as the r^{th} *infinitesimal neighborhood* of $M^\sim$ in M. The sequence

$$M^1 = M^\sim, M^2, M^3, \ldots$$

actually terminates with

$$M^{q+1} = M\,.$$

This is the same as saying that

$$\mathcal{J}^{q+1} = 0\,.$$

To see this, we can work locally and take $M = U^{p|q}$. The sections of $\mathcal{J}$ over an open subset V of U are elements of the form

$$\sigma = \sum_j s_j \theta^j\,,$$

where s_j are sections over V; it is obvious that if we take a product $\sigma_1 \cdots \sigma_r$ of such elements, the product is 0 if $r > q$. Notice, however, that the M^r are *not* supermanifolds; they are in general only superspaces in the sense of Manin.

Suppose now we have a morphism

$$\psi : M \longrightarrow N$$

of supermanifolds. Let ψ^* be the pullback $\mathcal{O}_N \longrightarrow \mathcal{O}_M$. If t is a section of $\mathcal{O}_N$ defined around $y = \psi(x)$ $(x \in M)$ and $s = \psi^*(t)$, then $s - s^\sim(x)$ is not invertible in any neighborhood of x, and so $t - s^\sim(x)$ is not invertible in any neighborhood of y, showing that

$$\psi^*(t)^\sim(x) = t^\sim(\psi(x))\,.$$

In particular,

$$\psi^* \mathcal{J}_N \subset \mathcal{J}_M\,.$$

This shows that we have a morphism

$$\psi^\sim : M^\sim \longrightarrow N^\sim$$

of classical manifolds associated to $\psi(M \longrightarrow N)$. Clearly, the assignment $\psi \longrightarrow \psi^\sim$ commutes with composition, and so the assignment

$$M \longrightarrow M^\sim$$

is functorial. More generally, for any fixed $r \geq 1$, the assignment

$$M \longrightarrow M^r$$

is also functorial, in view of the relation

$$\psi \mathcal{J}_N^r \subset \mathcal{J}_M^r .$$

REMARK. If $M = E^\flat$ where E is the exterior bundle of a vector bundle over a classical manifold N, the $\mathcal{O}(U)$ are actually *modules* over $C^\infty(U)$ for U open in N and so we have maps $C^\infty(U) \longrightarrow \mathcal{O}(U)$. This means that we have a map $M \longrightarrow M^\sim$ as well as the imbedding $M^\sim \longrightarrow M$. In other words, we have a *projection* $M \longrightarrow M^\sim$. This makes it clear why this is such a special situation.

Construction of Supermanifolds by Gluing. It is clear from the definition of supermanifolds that general supermanifolds are obtained by gluing superdomains. However, the gluing has to be done more carefully than in the classical case because the rings of the sheaf are not function rings, and so the gluing data have to be sheaf isomorphisms that have to be specified and do not come automatically.

Let X be a topological space, let $X = \bigcup_i X_i$ where each X_i is open, and let $\mathcal{O}_i$ be a sheaf of rings on X_i for each i. Write $X_{ij} = X_i \cap X_j$, $X_{ijk} = X_i \cap X_j \cap X_k$. Let

$$f_{ij} : (X_{ji}, \mathcal{O}_j | X_{ji}) \longrightarrow (X_{ij}, \mathcal{O}_i | X_{ij})$$

be an isomorphism of sheafs with

$$f_{ij}^\sim = \mathrm{id}_{X_{ji}} = \text{the identity map on } X_{ji} = X_{ij} .$$

To say that we *glue the ringed spaces* $(X_i, \mathcal{O}_i)$ *through the* f_{ij} means the construction of a sheaf of rings $\mathcal{O}$ on X and for each i a sheaf isomorphism

$$f_i : (X_i, \mathcal{O}|X_i) \longrightarrow (X_i, \mathcal{O}_i|X_i) , \quad f_i^\sim = \mathrm{id}_{X_i} ,$$

such that

$$f_{ij} = f_i f_j^{-1}$$

for all i, j. The conditions, necessary and sufficient, for the existence of $(\mathcal{O}, (f_i))$ are the so-called *gluing conditions*:

(a) $f_{ii} = \mathrm{id}$ on $\mathcal{O}_i$.
(b) $f_{ij} f_{ji} = \mathrm{id}$ on $\mathcal{O}_i | X_{ij}$.
(c) $f_{ij} f_{jk} f_{ki} = \mathrm{id}$ on $\mathcal{O}_i | X_{ijk}$.

The proof of the sufficiency (the necessity is obvious) is straightforward. In fact, there is essentially only one way to define the sheaf $\mathcal{O}$ and the f_i. For any open set $U \subset X$, let $\mathcal{O}(U)$ be the set of all (s_i) such that

$$s_i \in \mathcal{O}_i(U \cap X_i) , \quad s_i = f_{ij}(s_j) ,$$

for all i, j. $\mathcal{O}(U)$ is a subring of the full direct product of the $\mathcal{O}_i(U \cap X_i)$. The f_i are defined by

$$f_i : (s_i) \longmapsto s_i$$

for all i. It is easy but a bit tedious to verify that $(\mathcal{O}, (f_i))$ satisfy the requirements. If $(\mathcal{O}', (f_i'))$ are a second system satisfying the same requirement, and $s_i' = f_i'^{-1}(s_i)$, the s_i' are restrictions of a section $s' \in \mathcal{O}'(U)$ and $(s_i) \longmapsto s'$ is an isomorphism. These isomorphisms give a sheaf isomorphism $\mathcal{O} \longrightarrow \mathcal{O}'$ compatible with the (f_i), (f_i'). The details are standard and are omitted. Notice that given the X_i, $\mathcal{O}_i$, f_{ij}, the data $\mathcal{O}$, (f_i) are unique up to unique isomorphism.

For brevity we shall usually refer to the f_{ij} as isomorphisms of super ringed spaces

$$f_{ij} : X_{ji} \simeq X_{ij}, \qquad X_{ij} = (X_{ij}, \mathcal{O}_i|X_i \cap X_j),$$

above the identity morphisms on $X_i \cap X_j$.

We now consider the case when the family (X_i) is closed under intersections. Suppose we have a class $\mathcal{R}$ of open subsets of X closed under intersections such that each $R \in \mathcal{R}$ has a sheaf of rings on it that makes it a ringed space and X is the union of the sets R. Then for these to glue to a ringed space structure on X the conditions are as follows. For each pair $R, R' \in \mathcal{R}$ with $R' \subset R$ there should be an isomorphism of ringed spaces

$$\lambda_{RR'} : R' \simeq R_{R'}$$

where $R_{R'}$ is the ringed space R' viewed as an open subspace of R, and these $\lambda_{R'R}$ should satisfy

$$\lambda_{RR''} = \lambda_{RR'}\lambda_{R'R''}, \qquad R'' \subset R' \subset R.$$

In this case if Y is a ringed space there is a natural bijection between the morphisms f of X into Y and families (f_R) of morphisms $R \longrightarrow Y$ such that

$$f_{R'} = f_R\lambda_{RR'}, \qquad R' \subset R.$$

The relation between f and the f_R is that f_R is the restriction of f to R. In the other direction, the morphisms from Y to X are described as follows. First of all, we must have a map $t(Y^\sim \longrightarrow X^\sim)$; then the morphisms g of S into X above t are in natural bijection with families (g_R) of morphisms from $Y_R := t^{-1}(R)$ into R such that

$$g_{R'} = \lambda_{RR'}g_R.$$

Example 1: Projective Superspaces. This can be done over both $\mathbf{R}$ and $\mathbf{C}$. We shall work over $\mathbf{C}$ and let X be the complex projective n-space with homogeneous coordinates z^i $(i = 0, 1, \dots, n)$. The superprojective space $Y = \mathbf{CP}^{n|q}$ can now be defined as follows: Heuristically we can think of it as the set of equivalence classes of systems

$$(z^1, \dots, z^{n+1}, \theta^1, \dots, \theta^q)$$

where equivalence is defined by

$$(z^1, \dots, z^{n+1}, \theta^1, \dots, \theta^q) \simeq \lambda(z^1, \dots, z^{n+1}, \theta^1, \dots, \theta^q)$$

whenever $\lambda \in \mathbf{C}$ is nonzero. For a more precise description, we take the reduced manifold to be X. For any open subset $V \subset X$ we look at the preimage V' of V in $\mathbf{C}^{n+1} \setminus \{0\}$ and the algebra $A(V') = H(V')[\theta^1, \dots, \theta^q]$ where $H(V')$ is the algebra of holomorphic functions on V'. Then $\mathbf{C}^\times$ acts on this superalgebra by

$$t : \sum_I f_I(z)\theta^I \longmapsto \sum_I t^{-|I|} f_I(t^{-1}z)\theta^I, \quad t \in \mathbf{C}^\times.$$

Let

$$\mathcal{O}_Y(V) = A(V')^{\mathbf{C}^\times}$$

be the subalgebra of elements invariant under this action. It is then immediately verified that $\mathcal{O}_Y$ is a sheaf of supercommuting **C**-algebras on X. Let X^i be the open set where $z^i \neq 0$, and let V above be a subset of X^i. Then V can be identified with an open subset V_1 of the affine subspace of $\mathbf{C}^{n+1}$ where $z^i = 1$. Then

$$A(V') \simeq H(V_1)[\theta^1, \ldots, \theta^q].$$

This shows that Y is a complex analytic supermanifold. This is the projective superspace $\mathbf{CP}^{n|q}$. For a deeper discussion of these and other Grassmannians and flag supermanifolds, see Manin.[2]

Products. The category of supermanifolds admits products. For this purpose we start with the category of ringed spaces and introduce the notion of *categorical products*. Let X_i $(1 \leq i \leq n)$ be spaces in the category. A ringed space X together with ("projection") maps $P_i : X \longrightarrow X_i$ is called a product of the X_i,

$$X = X_1 \times \cdots \times X_n ,$$

if the following is satisfied: for any ringed space Y, the map

$$f \longmapsto (P_1 \circ f, \ldots, P_n \circ f)$$

from $\mathrm{Hom}(Y, X)$ to $\prod_i \mathrm{Hom}(Y, X_i)$ is a bijection. In other words, the morphisms f from Y to X are identified with n-tuples $(f_1, \ldots, f_n)$ of morphisms $f_i(Y \longrightarrow X_i)$ such that $f_i = P_i \circ f$ for all i. It is easy to see that if a categorical product exists, it is unique up to unique isomorphism. Notice that this is another example of defining an object by giving the set of morphisms of an arbitrary object into it.

We shall now show that in the category of supermanifolds, (categorical) products exist. Let X_i $(1 \leq i \leq n)$ be supermanifolds. Let $X^\sim = X_1^\sim \times \cdots \times X_n^\sim$ be the product of the classical manifolds associated to the X_i. We wish to construct a supermanifold X and morphisms $P_i(X \longrightarrow X_i)$ such that $P_i^\sim$ is the projection $X^\sim \longrightarrow X_i^\sim$ and $(X, (P_i))$ is a product of the X_i. If $X_i = U_i^{p_i|q_i}$ with coordinates $(x_i^1, \ldots, x_i^{p_i}, \theta_i^1, \ldots, \theta_i^{q_i})$, then their product is $U^{p|q}$ where $p = \sum_i p_i, q = \sum_i q_i$, with coordinates (x_i^j, θ_i^m); for the projection P_i we have

$$P_i^* x_i^j = x_i^j , \quad P_i^* \theta_i^m = \theta_i^m .$$

Suppose now the X_i are arbitrary. Let $\mathcal{R}$ be the set of rectangles R in $X^\sim$, $R = U_{1R} \times \cdots \times U_{nR}$, such that the U_{iR} are isomorphic to coordinate superdomains; we choose some isomorphism for each of these. Then each $R \in \mathcal{R}$ can be viewed as a supermanifold with projections P_{iR}. Suppose now that $R' \subset R (R, R' \in \mathcal{R})$ and $P_{iR|R'}$ is the restriction of P_{iR} to R'; then $(R', (P_{iR|R'}))$ is also a product supermanifold structure on R'. Because of the uniquely isomorphic nature of the products, we have a *unique* isomorphism of supermanifolds

$$\lambda_{RR'} : R' \simeq R_{R'}$$

such that

$$P_{iR|R'} = \lambda_{RR'} P_{iR'} .$$

If now $R'' \subset R' \subset R$ we have

$$\lambda_{RR''} P_{iR''} = P_{iR|R''}$$

while

$$\lambda_{R'R} \lambda_{R'R''} P_{iR''} = \lambda_{RR'} P_{iR'|R''} = P_{iR|R''} .$$

Hence by the uniqueness of the λ's we get

$$\lambda_{RR'} \lambda_{R'R''} = \lambda_{RR''} .$$

The discussion above on gluing leads at once to the fact that the rectangles glue together to form a supermanifold X, the projections P_{iR} define projections $P_i(X \longrightarrow X_i)$, and $(X, (P_i))$ is a product of the X_i. We omit the easy details.

4.3. Morphisms

The fact that the category of supermanifolds is a very viable one depends on the circumstance that morphisms between them can be described (locally) *exactly as in the classical case*. Classically, a map from an open set in $\mathbf{R}^m$ to one in $\mathbf{R}^m$ is of the form

$$(x^1, \ldots, x^m) \longmapsto (y^1, \ldots, y^n)$$

where the y^i are smooth functions of the $x^1, \ldots, x^m$. In the super context the same description prevails. To illustrate what we have in mind, we shall begin by discussing an example. This example will also make clear the point we made earlier, namely, that a supermanifold should not be thought of simply as an exterior bundle of some vector bundle on a classical manifold.

A morphism $\mathbf{R}^{1|2} \longrightarrow \mathbf{R}^{1|2}$: What do we do when we describe a smooth map between two manifolds? We take local coordinates (x^i), (y^j) and then define the morphism as the map

$$(x^i) \longrightarrow (y^j)$$

where the y^j are smooth functions of the x^i. It is a fundamental fact of the theory of supermanifolds; in fact, it is what makes the theory reasonable, that the morphisms in the super category can also be described in the same manner. Before proving this we shall look at an example.

Let $M = \mathbf{R}^{1|2}$. We want to describe a morphism ψ of M into itself such that $\psi^\sim$ is the identity. Let ψ^* be the pullback. We use t, θ^1, θ^2 as the coordinates on M, and t as the coordinate on $M^\sim = \mathbf{R}$. Since $\psi^*(t)$ is an even section and $(\psi^\sim)^*(t) = t$, it follows that

$$\psi^*(t) = t + f\theta^1\theta^2$$

where f is a smooth function of t. Similarly,

$$\psi^*(\theta^j) = g_j\theta^1 + h_j\theta^2$$

where g_j, h_j are again smooth functions of t. However, it is not immediately obvious how ψ^* should be defined for an arbitrary section, although for sections of the form

$$a + b_1\theta^1 + b_2\theta^2$$

where a, b_1, b_2 are *polynomials* in t, the prescription is uniquely defined; we simply replace t by $\psi^*(t)$ in a, b_1, b_2. It is already reasonable to expect by Weierstrass's approximation theorem that ψ^* should be uniquely determined. To examine this, let us take the case where

$$\psi^*(t) = t + \theta^1\theta^2, \quad \psi^*(\theta^j) = \theta^j, \quad j = 1, 2.$$

If g is a smooth function of t on an open set $U \subset \mathbf{R}$, we want to define $\psi_U^*(g)$. Formally we should define it to be $g(t+\theta^1\theta^2)$, and this definition is even rigorous if g is a polynomial as we observed just now. For arbitrary g let us expand $g(t+\theta^1\theta^2)$ as a formal Taylor series (!) as

$$g(t + \theta^1\theta^2) = g(t) + g'(t)\theta^1\theta^2$$

wherein the series does not continue because $(\theta^1\theta^2)^2 = 0$. We shall now *define* $\psi_U^*(g)$ by the above formula. It is an easy verification that ψ_U^* is then a homomorphism

$$C^\infty(U) \longrightarrow C^\infty(U)[\theta^1, \theta^2].$$

If

$$g = g_0 + g_1\theta^1 + g_2\theta^2 + g_{12}\theta^1\theta^2,$$

then we must define

$$\psi_U^*(g) = \psi^*(g_0) + \psi^*(g_1)\theta^1 + \psi^*(g_2)\theta^2 + \psi^*(g_{12})\theta^1\theta^2.$$

It is then clear that ψ_U^* is a homomorphism

$$C^\infty(U)[\theta^1, \theta^2] \longrightarrow C^\infty(U)[\theta^1, \theta^2]$$

with

$$\psi_U^*(t) = t + \theta^1\theta^2, \quad \psi_U^*(\theta_j) = \theta^j, \quad j = 1, 2.$$

The family (ψ_U^*) then defines a morphism $\mathbf{R}^{1|2} \longrightarrow \mathbf{R}^{1|2}$. It is obvious that this method goes through in the general case also when f, g_1, g_2 are arbitrary instead of 1 as above.

To see that the pullback homomorphism ψ^* is uniquely defined, we must prove that $\psi_U^*(g) = g + g'\theta^1\theta^2$ for $g \in C^\infty(U)$. Now $\psi_U^*(g)$ must be even, and so we can write

$$\psi^*(g) = g + D(g)\theta^1\theta^2.$$

Clearly D is an endomorphism of $C^\infty(U)$. The fact that ψ^* is a homomorphism now implies that D is a derivation. But $D(t) = 1$, and so D and d/dt are two derivations of $C^\infty(U)$ that coincide for t. They are therefore identical. So $D = d/dt$, showing that $\psi_U^*(g) = g + g'\theta^1\theta^2$.

This example also shows that the supermanifold $\mathbf{R}^{1|2}$ admits more self-morphisms than the exterior bundle of rank 2 over $\mathbf{R}$. Thus the category of exterior bundles is not equivalent to the category of supermanifolds even in the differentiable case, as we have already observed. Automorphisms such as the one discussed above are the geometric versions of true Fermi-Bose symmetries characteristic of supersymmetry where the even and odd coordinates are thoroughly mixed.

The main result on morphisms can now be formulated.

THEOREM 4.3.1 *Let $U^{p|q}$ be an open submanifold of $\mathbf{R}^{p|q}$. Suppose M is a supermanifold and ψ is a morphism of M into $U^{p|q}$. If*

$$f_i = \psi^*(t^i)\,, \quad qg_j = \psi^*(\theta^j)\,, \quad 1 \le i \le p,\ 1 \le j \le q\,,$$

then the f_i (g_j) are even (odd) elements of $\mathcal{O}_M(M)$. Conversely, if $f_i, g_j \in \mathcal{O}_M(M)$ are given with f_i even and g_j odd, there is a unique morphism $\psi(M \longrightarrow U^{p|q})$ such that

$$f_i = \psi^*(t^i)\,, \quad g_j = \psi^*(\theta^j)\,, \quad 1 \le i \le p,\ 1 \le j \le q\,.$$

Only for the converse does one need a proof. In some sense at the heuristic level the uniqueness part of this theorem is not a surprise because if a morphism is given on the coordinates x^i, θ^j, then it is determined on all sections of the form $\sum_I p_I \theta^I$ where the p_I are *polynomials* in the x^i, and clearly some sort of continuity argument should imply that it is determined uniquely for the sections where the p_I are merely smooth. In fact, (as Berezin did in his original memoirs) this argument can be made rigorous by introducing a topology—the usual one on smooth functions—on the sections and showing first that morphisms are continuous. But we shall avoid the topological arguments in order to bring more sharply into focus the analogy with schemes by remaining in the algebraic framework throughout as we shall do (see the paper of Leĭtes[3]). In this approach the polynomial approximation is carried out using the Taylor series only up to terms of order q, and use is made of the principle that if two sections have the same Taylor series up to and including terms of order q at all points of an open set, the two sections are identical. So, before giving the formal proof of the theorem, we shall formulate and prove this principle.

Let M be a supermanifold and let $\mathcal{O} = \mathcal{O}_M$ be its structure sheaf. Let $m \in M^\sim$ be a fixed point. We can speak of germs of sections defined in a neighborhood of m. The germs form a supercommutative $\mathbf{R}$-algebra $\mathcal{O}_m = \mathcal{O}_{M,m}$. For any section f defined around m, let $[f]_m = [f]$ denote the corresponding germ. We have previously considered the ideal $\mathcal{J}_m = \mathcal{J}_{M,m}$ of germs $[f]$ such that $[f^\sim] = 0$. We now introduce the larger ideal $\mathcal{I}_m = \mathcal{I}_{M,m}$ of germs for which $f^\sim(m) = 0$, i.e.,

$$\mathcal{I}_m = \mathcal{I}_{M,m} = \left\{ [f]_m \mid f^\sim(m) = 0 \right\}.$$

By the definition of a supermanifold there is an isomorphism of an open neighborhood of m with $U^{p|q}$. Let x^i, θ^j denote the pullbacks of the coordinate functions of $U^{p|q}$. We may assume that $x^{i\sim}(m) = 0$ $(1 \le i \le p)$.

LEMMA 4.3.2 *We have the following*:

(i) *$\mathcal{I}_{M,m}$ is generated by $[x^i]_m, [\theta^j]_m$. Moreover, if $\psi(M \longrightarrow N)$ is a morphism, then for any $n \in N$ and any $k \ge 0$,*

$$\psi^*(\mathcal{I}^k_{N,n}) \subset \mathcal{I}^k_{M,m}\,, \quad m \in M,\ \psi(m) = n\,.$$

(ii) *If $k > q$ and f is a section defined around m such that $[f]_{m'} \in \mathcal{I}^k_{m'}$ for all m' in some neighborhood of m, then $[f]_m = 0$.*

(iii) *For any k and any section f defined around m, there is a polynomial $P = P_{k,f,m}$ in the $[x^i]$, $[\theta^j]$ such that*

$$f - P \in \mathcal{I}_m^k .$$

PROOF: All the assertions are local, and so we may assume that $M = U^{p|q}$ where U is a *convex* open set in $\mathbf{R}^p$ and $m = 0$. Let us first prove (i). By Taylor series where the remainder is given as an integral, we know that if g is a smooth function of the x^i defined on *any* convex open set V containing 0, then, for any $k \geq 0$,

$$g(x) = g(0) + \sum_j x^j (\partial_j g)(0) + \cdots$$
$$+ \frac{1}{k!} \sum_{j_1,\ldots,j_k} x^{j_1} \ldots x^{j_k} (\partial_{j_1} \ldots \partial_{j_k} g)(0) + R_k(x)$$

where

$$R_k(x) = \frac{1}{k!} \sum_{j_1,\ldots,j_{k+1}} x^{j_1} \ldots x^{j_{k+1}} g_{j_1 j_2 \cdots j_{k+1}} ,$$

the $g_{j_1 j_2 \cdots j_{k+1}}$ being smooth functions on V defined by

$$g_{j_1 j_2 \cdots j_{k+1}}(x) = \int_0^1 (1-t)^k (\partial_{j_1} \ldots \partial_{j_{k+1}} g)(tx) dt .$$

Take first $k = 0$ and let $g(0) = 0$. Then

$$g = \sum_j x^j g_j .$$

If now $f = f_0 + \sum_I f_I \theta^I$ is in $\mathcal{O}(V)$ and $f_0(0) = 0$, then taking $g = f_0$ we obtain the first assertion in (i). For the assertion about ψ we have already seen that it is true for $k = 1$ (the locality of morphisms). Hence it is true for all k.

Let us first remark that because any section h can be written as $\sum_I h_I \theta^I$, it makes sense to speak of the *evaluation* $h(n) = \sum_I h_I(n)\theta^I$ at any point n; this is not to be confused with $h^\sim(n)$, which is invariantly defined and lies in $\mathbf{R}$ while $h(n)$ depends on the coordinate system and lies in $\mathbf{R}[\theta^1, \ldots, \theta^q]$.

To prove (ii), let $k > q$ and let us consider $\mathcal{I}_0^k$. Any product of k elements chosen from $x^1, \ldots, x^p, \theta^1, \ldots, \theta^q$ is zero unless there is at least one x^j. So

$$\mathcal{I}_0^k \subset \sum_j [x^j] \mathcal{O}_0 .$$

Therefore, if $[f] \in \mathcal{I}_0^k$, then

$$(*) \qquad f(0) = 0$$

where $f(0)$ is the evaluation of the section at 0. Suppose now f is in $\mathcal{O}$ and $[f]_n$ lies in $\mathcal{I}_n^k$ for all n in some open neighborhood N of 0. Then $(*)$ is applicable with 0 replaced by n. Hence

$$f(n) = 0, \quad n \in N .$$

This proves that the germ $[f]$ is 0.

To prove (iii) we take any section defined around 0, say $f = \sum_I f_I\theta^I$. Fix $k \geq 1$. Writing $f_I = g_I + R_I$ where g_I is the Taylor expansion of f_I at 0 and R_I is in the ideal generated by the monomials in the x^j of degree k, it follows at once that for $P = \sum g_I\theta^I$ and $R = \sum_I R_I\theta^I$ we have $f = P + R$. Going over to germs at 0, we see that $[P]$ is a polynomial in the $[x]$'s and $[\theta]$'s while $[R]$ is in the ideal $\mathcal{I}_0^k$. This proves (iii). □

PROOF OF THEOREM 4.3.1: We are now in a position to prove (the converse part of) Theorem 4.3.1.

Uniqueness. Let ψ_i $(i = 1, 2)$ be two morphisms such that $\psi_1^*(u) = \psi_2^*(u)$ for $u = x^i, \theta^j$. This means that $\psi_1^\sim = \psi_2^\sim$. We must prove that $\psi_1^*(u) = \psi_2^*(u)$ for all $u \in C^\infty(U)[\theta^1, \ldots, \theta^q]$. This equality is true for all polynomials in t^i, θ^j. Let $u \in \mathcal{O}(V)$ where V is an open set contained in U. Write $g = \psi_1^*(u) - \psi_2^*(u)$. Let $k > n$ where M has dimension $m|n$. Let $x \in M$ and let $y = \psi_1^\sim(x) = \psi_2^\sim(x) \in U$. By (iii) of the lemma, we can find a polynomial P in the t^i, θ^j such that $[u]_y = [P]_y + [R]_y$ where $[R]_y$ is in $\mathcal{I}_y^k$. Applying ψ_i^* to this relation, noting that $\psi_1^*([P]_y) = \psi_2^*([P]_y)$, we obtain, in view of (i) of the lemma, that $[g]_x \in \mathcal{I}_{M,x}^k$. Since x is arbitrary, $g = 0$ by (ii) of the lemma.

Existence. We write M as a union of open sets W, each of which has coordinate systems. In view of the uniqueness it is enough to construct the morphism $W \longrightarrow U$ and so we can take $M = W$. We follow the method used in the example of the morphism $\mathbf{R}^{1|2} \longrightarrow \mathbf{R}^{1|2}$ discussed earlier. It is further enough, as in the example above, to construct a homomorphism $C^\infty(U) \longrightarrow \mathcal{O}(W)_0$ taking x^i to f_i; such a homomorphism extends at once to a homomorphism of $C^\infty(U)[\theta^1, \ldots, \theta^q]$ into $\mathcal{O}(W)$, which takes θ^j to g_j. Write $f_i = r_i + n_i$ where $r_i \in C^\infty(W)$ and $n_i = \sum_{|I|\geq 1} n_{iI}\varphi^I$ (here y^r, φ^s are the coordinates on W). If $g \in C^\infty(U)$ we define $\psi^*(g)$ by the *formal* Taylor expansion

$$\psi^*(g) = g(r_1 + n_1, \ldots r_p + n_p) := \sum_\gamma \frac{1}{\gamma!}(\partial^\gamma g)(r_1, \ldots, r_p)n^\gamma$$

the series being finite because of the nilpotency of the n_i. To verify that $g \longmapsto \psi^*(g)$ is a homomorphism, we think of this map as a composition of three homomorphisms. The first of these is

$$\alpha : g \longmapsto \sum_\gamma \frac{1}{\gamma!}(\partial^\gamma g)T^\gamma\,,$$

which is a homomorphism of $C^\infty(U)$ into $C^\infty(U)[[T^1, \ldots, T^p]]$, the T^i being indeterminates and $[[\ldots]]$ denotes the formal power series ring; the homomorphism property of α follows from the standard Leibniz formula

$$\partial^\gamma(gh) = \sum_\delta \binom{\gamma}{\delta}(\partial^\delta g)(\partial^{\gamma-\delta}h)\,.$$

The second homomorphism is

$$\beta : C^\infty(U)[[T^1, \ldots, T^p]] \longrightarrow C^\infty(W)[[T^1, \ldots, T^p]]\,,$$

which extends the homomorphism $C^\infty(U) \longrightarrow C^\infty(W)$ induced by the map $W \longrightarrow U$. The third is the homomorphism

$$\gamma : C^\infty(W)[[T^1, \dots, T^p]] \longrightarrow C^\infty(W)[\varphi^1, \dots, \varphi^m],$$

which is the identity on $C^\infty(W)$ and takes T^i to φ^i, $1 \le i \le p$. Then $\psi^* = \gamma\beta\alpha$. The theorem is fully proven. □

REMARK. This theorem shows that morphisms between supermanifolds can be written in local coordinates in the form

$$x^1, \dots, x^m, \theta^1, \dots, \theta^n \longmapsto y^1, \dots, y^p, \varphi^1, \dots, \varphi^q$$

where y^i, φ^j are even and odd sections, respectively. The theory of supermanifolds thus becomes very close to the theory of classical manifolds and hence very reasonable. Also, the fact that Taylor series of arbitrary order were used in the proof suggests that it is not possible to define supermanifolds in the C^k category for finite k unless one does artificial things like coupling the number of odd coordinates to the degree of smoothness.

The Symbolic Way of Calculation. This theorem on the determination of morphisms is the basis of what one may call the *symbolic way of calculation*. Thus, if M, N are supermanifolds where (x^i, θ^j) are coordinates on M and (y^r, φ^s) are coordinates on N, we can think of a morphism $\psi(M \longrightarrow N)$ symbolically as

$$(x, \theta) \longrightarrow (y, \varphi), \quad y = y(x, \theta),\ \varphi = \varphi(x, \theta),$$

which is an abuse of notation for the map ψ^* such that

$$\psi^*(y^r) = y^r(x, \theta) \in \mathcal{O}_M(M)_0, \quad \psi^*(\varphi^s) = \varphi^s(x, \theta) \in \mathcal{O}_M(M)_1.$$

We shall see later in Section 4.6 how useful this symbolic point of view is in making calculations free of pedantic notation.

4.4. Differential Calculus

The fundamental result is the differential criterion for a system of functions to form a coordinate system at a point. This leads as usual to results on the local structure of isomorphisms, immersions, and submersions.

4.4.1. Derivations and Vector Fields. Let us first look at derivations. Recall that a homogeneous derivation of a superalgebra B over k (k a field of characteristic 0) is a k-linear map $D : B \longrightarrow B$ such that

$$D(ab) = (Da)b + (-1)^{p(D)p(a)} a(Db), \quad a, b \in B.$$

Let R be a commutative k-algebra with unit element and let $A = R[\theta^1, \dots, \theta^q]$ as usual. Then A is a supercommutative k-algebra and one has the space of derivations of A. If ∂ is a derivation of R, it extends uniquely as an even derivation of A that vanishes for all the θ^i. We denote this by ∂ again. On the other hand, if we fix i, there is a unique odd derivation of A that is 0 on A and takes θ^j to $\delta_{ij}\theta^i$. We denote this by $\partial/\partial\theta^i$. Thus

$$\partial \sum_I f_I\theta^I = \sum_I (\partial f_I)\theta^I\,, \qquad \frac{\partial}{\partial\theta^j}\Big(\sum_{j\notin I} f_I\theta^I + \sum_{j\notin I} f_{j,I}\theta^j\theta^I\Big) = \sum_{j\notin I} f_{j,I}\theta^I\,.$$

If M is a supermanifold, one can then define *vector fields* on M as derivations of the sheaf $\mathcal{O}_M$. More precisely, they are families of derivations $(D_U) : \mathcal{O}(U) \longrightarrow \mathcal{O}(U)$ that are compatible with restrictions. The derivations form a sheaf of modules over the structure sheaf $\mathcal{O}$. It is called the *tangent sheaf* of M in analogy with what happens in the classical case. Let us denote it by $\mathcal{T}M$.

To see what its local structure is, let us now consider the case $M = U^{p|q}$. If $R = C^\infty(U)$, we thus have derivations

$$\frac{\partial}{\partial x^i}, \frac{\partial}{\partial\theta^j}$$

on $\mathcal{O}(U)$. We shall now show by the technique of polynomial approximation used earlier that the derivations of $\mathcal{O}(U)$ form a module isomorphic to the free module $A^{p|q}$ where $A = \mathcal{O}(U)$, with the partials listed above as a homogeneous basis. Indeed, let D be any derivation (even or odd) of $\mathcal{O}(U)$, and let us write $y^1, \ldots, y^m$ for the entire set of coordinates $x^1, \ldots, \theta^q$ $(m = p + q)$. Let $a_j = Dy^j$; we wish to show that $D = \sum_j a_j\partial/\partial y^j$ (the freeness is clear since this derivation must take y^j to a_j and so is 0 only if the a_j are all 0). Let D' be the derivation $D - \sum_j a_j\partial/\partial y^j$. Then $D'y^j = 0$ for all j, and so, by the derivation property $D'P = 0$ for all polynomials in the y^j. Suppose now that $f \in \mathcal{O}(U)$ and $u \in U$. Then there is a polynomial P_k in the y^j such that for $g = f - P_k$, $[g]_u \in \mathcal{I}_u^k$. Hence $[D'f]_u = [D'g]_u$. But $[D'g]_u \in \mathcal{I}_u^{k-1}$ and so, if $k > q + 1$, we can conclude that $[D'f]_u \in \mathcal{I}_k^{q+1}$. Since $u \in U$ is arbitrary, we have $D'f = 0$.

Thus the tangent sheaf $\mathcal{T}M$ on M is locally isomorphic to the free module $\mathcal{O}(U)^{p|q}$. It is thus an example of a *vector bundle* on the supermanifold on M, i.e., a sheaf of $\mathcal{O}$-modules on M that is locally isomorphic to $\mathcal{O}^{r|s}$ for suitable r, s.

Once the partial derivatives with respect to the coordinate variables are defined, the differential calculus on supermanifolds takes almost the same form as in the classical case except for the slight but essential differences originating from the presence of odd derivations. For explicit formulas we have

$$\frac{\partial}{\partial x^i}\sum f_I\theta^I = \sum_I \frac{\partial f}{\partial x^i}\theta^I\,, \qquad \frac{\partial}{\partial\theta^j}\Big(\sum_{j\notin I} f_I\theta^I + \sum_{j\notin I} f_{j,I}\theta^j\theta^I\Big) = \sum_{j\notin I} f_{j,I}\theta^I\,.$$

Tangent Space and the Tangent Map of a Morphism. Let M be a supermanifold and let $m \in M$. Then as in the classical case we define a *tangent vector* to M at m as a derivation of the stalk $\mathcal{O}_m$ into $\mathbf{R}$. More precisely, a homogeneous tangent vector ξ at m is a linear map

$$\xi : \mathcal{O}_m \longrightarrow \mathbf{R}$$

such that

$$\xi(fg) = \xi(f)g(m) + (-1)^{p(\xi)p(f)} f(m)\xi(g)\,, \qquad f, g \in \mathcal{O}_m\,.$$

If x^i, θ^j are local coordinates for M at some point, the tangent space has

$$\left(\frac{\partial}{\partial x^i}\right)_m, \left(\frac{\partial}{\partial \theta^j}\right)_m$$

as a basis and so is a super vector space of dimension $p|q$; this is done in the same way as we did in the case of vector fields by polynomial approximation. This is thus true in general. We denote by $T_m(M)$ the tangent space of M at m. If $\psi(M \longrightarrow N)$ is a morphism of supermanifolds and $m \in M, n = \psi(m) \in N$, then

$$\xi \longmapsto \xi \circ \psi^*$$

is a morphism of super vector spaces from $T_m(M)$ to $T_n(N)$, denoted by $d\psi_m$:

$$d\psi_m : T_m(M) \longrightarrow T_n(N).$$

This is called the *tangent map* of ψ at m. It is obvious that the assignment

$$\psi \longmapsto d\psi_m$$

preserves composition in the obvious sense. In local coordinates this is a consequence of the chain rule, which we shall derive presently in the super context.

Let us now derive the chain rule. Let

$$\psi : U^{p|q} \longrightarrow V^{m|n}$$

be a morphism and let (y^j) and (z^k) be the coordinates on $U^{p|q}$ $(V^{m|n})$ where we are including both the even and odd coordinates in this notation. Then for any $f \in \mathcal{O}(V)$ we have

(chain rule) $$\frac{\partial \psi^*(f)}{\partial y^i} = \sum_k \frac{\partial \psi^*(z^k)}{\partial y^i} \psi^*\left(\frac{\partial f}{\partial z^k}\right).$$

If we omit reference to ψ^* as we usually do in classical analysis, this becomes the familiar

$$\frac{\partial}{\partial y^i} = \sum_k \frac{\partial z^k}{\partial y^i} \frac{\partial}{\partial z^k}.$$

This is proven as before. Let D be the difference between the two sides. Then D is a derivation from $\mathcal{O}(V)$ to $\mathcal{O}(U)$ in the sense that

$$D(fg) = (Df)g + (-1)^{p(D)p(f)} f(Dg),$$

where $p(D)$ is just the parity of y^i, and it is trivial that $Dz^k = 0$ for all k. Hence $D = 0$. The argument is again by polynomial approximation.

In the above formula the coefficients have been placed to the left of the derivations. This will, of course, have sign consequences when we compose two morphisms. Let

$$\psi : U \longrightarrow V, \quad \varphi : V \longrightarrow W, \quad \tau = \varphi\psi.$$

Let (y^k), (z^r), (t^m) be the coordinates on U, V, W, respectively. If we write $p(y^k)$ for the parity of y^k and so on, then the parity of $\partial z^r/\partial y^k$ is $p(z^r) + p(y^k)$. The chain rule gives

$$\frac{\partial t^m}{\partial y^k} = \sum_r (-1)^{p(z^r)(p(y^k)+p(t^m)+1)+p(y^k)p(t^m)} \frac{\partial t^m}{\partial z^r} \frac{\partial z^r}{\partial y^k}$$

if we remember that $p(z^r)^2 = p(z^r)$. Hence if we define

$$z^r_{,k} = (-1)^{(p(z^r)+1)p(y^k)} \frac{\partial z^r}{\partial y^k},$$

and also

$$t^m_{,r} = (-1)^{(p(t^m)+1)p(z^r)} \frac{\partial t^m}{\partial z^r}, \quad t^m_{,k} = (-1)^{(p(t^m)+1)p(y^k)} \frac{\partial t^m}{\partial y^k},$$

then we have

$$t^m_{,k} = \sum_r t^m_{,r} z^r_{,k}.$$

So if we write

$$J\psi = \left(z^r_{,k}\right),$$

then composition corresponds to matrix multiplication. In terms of even and odd coordinates x^i, θ^j for U and y^s, φ^n for V with

$$\psi^*(y^s) = f_s, \quad \psi^*(\varphi^n) = g_n,$$

we obtain

$$J\psi = \begin{pmatrix} \frac{\partial f}{\partial x} & -\frac{\partial f}{\partial \theta} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial \theta} \end{pmatrix}.$$

For computations involving the tangent map, this refinement has no effect. In fact, with respect to the bases

$$\left(\frac{\partial}{\partial x^i}\right)_m, \left(\frac{\partial}{\partial \theta^j}\right)_m, \left(\frac{\partial}{\partial y^r}\right)_n, \left(\frac{\partial}{\partial \varphi^k}\right)_n,$$

the matrix of $d\psi_m$ is

$$\begin{pmatrix} \frac{\partial f}{\partial x}^{\sim}(m) & 0 \\ 0 & \frac{\partial g}{\partial \theta}^{\sim}(m) \end{pmatrix}$$

as it should be, since $d\psi_m$ is an even map.

Differential Criteria. Let us work with a supermanifold M and let $m \in M$. Let

$$f_1, \dots, f_p, g_1, \dots, g_q$$

be sections of $\mathcal{O}$ with f_i even and g_j odd, defined around m. Then there is an open neighborhood V of m and a unique morphism ψ of the supermanifold V into $\mathbf{R}^{p|q}$ such that

$$\psi^*(x^i) = f_i, \quad \psi^*(\theta^j) = g_j, \quad 1 \le i \le p, \ 1 \le j \le q.$$

We say that the (f_i, g_j) *form a coordinate system for M at m* if ψ is an isomorphism of a neighborhood of m with an open submanifold of $\mathbf{R}^{p|q}$.

THEOREM 4.4.1 *The following are equivalent*:

(i) *The (f_i, g_j) form a coordinate system for M at m.*

(ii) *ψ is an isomorphism of supermanifolds from a neighborhood of m in M to a neighborhood of $\psi(m)$ in $\mathbf{R}^{p|q}$.*
(iii) *$d\psi_m$ is a linear isomorphism of $T_m(M)$ with $T_{\psi(m)}(N)$.*
(iv) *We have*

$$\det\left(\frac{\partial f}{\partial x}\right)^{\sim}(m)\det\left(\frac{\partial g}{\partial \theta}\right)^{\sim}(m)\neq 0\,.$$

PROOF: The equivalence (i) $\Longleftrightarrow$ (ii) is just the definition. Also, it is obvious that (iii) $\Longleftrightarrow$ (iv). The implication (ii)$\Longrightarrow$(iii) is also easy; if $n=\psi(m)$ and φ is the inverse of ψ, then $d\varphi_n d\psi_m = d\psi_m d\varphi_n = 1$, which shows that $d\psi_m$ is a linear isomorphism. So it remains to prove that (iii)$\Longrightarrow$(ii). In this proof we shall use either (iii) or (iv) interchangeably. We also suppose that $M=U^{p|q}$.

Since $\det(\partial f^{\sim}/\partial x)^{\sim}(m)\neq 0$, we know that $(f_1^{\sim},\dots,f_p^{\sim})$ form a system of coordinates for the classical manifold $M^{\sim}$ near m and so $f_1^{\sim},\dots,f_p^{\sim},\theta^1,\dots,\theta^q$ is a system of coordinates for $U^{p|q}$ at m. So we may assume (after shrinking U) that

$$f_i\equiv x_i(\mathcal{J})$$

where $\mathcal{J}$ is the ideal generated by the θ^j. Now

$$g_j=\sum_k f_{jk}\theta^k+\sum_{km} f_{jkm}\theta^k\theta^m+\cdots$$

where f_{jk}, f_{jkm}, etc., are smooth functions defined near m. By assumption the matrix $(f_{jk}^{\sim})$ is invertible at m and hence near m. So

$$x^1,\dots,x^p,\varphi^1,\dots,\varphi^q\,,\quad \varphi^j=\sum_k f_{jk}\theta^k\,,$$

is again a system of coordinates. So we may assume that

$$g_j\equiv\theta^j(\mathcal{J}^2)\,.$$

So we have a morphism

$$\psi: U\longrightarrow V$$

such that

$$\psi^*(y^i)=f_i\equiv x^i(\mathcal{J})\,,\quad \psi^*(\varphi^j)\equiv\theta^j(\mathcal{J}^2)\,,$$

and we wish to prove that ψ^* is an isomorphism on a suitably small neighborhood of m. Note that the reduced morphism is the identity so that $U=V$. Let μ be the morphism $V\longrightarrow U$ such that $\mu^*(x^i)=y^i$, $\mu^*(\theta^j)=\varphi^j$. The morphism μ is not the inverse of ψ that we are after but is like a *parametrix*; i.e., $\mu\psi$ is close to the identity in some sense. Actually,

$$\psi^*\mu^*=1+N$$

where N is *nilpotent*. We shall in fact show that $N^{q+1}=0$. Let τ be the morphism $\mu\psi$ from U to U so that $\tau^*=\psi^*\mu^*=1+N$. Clearly, $N1=0$ while

$$Nx^i\equiv 0(\mathcal{J})\,,\quad N\theta^j\equiv 0(\mathcal{J}^2)\,.$$

Since $\tau^*(\theta^j)\equiv\theta^j(\mathcal{J}^2)$, it follows that $\tau^*(\theta^j)\in\mathcal{J}$ and hence, because τ^* is a homomorphism, $\tau^*(\mathcal{J})\subset\mathcal{J}$. Thus $\tau^*(\mathcal{J}^k)\subset\mathcal{J}^k$ for all $k\geq 1$. By definition $\tau^{\sim}=$

$\mu^\sim\psi^\sim$ is the identity, and so it is clear that $\tau^*(g) \equiv g(\mathcal{J})$ for all $g \in \mathcal{O}(V)$. We now claim that N maps $\mathcal{J}^k$ into $\mathcal{J}^{k+1}$ for all $k \geq 1$. Since N is *not* a homomorphism, we have to do this for each k. This means showing that $\tau^*(g) \equiv g(\mathcal{J}^{k+1})$ if $g \in \mathcal{J}^k$. Take $g = h\theta^J$ where $|J| \geq k$. Then $\tau^*(g) = \tau^*(h)\tau^*(\theta^J)$. Now

$$\tau^*(\theta^{j_1} \cdots \theta^{j_r}) = (\theta^{j_1} + \beta_1) \cdots (\theta^{j_r} + \beta_r)$$

where the $\beta_j \in \mathcal{J}^2$, and so

$$\tau^*(\theta^J) \equiv \theta^J(\mathcal{J}^{r+1}).$$

Hence, if $|J| = r \geq k$,

$$\tau^*(g) = \tau^*(h)\tau^*(\theta^J) = (h + w)(\theta^J + \xi)$$

where $w \in \mathcal{J}$ and $\xi \in \mathcal{J}^{k+1}$ so that

$$\tau^*(g) \equiv g(\mathcal{J}^{k+1}), \quad g \in \mathcal{J}^k.$$

Thus N maps $\mathcal{J}^k$ into $\mathcal{J}^{k+1}$ for all k; hence $N^{q+1} = 0$.

The fact that $N^r = 0$ for $r > q$ implies that $1 + N$ is invertible; indeed, $(1 + N)^{-1} = \sum_{s \geq 0}(-1)^s N^s$. Let ν^* be the inverse of τ^*. Thus $\psi^*\mu^*\nu^* = 1$, showing that ψ^* has a right inverse. So there is a morphism φ from V to U such that $\varphi\psi = 1_U$. On the other hand, because the invertibility of $d\varphi$ follows from the above relation, we can apply the preceding result to φ to find a morphism ψ' such that $\psi'\varphi = 1_V$. So $\psi' = \psi'\varphi\psi = \psi$. Thus ψ is an isomorphism. □

COROLLARY 4.4.2 *If $\psi(M \longrightarrow N)$ is a morphism such that $d\psi_m$ is bijective everywhere, then $\psi^\sim$ maps M onto an open subspace N' of N; if $\psi^\sim$ is also one-to-one, then ψ is an isomorphism of M with N' as supermanifolds.*

Local Structure of Morphisms. The above criterion makes it possible, exactly as in classical geometry, to determine the canonical forms of immersions and submersions. The general problem is as follows: Let M, N be supermanifolds and let $m \in M, n \in N$ be fixed. Let ψ be a morphism from M to N with $\psi(m) = n$. If γ (γ') is a local automorphism of $M(N)$ fixing $m(n)$, then $\psi' = \gamma' \circ \psi \circ \gamma$ is also a morphism from M to N taking m to n. We then say that $\psi \simeq \psi'$. The problem is to describe the equivalence classes. The representatives of the equivalence classes are called *local models*. Clearly, the linear maps $d\psi_m$ and $d\psi'_m$ are equivalent in the sense that $d\psi'_m = g' d\psi_m g$ where g (g') is an automorphism of $T_m(M)$ $(T_n(N))$. The even and odd ranks and nullities involved are thus invariants. The morphism ψ is called an *immersion at m* if $d\psi_m$ is injective, and a *submersion at m* if $d\psi_m$ is surjective. We shall now show that the local models for an immersion are

$$M = U^{p|q}, \quad (x^i, \theta^j), \quad N = M \times V^{r|s}(0 \in V), \quad (x^i, y^t, \theta^j, \varphi^k),$$

with

$$\psi^\sim : m \longmapsto (m, 0),$$
$$\psi^* : x^i \mapsto x^i, \ \theta^j \mapsto \theta^j, \ y^t, \varphi^k \mapsto 0.$$

We shall also show that for the submersions the local models are *projections*, namely,

$$N = U^{p|q}\,, \quad (x^i, \theta^j)\,, \quad M = N \times V^{r|s}\,, \quad (x^i, y^t, \theta^j, \varphi^k)\,,$$

with

$$\psi^{\sim} : (m, v) \longmapsto m\,,$$
$$\psi^* : x^i \mapsto x^i,\ \theta^j \mapsto \theta^j\,.$$

THEOREM 4.4.3 *The above are local models for immersions and submersions.*

PROOF: *Immersions.* Let $\psi(U^{p|q} \longrightarrow V^{p+r|q+s})$ be an immersion at $0 \in U$, with (x^i, θ^j) as coordinates for U and (u^a, ξ^b) as coordinates for V. Write $\psi^*(g) = g^*$. Since $d\psi_m$ is separately injective on $T_m(M)_0$ and $T_m(M)_1$, we see that the matrices

$$\left(\frac{\partial u^{a*}}{\partial x^i}\right)_{1\le i\le p,1\le a\le p+r}, \quad \left(\frac{\partial \xi^{b*}}{\partial \theta^j}\right)_{1\le j\le q,1\le b\le q+s},$$

have ranks p and q at m, respectively. By permuting the u^a and ξ^b, we may therefore assume that the matrices

$$\left(\frac{\partial u^{a*}}{\partial x^i}\right)_{1\le i\le p,1\le a\le p}, \quad \left(\frac{\partial \xi^{b*}}{\partial \theta^j}\right)_{1\le j\le q,1\le b\le q},$$

which are composed of the first p columns of the first matrix and the first q columns of the second matrix, are invertible at m. This means that

$$u^{1*}, \ldots, u^{p*}, \xi^{1*}, \ldots, \xi^{q*}$$

form a coordinate system for $U^{p|q}$ at m. We may therefore assume that

$$u^{r*} = x^r, \ \ 1 \le r \le p\,, \quad \xi^{s*} = \theta^q\,.$$

However, u^{a*} $(a > p)$, ξ^{b*} $(b > q)$ may not map to 0 as in the local model. Let

$$u^{a*} = \sum_I g_{aI}\theta^I\,, \quad \xi^{b*} = \sum_J h_{bJ}\theta^J\,,$$

where g_{aI}, h_{bJ} are C^∞-functions of $x^1, \ldots, x^p$. Let

$$w^a = \sum_I g_{aI}(u^1, \ldots, u^p)\xi^I\,, \ \ a > p\,, \quad \eta^b = \sum_J h_{bJ}(u^1, \ldots, u^p)\xi^J\,, \ \ b > q\,.$$

Then ψ^* maps $u'^a = u^a - w^a$ $(a > p)$ and $\xi'^b = \xi^b - \eta^b$ $(b > q)$ to 0. It is obvious that

$$u^1, \ldots, u^p, u'^{p+1}, \ldots, u'^m, \xi^1, \ldots, \xi^q, \xi'^{q+1}, \ldots, \xi'^n$$

is a coordinate system at $\psi(m)$. With this coordinate system the morphism ψ is in the form of the local model.

Submersions. Let ψ be a submersion of $V^{p+r|q+s}$ on $M = U^{p|q}$ with $m = 0, \psi(m) = n = 0$. Let (x^i, θ^j) be the coordinates for $U^{p|q}$ and (y^a, φ^b) the coordinates for $V^{p+r|q+s}$. The map $d\psi_0$ is surjective separately on $T_0(M)_0$ and $T_0(M)_1$. So the matrices

$$\left(\frac{\partial x^{i^*}}{\partial y^a}\right)_{1\leq a\leq p+r, 1\leq i\leq p}, \quad \left(\frac{\partial \theta^{j^*}}{\partial \varphi^b}\right)_{1\leq b\leq q+s, 1\leq j\leq q}$$

have ranks p and q at 0, respectively. We may therefore assume that the submatrices composed of the first p rows of the first matrix and of the first q rows of the second matrix are invertible at 0. This means that

$$x^{1^*}, \ldots, u^{p^*}, y^{p+1}, \ldots, y^{p+r}, \theta^{1^*}, \ldots, \varphi^{q^*}, \varphi^{q+1}, \ldots, \varphi^{q+s}$$

form a coordinate system for $V^{p+r,q+s}$ at 0. The morphism is in the form of the local model in these coordinates. This proves the theorem. □

4.5. Functor of Points

In algebraic geometry as well as in supergeometry, the basic objects are somewhat strange and the points of their underlying topological space do not have geometric significance. There is a second notion of points that is geometric and corresponds to our geometric intuition. Moreover, in the supergeometric context this notion of points is essentially the one that the physicists use in their calculations. The mathematical basis for this notion is the so-called *functor of points*.

Let us first consider affine algebraic geometry. The basic objects are algebraic varieties defined by polynomial equations

$$(*) \qquad p_r(z^1, \ldots, z^n) = 0, \quad r \in I,$$

where the polynomials p_r have coefficients in **C** and the z^i are complex variables. It is implicit that the solutions are from $\mathbf{C}^n$ and the set of solutions forms a variety with its Zariski topology and structure sheaf. However, Grothendieck focused attention on the fact that one can look for solutions in A^n where A is any commutative **C**-algebra with unit. Let $V(A)$ be the set of these solutions; $V(\mathbf{C})$ is the underlying set for the classical complex variety defined by these equations. The elements of $V(A)$ are called the *A-points* of the variety $(*)$. We now have an assignment

$$\mathbf{V} : A \longmapsto V(A)$$

from the category of commutative **C**-algebras with units into the category of sets. This is the *functor of points* of the variety $(*)$. That the above assignment is functorial is clear: if B is a **C**-algebra with a map $A \longrightarrow B$, then the map $A^n \longrightarrow B^n$ maps $V(A)$ into $V(B)$. It turns out that *the functor V contains the same information* as the classical complex variety, and the set of morphisms between two affine varieties is bijective with the set of natural maps between their functors of points. This follows from Yoneda's lemma. The set $V(A)$ itself can also be described as $\mathrm{Hom}(\mathbf{C}[V], A)$. Obviously an arbitrary functor from **C**-algebras to sets will not rise as the functor points of an affine variety or the algebra of polynomial functions on such a variety (by Hilbert's zeros theorem these are the algebras that are finitely generated over **C** and reduced in the sense that they have no nonzero nilpotents).

If a functor has this property it is called *representable*. Thus affine algebraic geometry is the same as the theory of representable functors. Notice that the sets $V(A)$ *have no structure*; it is their *functorial property* that contains the information residing in the classical variety.

Now the varieties one encounters in algebraic geometry are not always affine; the projective ones are obtained by gluing affine ones. In the general case they are schemes. The duality between varieties and algebras makes it clear that for a given scheme X, one has to understand by its points any morphism from an *arbitrary scheme* into X. In other words, given a scheme X, the functor

$$S \longmapsto \mathrm{Hom}(S, X) \quad (S \text{ an arbitrary scheme})$$

is called the *functor of points of* X; $\mathrm{Hom}(S, X)$ is denoted by $X(S)$ and is called the set of S-points of X. Heuristically we may think of $X(S)$ as points of X *parametrized by* S. This notion of points is much closer to the geometric intuition than the points of the underlying space of a scheme. For example, the underlying topological space of the product of two schemes X, Y is *not* the product of X and Y; however, this is true for S-points: $(X \times Y)(S) \simeq X(S) \times Y(S)$ canonically. A functor from schemes to sets is called *representable* if it is naturally isomorphic to the functor of points of a scheme; the scheme is then uniquely determined up to isomorphism and is said to *represent* the functor. In many problems, especially in the theory of moduli spaces, it is most convenient to define first the appropriate functor of points and then prove its representability.

We take over this point of view in supergeometry. The role of schemes is played by supermanifolds and the role of affine schemes or their coordinate rings is played by supercommutative algebras. If X is a supermanifold, its functor points are

$$S \longmapsto X(S) \quad (S \text{ a supermanifold})$$

where

$$X(S) = \mathrm{Hom}(S, X) = \text{set of morphisms } S \longrightarrow X\,.$$

$X(S)$ is the set of S-points of X. If X, Y are supermanifolds, then $(X \times Y)(S) \simeq X(S) \times Y(S)$ canonically. A morphism ψ from $\mathbf{R}^{0|0}$ into X is really a point of $X^\sim$ in the classical sense; indeed, if U is open in $X^\sim$, the odd elements of $\mathcal{O}(U)$ must map to 0 under ψ^*, and so ψ^* factors through to a homomorphism of $\mathcal{O}^\sim$ into $\mathbf{R}$. To define morphisms that see the odd structure of X, we must use supermanifolds themselves as domains for the morphisms. Later on, when we treat super Lie groups, we shall see the usefulness of this point of view.

Consider the simplest example, namely, $\mathbf{R}^{p|q}$. If S is a supermanifold, the S-points of $\mathbf{R}^{p|q}$ are systems $(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q)$ where $x^i \in \mathcal{O}_S(S)_0$, $\theta^j \in \mathcal{O}_S(S)_1$. This is not any different from the heuristic way of thinking of $\mathbf{R}^{p|q}$ as the set of all systems $(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q)$ where the x^i are even variables and the θ^j are odd variables. One can think of $\mathbf{R}^{p|q}$ as a "group" with the group law

$$(x, \theta) + (x', \theta') \longrightarrow (x + x', \theta + \theta').$$

At the level of S-points, this is *exactly* a group law; the symbols denote elements of $\mathcal{O}_S(S)$ of the appropriate parity. Thus the informal or symbolic way of thinking and writing about supermanifolds is essentially the same as the mode of operation furnished by the language of the functor of points.

4.6. Integration on Supermanifolds

Integration on supermanifolds consists of integrating with respect to both the even and odd variables. For the even variables it is the standard classical theory, but integration in anticommuting variables is new and was discovered by Berezin, who also discovered the change-of-variables formula.

The integral on an exterior algebra

$$A = \mathbf{R}[\theta^1, \ldots, \theta^q]$$

is a linear function

$$A \longrightarrow \mathbf{R}, \quad a \longmapsto \int a = \int a \, d^q\theta ,$$

uniquely determined by the following properties:

$$\int \theta^I = 0, \quad |I| < q, \qquad \int \theta^Q = 1, \quad Q = [1, \ldots, q].$$

We use the notation

$$Q = [1, \ldots, q]$$

to denote the ordered set $1, \ldots, q$ throughout this section. Thus integration is also differentiation, and

$$\int = \left(\frac{\partial}{\partial\theta^q}\right)\left(\frac{\partial}{\partial\theta^{q-1}}\right)\cdots\left(\frac{\partial}{\partial\theta^1}\right).$$

For a superdomain $U^{p|q}$ the integral is a linear form

$$\mathcal{O}_c(U) \longrightarrow \mathbf{R}, \quad s \longmapsto \int s = \int s \, d^px \, d^q\theta ,$$

where the suffix "c" means that the sections are *compactly supported*; the integral is evaluated by repeated integration. Thus

$$\int \sum_I s_I \theta^I = \int s_Q \, d^px \, \mathbf{M}.$$

Sometimes we write

$$\int s = \int_U s$$

to emphasize that the integration is over U. Thus *the integral picks out just the coefficient of θ^Q and integrates it in the usual way with respect to the even variables.* This might seem very peculiar till one realizes that any definition should be made in such a way that one has a nice formula for changing variables in the integral. Now the Berezinian is the replacement of the determinant in the super context, and we shall see that this definition of the integral is precisely the one for which one can prove a change-of-variables formula exactly analogous to the classical one, with Ber replacing det.

Statement of the Change-of-Variables Formula. Let

$$\psi : U^{p|q} \longrightarrow V^{p|q}$$

be an isomorphism of supermanifolds. In symbolic notation we write this transformation as

$$(x, \theta) \longmapsto (y, \varphi);$$

if (x, θ) are coordinates for U and (u, ξ) are coordinates for V, this means that

$$\psi^*(u^i) = y^i(x, \theta), \quad \psi^*(\xi^j) = \varphi^j(x, \theta).$$

We then have the modified tangent map with matrix

$$J\psi = \begin{pmatrix} \frac{\partial y}{\partial x} & -\frac{\partial y}{\partial \theta} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial \theta} \end{pmatrix}.$$

Notice that y is even and φ is odd so that this matrix is even, i.e., has even elements in the diagonal blocks and odd elements in the off-diagonal blocks. It is also invertible because ψ is a super diffeomorphism. Hence its Berezinian makes sense. We then have the following theorem.

THEOREM 4.6.1 *For all compactly supported sections $s \in \mathcal{O}_V(V)$, we have*

$$\int_V s = \int_U \psi^*(s)\,\mathrm{Ber}(J\psi).$$

The proof of this remarkable formula is a little involved. It is mostly a question of accommodating the odd variables in the classical formula for change of variables. The method of proving this is to exhibit the diffeomorphism ψ as a composition of simpler diffeomorphisms and then use the multiplicative property of both $J\psi$ and Ber to reduce the proof to the case of the simpler diffeomorphisms.

We can already make a simplification. Since $\psi^\sim$ is a diffeomorphism of the reduced manifolds associated to $U^{p|q}$ and $V^{p|q}$, we can introduce the diffeomorphism τ from $U^{p|q}$ to $V^{p|q}$, which is defined by

$$\tau : (x, \theta) \longrightarrow (y^\sim, \theta).$$

For this change of variables the theorem is just the classical change-of-variables formula, and because $\tau^{-1}\psi$ is an isomorphism of $U^{p|q}$ with itself, we may replace ψ by $\tau^{-1}\psi$. Thus we may assume that

$$U = V, \quad y(x, \theta) \equiv x \ (\mathrm{mod}\ \mathcal{J}).$$

Here we recall that $\mathcal{J}$ is the ideal in $\mathcal{O}_U(U)$ generated by the θ^j.

The Purely Odd Case. We first deal with the case $p = 0$. Thus we are dealing with isomorphisms of $\mathbf{R}^{0|q}$ with itself, i.e., automorphisms of the exterior algebra $A = \mathbf{R}[\theta^1, \ldots, \theta^q]$. In the general case of such a transformation $\theta \longrightarrow \varphi$, we have

$$\varphi^i \equiv \sum_j c_{ij}\theta^j \ (\mathrm{mod}\ \mathcal{J}^3)$$

where the matrix (c_{ij}) is invertible. By a linear transformation we can make it the identity, and so we may assume that

$$\varphi^i \equiv \theta^i \ (\mathrm{mod}\ \mathcal{J}^3), \quad 1 \le i \le q .$$

We remark that any morphism $\theta \longrightarrow \varphi$ satisfying $\varphi^i \equiv \theta^i (\mathrm{mod}\ \mathcal{J}^3)$ for all i is an automorphism by the differential criterion of Theorem 4.4.1. The ideals $\mathcal{J}^r$ are invariant under all automorphisms.

Consider first the case in which ψ changes just one of the coordinates, say θ^1. Thus we have

$$\psi : \theta \longrightarrow \varphi, \quad \varphi^1 = \theta^1 + \alpha, \quad \varphi^j = \theta^j, \ j > 1 .$$

Then $\partial\alpha/\partial\theta^1$ is even and lies in $\mathcal{J}^2$. Write

$$\alpha = \theta^1\beta + \gamma, \quad \beta, \gamma \in \mathbf{R}[\theta^2, \ldots, \theta^q] .$$

Then

$$\alpha_{,1} := \frac{\partial\alpha}{\partial\theta^1} = \beta \quad \text{and} \quad \mathrm{Ber}(J\psi) = (1 + \alpha_{,1})^{-1} = (1+\beta)^{-1} .$$

Notice the inverse here; the formula for the Berezinian involves the *inverse* of the matrix corresponding to the odd-odd part. Thus we have to prove that

$$\int u = \int \psi^*(u)(1 + \alpha_{,1})^{-1} .$$

This amounts to showing that

$$\int \varphi^I (1+\beta)^{-1} = \begin{cases} 0 & \text{if } |I| < q \\ 1 & \text{if } I = Q. \end{cases}$$

We must remember that Ber is even and so commutes with everything, and φ^I is the expression obtained by making the substitution $\theta^j \longmapsto \varphi^j$. If $j \notin I$ we have

$$\int \theta^2 \cdots \theta^r (1+\beta)^{-1} = 0$$

because the integrand does not involve θ^1. Suppose $|I| < q$ and I contains the index 1, say, $I = \{1, \ldots, r\}$ with $r < q$. Then, with $\gamma_1 = \gamma(1+\beta)^{-1}$, we have

$$\begin{aligned} \int (\theta^1(1+\beta) + \gamma)\theta^2 \cdots \theta^r (1+\beta)^{-1} &= \int (\theta^1 + \gamma_1)\theta^2 \cdots \theta^r \\ &= \int \theta^1 \cdots \theta^r + \int \gamma_1 \theta^2 \cdots \theta^r \\ &= 0, \end{aligned}$$

the last equality following from the fact that the first term involves only $r < q$ odd variables and the second does not involve θ^1. For the case θ^Q the calculation is essentially the same. We have

$$\int (\theta^1(1+\beta) + \gamma)\theta^2 \cdots \theta^q (1+\beta)^{-1} = \int (\theta^1 + \gamma_1)\theta^2 \cdots \theta^q = 1 .$$

Clearly, this calculation remains valid if the transformation changes just one odd variable, not necessarily the first. Let us say that such transformations are of level 1. A transformation of level r then changes exactly r odd variables. We shall establish the result for transformations of level r by induction on r, starting from the case $r = 1$ proven just now. The induction step is carried out by exhibiting any transformation of level $r + 1$ as a composition of a transformation of level 1 and one of level r.

Suppose that we have a transformation of level $r + 1$ of the form

$$\theta \longrightarrow \varphi, \quad \varphi^i = \theta^i + \gamma^i,$$

where $\gamma^i \in \mathcal{J}^3$ and is 0 for $i > r + 1$. We write this as a composition

$$\theta \longrightarrow \tau \longrightarrow \varphi$$

where

$$\tau^i = \begin{cases} \theta^i + \gamma^i & \text{if } i \leq r \\ \theta^i & \text{if } i > r, \end{cases} \qquad \varphi^i = \begin{cases} \tau^i & \text{if } i \neq r+1 \\ \tau^{r+1} + \gamma' & \text{if } i = r+1, \end{cases}$$

with a suitable choice of γ'. The composition is then the map

$$\theta \longrightarrow \varphi$$

where

$$\varphi^i = \begin{cases} \theta^i + \gamma^i & \text{if } i \leq r \\ \theta^{r+1} + \gamma'(\tau) & \text{if } i = r+1 \\ \theta^i & \text{if } i > r+1. \end{cases}$$

Since $\theta \longrightarrow \tau$ is an even automorphism of the exterior algebra, it preserves $\mathcal{J}^3$ and is an automorphism on it, and so we can choose γ' such that $\gamma'(\tau) = \gamma^{r+1}$. The induction step argument is thus complete, and the result established in the purely odd case, i.e., when $p = 0$.

The General Case. We consider the transformation

$$(x, \theta) \longrightarrow (y, \varphi), \quad y \equiv x \ (\mathrm{mod}\ \mathcal{J}^2).$$

This can be regarded as the composition

$$(x, \theta) \longrightarrow (z, \tau) \longrightarrow (y, \varphi)$$

where

$$z = x, \quad \tau = \varphi, \qquad \text{and} \qquad y = y(z, \varphi), \quad \varphi = \tau.$$

So it is enough to treat these two cases separately. Note that, by Theorem 4.4.1 and Corollary 4.4.2, any morphism of either of these types is an isomorphism.

Case 1. $(x, \theta) \longrightarrow (x, \varphi)$. If σ denotes this map, then we can think of σ as a *family* (σ_x) of x-dependent automorphisms of $\mathbf{R}[\theta^1, \ldots, \theta^q]$. Clearly

$$\mathrm{Ber}(J\sigma)(x) = \mathrm{Ber}(J\sigma_x),$$

and so the result is immediate from the result for the purely odd case proven above.

Case 2. $(x, \theta) \longrightarrow (y, \theta)$ with $y \equiv x \ (\mathrm{mod}\ \mathcal{J}^2)$. Exactly as in the purely odd case we introduce the *level* of the transformation and show that any transformation of this type of level $r+1$ is the composition of a transformation of level 1 with one of level r. Indeed, the key step is the observation that if τ is a transformation of level r, it induces an automorphism of $\mathcal{J}^2$, and so, given any $\gamma \in \mathcal{J}^2$, we can find a $\gamma' \in \mathcal{J}^2$ such that $\gamma = \gamma'(\tau)$. We are thus reduced to the case of level 1. So we may assume that

$$y^1 = x^1 + \gamma(x, \theta), \quad y^i = x^i, \; i > 1, \; \varphi^j = \theta^j .$$

In this case the argument is a little more subtle. Let ψ denote this transformation. Then

$$\mathrm{Ber}(J\psi) = 1 + \frac{\partial \gamma}{\partial x^1} =: 1 + \gamma_{,1} .$$

Note that there is no inverse here unlike the purely odd case. We want to prove that for a compactly supported smooth function f one has the formula

$$\int f(x^1 + \gamma, x^2, \ldots, x^p)\theta^I(1 + \gamma_{,1})d^px\, d^q\theta = \int f(x)\theta^I d^px\, d^q\theta .$$

Clearly, it is enough to prove that

$$(*) \qquad \int f(x^1 + \gamma, x^2, \ldots, x^p)\left(1 + \frac{\partial \gamma}{\partial x^1}\right)d^px = \int f(x)d^px .$$

The variables other than x^1 play no role in $(*)$, and so we need to prove it only for $p = 1$. Write $x = x^1$. Thus we have to prove that

$$\int f(x + \gamma)(1 + \gamma')dx = \int f(x)dx , \quad \gamma' = \frac{d\gamma}{dx} .$$

We expand $f(x+\gamma)$ as a Taylor series (which terminates because $\gamma \in \mathcal{J}^2$). Then,

$$\begin{aligned}
&\int f(x + \gamma)(1 + \gamma')dx \\
&= \sum_{r \geq 0} \frac{1}{r!} \int f^{(r)}\gamma^r(1 + \gamma')dx \\
&= \int f dx + \sum_{r \geq 0} \frac{1}{(r+1)!} \int f^{(r+1)}\gamma^{r+1}\, dx + \sum_{r \geq 0} \frac{1}{r!} \int f^{(r)}\gamma^r\gamma'\, dx \\
&= \int f dx + \sum_{r \geq 0} \frac{1}{(r+1)!} \int (f^{(r+1)}\gamma^{r+1} + (r+1)f^{(r)}\gamma^r\gamma')dx \\
&= 0
\end{aligned}$$

because

$$\int (f^{(r+1)}\gamma^{r+1} + (r+1)f^{(r)}\gamma^r\gamma')dx = \int (f^{(r)}\gamma^{(r+1)})'\, dx = 0 .$$

This completes the proof of Theorem 4.6.1.

There is no essential difficulty in now carrying over the theory of integration to an arbitrary supermanifold M whose reduced part is orientable. One can introduce the so-called *Berezinian bundle*, which is a line bundle on M such that its sections are *densities* that are the objects to be integrated over M. Concretely, one can define a density given an atlas of coordinate charts (x, θ) covering M as a choice of a density

$$\delta(x, \theta) d^p x \, d^q \theta$$

for each chart, so that on the overlaps they are related by

$$\delta(y(x, \theta), \varphi(x, \theta)) \,\mathrm{Ber}(J\psi) = \delta(y, \varphi)$$

where ψ denotes the transformation

$$\psi : (x, \theta) \longrightarrow (y, \varphi) .$$

We do not go into this in more detail. For a more fundamental way of proving the change-of-variables formula, see the work by Deligne and Morgan;[4] see also Leĭtes[3] and Berezin.[5]

4.7. Submanifolds: Theorem of Frobenius

Let M be a supermanifold. Then a *submanifold* of M (subsupermanifold) is a pair (N, j) where N is a supermanifold, $j(N \longrightarrow M)$ is a morphism such that $j^\sim$ is an imbedding of $N^\sim$ onto a closed or locally closed submanifold of $M^\sim$, and j itself is an immersion of supermanifolds. From the local description of immersions it follows that if $n \in N$ then the morphisms from a given supermanifold S into N are precisely the morphisms f from S to M with the property that $f^\sim(S^\sim) \subset j^\sim(N^\sim)$. Let $M = U^{p|q}$ with $0 \in U$, and let

$$f_1, \ldots, f_r, g_1, \ldots, g_s$$

be sections on U such that

(1) the f_i are even and the g_j are odd and
(2) the matrices

$$\left(\frac{\partial f_a}{\partial x^i}\right), \quad \left(\frac{\partial g_b}{\partial \theta^j}\right),$$

have ranks r and s, respectively, at 0.

This is the same as requiring that there are even $f_{r+1}, \ldots, f_p$ and odd $g_{s+1}, \ldots, g_q$ such that

$$f_1, \ldots, f_p, g_1, \ldots, g_q$$

form a coordinate system at 0. Then

$$f_1 = \cdots = f_r = g_1 = \cdots = g_s = 0$$

defines a submanifold of $U^{p|q}$.

We do not go into this in more detail. The local picture of immersions makes it clear what submanifolds are like locally.

Theorem of Frobenius. We shall now discuss the super version of the classical *local* Frobenius theorem. Let M be a supermanifold and let $\mathcal{T}$ be the tangent sheaf. We start with the following definition.

DEFINITION A *distribution* over M is a graded subsheaf $\mathcal{D}$ of $\mathcal{T}$ that is locally a direct factor.

To say that $\mathcal{D}$ is locally a direct factor means the following: For each $x \in M$ there is an open set U containing x and a subsheaf $\mathcal{E}$ of the tangent sheaf $\mathcal{T}_U$ of U such that $\mathcal{D}_y \oplus \mathcal{E}_y = \mathcal{T}_y$ for all $y \in U$. Note also that since the notion of a distribution is local, we may assume in what follows that M is connected, i.e., that M_{red} is connected. If $\mathcal{D}$ is a subsheaf of $\mathcal{T}$, $x \in M$, and X is a vector field defined around x, we write $X \in \mathcal{D}_x$ to mean that the germ of X at x lies in $\mathcal{D}_x$.

This definition needs amplification since it differs in appearance from the usual definition of a distribution on a classical manifold. If M is classical and $T(M)$ is its tangent bundle, a distribution is just a subbundle. Since vector fields in the super case are not determined by the families of tangent vectors that they induce at the points of the supermanifold, it is necessary to define the notion of a subbundle in terms of the sheaf of sections of it, i.e., in terms of a subsheaf of the tangent sheaf. The following discussion, while amplifying the notion of a distribution, will also make clear why it is the correct generalization of the classical concept. We begin with a preparatory lemma.

LEMMA 4.7.1 (Nakayama's Lemma, Super Version) *Let A be a local supercommutative ring with maximal homogeneous ideal $\mathfrak{m}$. Let E be a finitely generated module for the ungraded ring A. We then have the following*:

(i) *If $\mathfrak{m}E = E$, then $E = 0$; more generally, if H is a submodule of E such that $E = \mathfrak{m}E + H$, then $E = H$.*
(ii) *Let $(v_i)_{1 \le i \le p}$ be a basis for the k-vector space $E/\mathfrak{m}E$ where $k = A/\mathfrak{m}$. Let $e_i \in E$ be above v_i. Then the e_i generate E. If E is a supermodule for the super ring A and v_i are homogeneous elements of the supervector space $E/\mathfrak{m}E$, we can choose the e_i to be homogeneous also (hence of the same parity as the v_i).*
(iii) *Suppose E is projective, i.e., there is a A-module F such that $E \oplus F = A^N$ where A^N is the free module for the ungraded ring A of rank N. Then E (hence also F) is free and the e_i above form a basis for E.*

PROOF: The proofs are easy extensions of the ones in the commutative case.[6] We begin the proof of (i) with the following observation: If B is a commutative local ring with $\mathfrak{n}$ as maximal ideal, then a square matrix R over B is invertible if (and only if) its reduction mod $\mathfrak{n}$ is invertible over the field $B/\mathfrak{n}$; in fact, if this is so, $\det(R) \notin \mathfrak{n}$ and so is a unit of B. This said, let u_i $(1 \le i \le N)$ generate E. If $E = \mathfrak{m}E$, we can find $m_{ij} \in \mathfrak{m}$ such that $e_i = \sum_j m_{ij} e_j$ for all i. Hence, if L is the matrix with entries $\delta_{ij} - m_{ij}$, then

$$L \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = 0 .$$

It is now enough to prove that L has a left inverse; i.e., for some $N \times N$ matrix P we have $PL = I$, for then, multiplying the above from the left by P, we get $u_i = 0$ for all i and so $E = 0$. It is even true that L is invertible; i.e., it has both a left and a right inverse (these two must then be the same). To show this, let us consider $B = A/J$ where J is the ideal generated by A_1. Since $J \subset \mathfrak{m}$, we have

$$A \longrightarrow B = A/J \longrightarrow k = A/\mathfrak{m} .$$

Let L_B (resp., L_k) be the reduction of L mod J (resp., mod $\mathfrak{m}$). Then B is local, its maximal ideal is $\mathfrak{m}/J$, and L_k is the reduction of $L_B \bmod \mathfrak{m}/J$. But B is commutative and $L_k = I$, and so L_B is invertible. But then L is invertible (see Section 3.6). If, more generally, we have $E = H + \mathfrak{m}E$, then $E/H = \mathfrak{m}(E/H)$ and so $E/H = 0$, i.e., $E = H$.

To prove (ii), let H be the submodule generated by the e_i. Then $E = \mathfrak{m}E + H$ and so $E = H$.

We now prove (iii). Clearly F is also finitely generated. We have $k^N = A^N/\mathfrak{m}^N = E/\mathfrak{m}E \oplus F/\mathfrak{m}F$. Let (w_j) be a basis of $F/\mathfrak{m}F$, and let f_j be elements of F above w_j. Then by (ii) the e_i, f_j form a basis of A^N while the e_i (resp., f_j) generate E (resp., F). Now there are exactly N of the e_i, f_j, and so, if X denotes the $N \times N$ matrix with columns $e_1, \ldots, f_1, \ldots$, then for some $N \times N$ matrix Y over A we have $XY = I$. So $X_B Y_B = I$ where the suffix B denotes reduction mod B. But B is commutative and so $Y_B X_B = I$. Hence X has a left inverse over A, which must be Y so that $YX = I$. If now there is a linear relation among the e_i, f_j and x is the column vector whose components are the coefficients of this relation, then $Xx = 0$; but then $x = YXx = 0$. In particular, E is a free module with basis (e_i). This proves the lemma. □

We shall now apply this version of Nakayama's lemma to supermanifolds. Let M be as above.

COROLLARY 4.7.2 *Let $x \in M$ and let $R^1, \ldots, R^a$ be vector fields defined around x such that the tangent vectors $R^1_x, \ldots, R^a_x$ are linearly independent. Then the vector fields define linearly independent elements of the $\mathcal{O}_x$-module $\mathcal{T}_x$.*

PROOF: Select vector fields S^j $(1 \le j \le b)$ defined around x such that $R^1_x, \ldots, R^a_x, S^1_x, \ldots, S^b_x$ form a basis of the tangent space to M at x. The above lemma implies that the germs of the vector fields $R^1, \ldots, R^a, S^1, \ldots, S^b$ form a free basis of $\mathcal{T}_x$. In particular, the R^i are linearly independent. □

LEMMA 4.7.3 *Let M be connected. Suppose $\mathcal{D}$ is a distribution. Then $\mathcal{D}_x$ is a free module for all $x \in M$, and the dimension of the supervector space $\mathcal{D}_x/\mathfrak{m}_x\mathcal{D}_x$ is independent of x. Let it be $r|s$ and let it be called the rank of $\mathcal{D}$. Then $\mathcal{D}$ is locally* *isomorphic to* $\mathcal{O}^{r|s}$. *If* $x \in M$ *and vector fields* $X^1, \dots X^r, Y^1, \dots, Y^s \in \mathcal{D}_x$ *are defined around* x *with* X^i *even and* Y^j *odd such that the tangent vectors of the* X^i, Y^j *are linearly independent at* x, *then for all* y *near* x, *the germs of* X^i, Y^j *form a basis for* $\mathcal{D}_y$. *Conversely, suppose that* $X^1, \dots, X^r, Y^1, \dots, Y^s$ *are vector fields with* X^i *even and* Y^j *odd and that their tangent vectors are linearly independent for all* $x \in M$; *then the sheaf generated by the* X^i, Y^j *is a distribution of rank* $r|s$.

PROOF: Let $\mathcal{D}$ be a distribution. Fix $x \in M$ and let $\dim \mathcal{D}_x/\mathfrak{m}_x\mathcal{D}_x = r|s$ where $r = r(x)$, $s = s(x)$. In what follows U will denote a generic open neighborhood of x. Let $X^i, Y^j \in \mathcal{D}_x$ be vector fields, even and odd, respectively, defined on U, such that their tangent vectors are a basis for $\mathcal{D}_x/\mathfrak{m}_x\mathcal{D}_x$. We select a subsheaf $\mathcal{E}$ of the tangent sheaf of U such that $\mathcal{T}_y = \mathcal{D}_y \oplus \mathcal{E}_y$ for all $y \in U$. Select vector fields $Z^k, T^n \in \mathcal{E}_x$, defined on U and even and odd, respectively, such that their tangent vectors at x form a basis of $\mathcal{E}_x/\mathfrak{m}_x\mathcal{E}_x$. Then the tangent vectors of X^i, Y^j, Z^k, T^n at x form a basis for $T_x(M)$ and hence this is true for all $y \in U$ also. By Lemma 4.7.1 the germs of these vector fields form a basis of $\mathcal{T}_y$ for all $y \in U$. In particular, if $\mathcal{D}'_y$ (resp., $\mathcal{E}'_y$) is the submodule of $\mathcal{T}_y$ generated by X^i, Y^j (resp., Z^k, T^n), then

$$\mathcal{D}'_y \subset \mathcal{D}_y\,, \quad \mathcal{E}'_y \subset \mathcal{E}_y\,, \quad \mathcal{D}'_y \oplus \mathcal{E}'_y = \mathcal{T}_y = \mathcal{D}_y \oplus \mathcal{E}_y\,,$$

so that $\mathcal{D}'_y = \mathcal{D}_y$, $\mathcal{E}'_y = \mathcal{E}_y$ for $y \in U$. This shows that $r(y) = r$, $s(y) = s$ for $y \in U$, and hence that r, s are locally constant, thus constant, and further that X^i, Y^j form a basis of $\mathcal{D}_y$ for $y \in U$.

For the converse, since the assertion is local we may assume that there are vector fields Z^k, T^n, even and odd, respectively, such that the tangent vectors of X^i, Y^j, Z^k, T^n are linearly independent at all points of M. These vector fields then form a basis of $\mathcal{T}_y$ for all $y \in M$. If $\mathcal{D}$, $\mathcal{E}$ are the sheaves generated by (X^i, Y^j), (Z^k, T^n), respectively, then $\mathcal{T} = \mathcal{D} \oplus \mathcal{E}$. Hence $\mathcal{D}$ is a distribution. The lemma is completely proven. ☐

DEFINITION A distribution $\mathcal{D}$ is *involutive* if $\mathcal{D}_m$ is a (super) Lie algebra for each point $m \in M$. In other words, if vector fields $X, Y \in \mathcal{D}_m$ are defined around m, then $[X, Y] \in \mathcal{D}_m$.

THEOREM 4.7.4 *A distribution is involutive if and only if at each point there is a coordinate system* (x, θ) *such that* $\mathcal{D}_m$ *is spanned by* $\partial/\partial x^i, \partial/\partial\theta^j$ $(1 \le i \le r, 1 \le j \le s)$.

The "if" part is trivial. So we need to prove that if $\mathcal{D}$ is involutive, it has the local structure described in the theorem.

Some Lemmas on the Local Structure of an Involutive Distribution. We need some lemmas of a local nature before we can prove the theorem. We assume that $M = U^{p|q}$ with coordinates (x, ξ) and $m = 0$.

LEMMA 4.7.5 *Let* X *be an even vector field whose value is a nonzero tangent vector at the point* m. *Then there is a coordinate system* (z, ζ) *at* m *in which* $X = \partial/\partial z^1$.

PROOF: Assume that $M = U^{p|q}$ with $m = 0$, the coordinates being (x, ξ). If there are no odd variables the result is classical, and so going over to the reduced manifold we may assume that

$$X = \frac{\partial}{\partial x^1} + \sum_j a_j \frac{\partial}{\partial x^j} + \sum_\rho \beta_\rho \frac{\partial}{\partial \xi^\rho}$$

where a_j are even, β_ρ are odd, and they are all in $\mathcal{J}$. Here and in the rest of the section we use the same symbol $\mathcal{J}$ to denote the ideal sheaf generated by the odd elements of $\mathcal{O}_U$ in any coordinate system. The evenness of a_j then implies that $a_j \in \mathcal{J}^2$. Moreover, we can find an even matrix $b = (b_{\rho\tau})$ such that $\beta_\rho \equiv \sum_\tau b_{\rho\tau}\xi^\tau$ (mod $\mathcal{J}^2$). Thus mod $\mathcal{J}^2$ we have

$$X \equiv \frac{\partial}{\partial x^1} + \sum_{\rho\tau} b_{\rho\tau}\xi^\tau \frac{\partial}{\partial \xi^\rho}\,.$$

We now make a transformation $U^{p|q} \longrightarrow U^{p|q}$ given by

$$(x, \xi) \longrightarrow (y, \eta)$$

where

$$y = x\,, \quad \eta = g(x)\xi\,, \quad g(x) = (g_{\rho\tau}(x))\,,$$

and g is an invertible matrix of smooth functions to be chosen suitably. Then we have a diffeomorphism and a simple calculation shows that

$$X \equiv \frac{\partial}{\partial y^1} + \sum_\rho \gamma_\rho \frac{\partial}{\partial \eta^\rho} \ (\text{mod}\ \mathcal{J}^2) \quad \text{and} \quad \gamma_\rho = \sum_\tau \xi^\tau \left(\frac{\partial g_{\rho\tau}}{\partial x^1} + \sum_\sigma g_{\rho\sigma} b_{\sigma\tau} \right).$$

We choose g so that it satisfies the matrix differential equations

$$\frac{\partial g}{\partial x^1} = -gb\,, \quad g(0) = I\,.$$

It is known that this is possible and that g is invertible. Hence

$$X \equiv \frac{\partial}{\partial y^1} \ (\text{mod}\ \mathcal{J}^2)\,.$$

We may therefore suppose that

$$X \equiv \frac{\partial}{\partial x^1} \quad (\text{mod}\ \mathcal{J}^2)\,.$$

We now show that one can choose in succession coordinate systems such that X becomes $\equiv \partial/\partial x^1$ (mod $\mathcal{J}^k$) for $k = 3, 4, \ldots$. This is done by induction on k. Assume that $k \geq 2$ and $X \equiv \partial/\partial x^1$ (mod $\mathcal{J}^k$) in a coordinate system (x, ξ). We shall then show that if we choose a suitable coordinate system (y, η) defined by

$$(x, \xi) \longrightarrow (y, \eta)\,, \quad y^i = x^i + a_i\,, \ \eta^\rho = \xi^\rho + \beta_\rho\,,$$

where $a_i, \beta_\rho \in \mathcal{J}^k$ are suitably chosen, then $X \equiv \partial/\partial y^1$ (mod $\mathcal{J}^{k+1}$). Let

$$X = \frac{\partial}{\partial x^1} + \sum_j g_j \frac{\partial}{\partial x^j} + \sum_\rho \gamma_\rho \frac{\partial}{\partial \xi^\rho}$$

where the $g_j, \gamma_\rho \in \mathcal{J}^k$. Then in the new coordinate system

$$\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j} + \sum_k \left(\frac{\partial a_k}{\partial x^j}\right)\frac{\partial}{\partial y^k} + \sum_\tau \left(\frac{\partial \beta_\tau}{\partial x^j}\right)\frac{\partial}{\partial \eta^\tau}.$$

Similarly,

$$\frac{\partial}{\partial \xi^\rho} = \frac{\partial}{\partial \eta^\rho} + \sum_k \left(\frac{\partial a_k}{\partial \xi^\rho}\right)\frac{\partial}{\partial y^k} + \sum_\tau \left(\frac{\partial \beta_\tau}{\partial/\partial \xi^\rho}\right)\frac{\partial}{\partial \eta^\tau}.$$

Hence, because $2k - 1 \geq k + 1$, we have

$$X = \frac{\partial}{\partial y^1} + \sum_j \left(g_j + \frac{\partial a_j}{\partial x^1}\right)\frac{\partial}{\partial y^j} + \sum_\tau \left(\gamma_\tau + \frac{\partial \beta_\tau}{\partial x^1}\right)\frac{\partial}{\partial \eta^\tau} + Z$$

where $Z \equiv 0 \ (\mathrm{mod}\ \mathcal{J}^{k+1})$. If we now choose, as is clearly possible, the a_j, β_τ such that

$$\frac{\partial a_j}{\partial x^1} = -g_j\,, \qquad \frac{\partial \beta_\tau}{\partial x^1} = -\gamma_\tau\,,$$

we see that $X \equiv 0 \ (\mathrm{mod}\ \mathcal{J}^{k+1})$. This finishes the proof. □

LEMMA 4.7.6 *Let Y be an odd vector field such that $Y^2 = 0$ and $Y_m \neq 0$. Then in some coordinate system we have $Y = \partial/\partial\theta^1$.*

PROOF: Note that the conditions $Y^2 = 0$, $Y_m \neq 0$ are both necessary for Y to be $\partial/\partial\theta^1$ in some coordinate system. Recall also that $Y_m \neq 0$ means that Y generates, at least locally around m, a distribution. The proof is patterned after the classical proof where a single vector field is considered. There the corresponding differential equations are written down and solved for arbitrary initial conditions in the "time" variable t, the initial conditions corresponding to $t = 0$. Here we do the same thing, with an odd variable θ^1 in place of t and with initial conditions at $\theta^1 = 0$. If we write

$$Y = \sum_i \alpha_i(z, \eta)\frac{\partial}{\partial z^i} + \sum_\rho a_\rho(z, \eta)\frac{\partial}{\partial \eta^\rho}\,,$$

then the condition $Y_m \neq 0$ may be taken to be

$$a_1(0, 0) \neq 0\,.$$

We now consider a map

$$\mathbf{R}^{0|1} \times U^{p|q-1} \longrightarrow U^{p|q}$$

where we use θ^1 as coordinate for $\mathbf{R}^{0|1}$, $(x, \theta^2, \ldots, \theta^q)$ for coordinates on $U^{p|q-1}$. The map is given by

$$\begin{aligned} z^i &= x^i + \theta^1\alpha_i(x, 0, \theta')\,, \\ \eta^1 &= \theta^1 a_1(x, 0, \theta')\,, \\ \eta^\rho &= \theta^\rho + \theta^1 a_\rho(x, 0, \theta')\,, \quad \rho \geq 2\,. \end{aligned}$$

Here $\theta' = (\theta^2, \dots, \theta^q)$. At $x = 0$, the tangent map of this map has the matrix

$$\begin{pmatrix} I_p & * & 0 \\ 0 & a_1(0,0) & 0 \\ 0 & * & I_{q-1} \end{pmatrix},$$

which has nonzero determinant because $a_1(0,0) \neq 0$. So we have a local isomorphism that we assume is defined on U by shrinking U. Under this isomorphism the vector field $\partial/\partial\theta^1$ corresponds to the vector field

$$\sum_i \alpha'_i \frac{\partial}{\partial z^i} + \sum_\rho a'_\rho \frac{\partial}{\partial \eta^\rho}$$

on $U^{p|q}$ where

$$\alpha'_i = \alpha_i(x, 0, \theta'), \quad a'_\rho = a_\rho(x, 0, \theta').$$

But

$$\alpha_i(z, \eta) = \alpha_i\big(\dots, x^i + \theta^1\alpha'_i, \dots, \theta^1 a'_1, \dots, \theta^\rho + \theta^1 a'_\rho, \dots\big).$$

Hence by Taylor expansion (terminating) we get, for a suitable β_i,

$$\alpha_i = \alpha'_i + \theta^1\beta_i.$$

Similarly, we have, for a suitable b_ρ,

$$a_\rho = a'_\rho + \theta^1 b_\rho.$$

Hence $\partial/\partial\theta^1$ goes over to a vector field of the form $Y - \theta^1 Z$ where Z is an even vector field and we have to substitute for θ^1 its expression in the coordinates (z, η).

Let V be the vector field in the (x, θ)–coordinates that corresponds to Z. Then

$$\frac{\partial}{\partial\theta^1} + \theta^1 V \longrightarrow Y$$

where $\longrightarrow$ means that the vector fields correspond under the isomorphism being considered. Since $Y^2 = 0$ we must have $(\partial/\partial\theta^1 + \theta^1 V)^2 = 0$. But a simple computation shows that

$$\left(\frac{\partial}{\partial\theta^1} + \theta^1 V\right)^2 = V - \theta^1 W = 0$$

where W is an odd vector field. Hence $V = \theta^1 W$. But then

$$\frac{\partial}{\partial\theta^1} + \theta^1 V = \frac{\partial}{\partial\theta^1} \longrightarrow Y$$

as we wanted to show. □

LEMMA 4.7.7 *The even part of $\mathcal{D}_m$ has a basis consisting of commuting (even) vector field germs.*

PROOF: Choose a coordinate system (z^i, η^ρ) around m. Let X^i $(1 \le i \le r)$ be even vector fields whose germs at m form a basis for the even part $\mathcal{D}_m$. Then the matrix of coefficients of these vector fields has the form

$$T = (a \;\; \alpha)$$

where a is an even $r \times p$ matrix of rank r, while α is odd. Multiplying from the left by invertible matrices of function germs changes the given basis into another and so we may assume, after a suitable reordering of the even coordinates z, that

$$a = (I_r \;\; a' \;\; \beta)\,.$$

So we have a new basis for the even part of $\mathcal{D}_m$ (denoted again by X^i) consisting of vector fields of the following form:

$$X^i = \frac{\partial}{\partial z^i} + \sum_{k>r} a'_{ik} \frac{\partial}{\partial z^k} + \sum_{\rho} \beta_{i\rho} \frac{\partial}{\partial \eta^\rho}\,, \qquad 1 \le i \le r\,.$$

The commutator $[X^i, X^j]$ must be a combination $\sum_{t \le r} f_t X^t$, and so f_t is the coefficient of $\partial/\partial z^t$ in the commutator. But it is clear from the above formulae that the commutator in question is a linear combination of $\partial/\partial z^k$ $(k > r)$ and the $\partial/\partial \eta^\rho$. Hence all the f_t are 0 and so the X^i commute with each other. □

LEMMA 4.7.8 *There is a coordinate system* (z, θ) *such that the even part of* $\mathcal{D}_m$ *is spanned by the* $\partial/\partial z^i$ $(1 \le i \le r)$.

PROOF: Let $(X^i)_{1 \le i \le r}$ be commuting vector fields spanning the even part of $\mathcal{D}_m$. We shall prove first that there is a coordinate system (z, η) in which the X^i have the triangular form, i.e.,

$$X^i = \frac{\partial}{\partial z^i} + \sum_{j<i} a_{ij} \frac{\partial}{\partial z^j}\,.$$

We use induction on r. The case $r = 1$ is just Lemma 4.7.5. Let $r > 1$ and assume the result for $r - 1$ commuting even vector fields. Then for suitable coordinates we may assume that

$$X^i = \frac{\partial}{\partial z^i} + \sum_{j<i} a_{ij} \frac{\partial}{\partial z^j}\,, \qquad i < r\,.$$

Write

$$X^r = \sum_t f_t \frac{\partial}{\partial z^t} + \sum_\rho g_\rho \frac{\partial}{\partial \eta^\rho}\,.$$

Then, for $j < r$,

$$[X^j, X^r] = \sum_t (X^j f_t) \frac{\partial}{\partial z^t} + \sum_\rho (X^j g_\rho) \frac{\partial}{\partial \eta^\rho} = 0\,.$$

Hence

$$X^j f_t = 0\,, \qquad X^j g_\rho = 0\,.$$

The triangular form of the X^j now implies that these equations are valid with $\partial/\partial z^j$ replacing X^j for $j \le r - 1$. Hence the f_t and g_ρ *depend only on the variables* z^k $(k \ge r, \eta^\tau)$. So we can write

$$X^r = \sum_{t \le r-1} f_t \frac{\partial}{\partial z^t} + X'\,, \quad X' = \sum_{t \ge r} f_t \frac{\partial}{\partial z^t} + \sum_\rho g_\rho \frac{\partial}{\partial \eta^\rho}\,.$$

Thus X' is an even vector field in the supermanifold with coordinates z^k $(k \geq r), \eta^\sigma$. By Lemma 4.7.5 we can change z^k $(k \geq r), \eta^\sigma$ to another coordinate system (w^k $(k \geq r)$, ζ^σ) such that X' becomes $\partial/\partial w^r$. If we make the change of coordinates

$$z, \eta \longrightarrow z^1, \ldots, z^{r-1}, w^r, w^{r+1}, \ldots, \zeta^\sigma \quad (k \geq r)$$

it is clear that the $\partial/\partial z^i$ for $i \leq r-1$ remain unchanged while X^r goes over to

$$\frac{\partial}{\partial w^r} + \sum_{t \leq r-1} k_t \frac{\partial}{\partial z^t},$$

which proves what we claimed. The triangular form of the X^i now shows that they span the same distribution as the $\partial/\partial z^i$. This proves the lemma. □

LEMMA 4.7.9 *In a suitable coordinate system at m, there is a basis for $\mathcal{D}_m$ of the form*

$$\frac{\partial}{\partial z^i}\ (1 \leq i \leq r), \quad Y^\rho,$$

where the vector fields Y^ρ are odd and have the form

$$(*) \qquad Y^\rho = \frac{\partial}{\partial \eta^\rho} + \sum_{j>r} \gamma_{\rho j} \frac{\partial}{\partial z^j} + \sum_{\tau > s} c_{\rho\tau} \frac{\partial}{\partial \eta^\tau}.$$

In particular, these vector fields supercommute with each other.

PROOF: Take a coordinate system (z, η) in which

$$\frac{\partial}{\partial z^i}\ (1 \leq i \leq r), \quad Y^\rho\ (1 \leq \rho \leq s),$$

span $\mathcal{D}_m$ where the Y^ρ are odd vector fields. The matrix of coefficients has the form

$$\begin{pmatrix} I_r & 0 & 0 \\ \beta_1 & \beta_2 & b \end{pmatrix}$$

where b is an even $s \times q$ matrix of rank s. Multiplying from the left and reordering the odd variables if necessary, we may assume that

$$b = (I_s, b').$$

Thus

$$Y^\rho = \frac{\partial}{\partial \eta^\rho} + \sum \gamma_{\rho j} \frac{\partial}{\partial z^j} + \sum_{\tau > s} c_{\rho\tau} \frac{\partial}{\partial \eta^\tau}.$$

Since the $\partial/\partial z^j$ for $j \leq r$ are already in $\mathcal{D}_m$, we may remove the corresponding terms, and so we may assume that

$$(*) \qquad Y^\rho = \frac{\partial}{\partial \eta^\rho} + \sum_{j>r} \gamma_{\rho j} \frac{\partial}{\partial z^j} + \sum_{\tau > s} c_{\rho\tau} \frac{\partial}{\partial \eta^\tau}.$$

The commutators

$$\left[\frac{\partial}{\partial z^i}, Y^\rho\right]\ (i \leq r), \quad [Y^\sigma, Y^\tau],$$

must be of the form

$$\sum_{t\le r} f_t \frac{\partial}{\partial z^t} + \sum_{\rho\le s} g_\rho Y^\rho$$

and so f_t, g_ρ are the coefficients of $\partial/\partial z^t$ and $\partial/\partial\eta^\rho$ in the commutators. But these coefficients are 0, and so the commutators must vanish. This finishes the proof. □

REMARK. It should be noted that the supercommutativity of the basis follows as soon as the vector fields Y^ρ are in the form $(*)$. We shall use this in the proof of Theorem 4.7.4.

PROOF OF THEOREM 4.7.4: For $s = 0$, namely, a purely even distribution, we are already done by Lemma 4.7.8. So let $s > 1$ and let the result be assumed for distributions of rank $r|s-1$. Let us work in a coordinate system with the property of the preceding lemma. The span of

$$\frac{\partial}{\partial z^i}\ (1 \le i \le r), \quad Y^\rho\ (1 \le \rho \le s-1),$$

is also an involutive distribution, say $\mathcal{D}'$, because of the supercommutativity of these vector fields (the local splitting is true because $\mathcal{D} = \mathcal{D}' \oplus \mathcal{E}$ where $\mathcal{E}$ is the span of Y^s). We may therefore assume that $Y^\rho = \partial/\partial\eta^\rho$ $(1 \le \rho \le s-1)$. Then we have

$$Y^s = b\frac{\partial}{\partial\eta^s} + \sum_j \alpha_j \frac{\partial}{\partial z^j} + \sum_{\tau\ne s} a_\tau \frac{\partial}{\partial\eta^\tau}.$$

Since $\partial/\partial z^j$ $(1 \le j \le r)$ and $\partial/\partial\eta^\rho$ $(1 \le \rho \le s-1)$ are in $\mathcal{D}_m$, we may assume that in the above formula the index $j > r$ and the index $\tau > s$. We may assume that $b(m) \ne 0$, reordering the odd variables $\eta^\sigma (\sigma \ge s)$ if needed. Thus we may assume that $b = 1$. Hence we may suppose that

$$Y^s = \frac{\partial}{\partial\eta^s} + \sum_{j>r} \alpha_j \frac{\partial}{\partial z^j} + \sum_{\tau>s} a_\tau \frac{\partial}{\partial\eta^\tau}.$$

By the remark following Lemma 4.7.9 we then have

$$\left[\frac{\partial}{\partial z^i}, Y^s\right] = 0, \quad \left[\frac{\partial}{\partial\eta^\sigma}, Y^s\right] = 0\ (i \le r-1, \sigma \le s-1), \quad (Y^s)^2 = 0.$$

These conditions imply in the usual manner that the α_j, a_τ depend only on z^k $(k > r)$, $\eta^\sigma (\sigma \ge s)$. Lemma 4.7.6 now shows that we can change z^k $(k > r)$, $\eta^\tau (\tau \ge s)$ into a new coordinate system w^k $(k > r)$, ζ^τ $(\tau \ge s)$ such that in this system Y^s has the form $\partial/\partial\zeta^s$. Hence in the coordinate system

$$z^1, \dots, z^r, w^k\ (k > r), \eta^1, \dots, \eta^{r-1}, \zeta^s, \dots,$$

the vector fields

$$\frac{\partial}{\partial z^i}\ (i \le r), \quad \frac{\partial}{\partial\eta^\tau}\ (\tau \le r-1), \quad \frac{\partial}{\partial\zeta^s},$$

span $\mathcal{D}_m$. This finishes the proof. □

4.8. References

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CHAPTER 5

Clifford Algebras, Spin Groups, and Spin Representations

5.1. Prologue

This chapter and the next are devoted to the theory of spinors and the spin representations. The importance of spinors in supersymmetry arises from the fact that from the very beginning the physical interpretation required that the odd parts of the supertranslation algebras arising in such theories be spin modules.

The spin representations are very special representations of the orthogonal Lie algebras. They are the fundamental representations that are *not* representations of the orthogonal groups but rather their twofold covers, the so-called *spin groups*. It turns out that the spin groups can be imbedded inside the even part of the Clifford algebras, and then the spin representations can be identified as modules for these even parts. Thus the theory of the spin modules becomes a part of the theory of Clifford algebras.

Because of the complexity and diversity of the various questions in the theory of spinors that are of interest for physical applications, I have divided the presentation into two chapters. The present chapter deals with the foundations of Clifford algebras, the construction of the spin groups, and the identification of the spin modules as Clifford modules. In the next chapter I have discussed more specialized questions that are of importance in physics: the reality of the spin modules for the real spin groups, the existence and uniqueness of invariant forms on spin modules, and the structure of bilinear morphisms of the spin modules into the vector and exterior tensor representations. Among other things these questions relate to the construction of super Poincaré algebras and the definition of supersymmetric Lagrangians. In all of this we treat both the complex and real cases, and the real cases in all dimensions and signatures.

Let me begin with a quick introduction to the spin representations. E. Cartan classified simple Lie algebras over $\mathbf{C}$ in his thesis in 1894, a classification that is nowadays done through the (Dynkin) diagrams. In 1913 he classified the irreducible finite-dimensional representations of these algebras.[1] For any simple Lie algebra $\mathfrak{g}$, Cartan's construction yields an irreducible representation canonically associated to each node of its diagram. These are the so-called *fundamental representations* in terms of which all irreducible representations of $\mathfrak{g}$ can be constructed using $\otimes$ and subrepresentations. Indeed, if π_j $(1 \leq j \leq \ell)$ are the fundamental representations and m_j are integers ≥ 0, and if v_j is the highest vector of π_j, then the subrepresentation of

$$\pi = \pi_1^{\otimes m_1} \otimes \cdots \otimes \pi_\ell^{\otimes m_\ell}$$

generated by

$$v = v_1^{\otimes m_1} \otimes \cdots \otimes v_\ell^{\otimes m_\ell}$$

is irreducible with highest vector v, and every irreducible module is obtained in this manner uniquely. As is well-known, Harish-Chandra and Chevalley independently developed around 1950 a general method for obtaining the irreducible representations without relying on case-by-case considerations as Cartan did. For these and other aspects of representation theory, see the book by Varadarajan.[2]

If $\mathfrak{g} = \mathfrak{sl}(\ell+1)$ and $V = \mathbf{C}^{\ell+1}$, then the fundamental module π_j is $\Lambda^j(V)$, and all irreducible modules can be obtained by decomposing the tensor algebra over the defining representation V. Similarly, for the symplectic Lie algebras, the decomposition of the tensors over the defining representation gives all the irreducible modules. But Cartan noticed that this is *not* the case for the orthogonal Lie algebras. For these the fundamental representations corresponding to the right extreme node(s) (the nodes of higher norm are to the left) could not be obtained from the tensors over the defining representation. Thus for $\mathfrak{so}(2\ell)$ $(\ell \geq 2)$, with diagram

$$\circ - \circ - \circ \quad \cdots \quad \circ - \circ < {}^{\circ}_{\circ} \qquad (\ell \text{ vertices})$$

there are two of these, denoted by $S^\pm$, of dimension $2^{\ell-1}$, and for $\mathfrak{so}(2\ell+1)$ $(\ell \geq 1)$ with diagram

$$\circ - \circ - \circ \quad \cdots \quad \circ - \circ \Rightarrow \circ \qquad (\ell \text{ vertices})$$

there is one such, denoted by S, of dimension 2^ℓ. These are the so-called *spin representations*; the $S^\pm$ are also referred to as *semispin representations*. The case $\mathfrak{so}(3)$ is the simplest. In this case the defining representation is SO(3) and its universal cover is SL(2). The tensors over the defining representation yield only the odd-dimensional irreducibles; the spin representation is the two-dimensional representation $D^{1/2} = \mathbf{2}$ of SL(2). The weights of the tensor representations are integers, while $D^{1/2}$ has the weights $\pm\frac{1}{2}$, revealing clearly why it cannot be obtained from the tensors. However, $D^{1/2}$ generates all representations; the representation of highest weight $j/2$ (j an integer ≥ 0) is the j-fold symmetric product of $D^{1/2}$, namely $\mathrm{Symm}^{\otimes j}\, D^{1/2}$. In particular, the vector representation of SO(3) is $\mathrm{Symm}^{\otimes 2}\, D^{1/2}$. In the other low-dimensional cases, the spin representations are as follows:

SO(4). Here the diagram consists of two unconnected nodes;

$${}^{\circ}_{\circ}$$

the Lie algebra $\mathfrak{so}(4)$ is not simple but semisimple and splits as the direct sum of two $\mathfrak{so}(3)$'s. The group SO(4) is not simply connected and SL(2)×SL(2) is its universal cover. The spin representations are the representations $D^{1/2,0} = \mathbf{2} \times \mathbf{1}$ and $D^{0,1/2} = \mathbf{1} \times \mathbf{2}$. The defining *vector* representation is $D^{1/2,0} \times D^{0,1/2}$.

SO(5). Here the diagram is the same as the one for Sp(4):

$$\circ \Rightarrow \circ \quad = \quad \circ \Leftarrow \circ$$

The group SO(5) is not simply connected, but Sp(4), which is simply connected, is therefore the universal cover of SO(5). The defining representation **4** is the spin representation. The representation $\Lambda^2\mathbf{4}$ is of dimension 6 and contains the trivial representation, namely, the line defined by the element that corresponds to the invariant symplectic form in $\mathbf{4}$. The quotient representation is 5-dimensional and is the defining representation for SO(5).

SO(6). We have come across this in our discussion of the Klein quadric. The diagrams for $\mathfrak{so}(6)$ and $\mathfrak{sl}(4)$ are the same,

$$\circ - \circ - \circ = \circ < {}^{\circ}_{\circ}$$

and so the universal covering group for SO(6) is SL(4). The spin representations are the defining representation $\mathbf{4}$ of SL(4) and its dual $\mathbf{4}^*$, corresponding to the two extreme nodes of the diagram. The defining representation for SO(6) is $\Lambda^2\mathbf{4} \simeq \Lambda^2\mathbf{4}^*$.

SO(8). This case is of special interest. The diagram has three extreme nodes:

$$\circ - \circ < {}^{\circ}_{\circ}$$

and the group $\mathfrak{S}_3$ of permutations in three symbols acts transitively on it. This means that $\mathfrak{S}_3$ is the group of automorphisms of SO(8) modulo the group of inner automorphisms, and so $\mathfrak{S}_3$ acts on the set of irreducible modules also. The *vector representation* $\mathbf{8}$ *as well as the spin representations* $\mathbf{8}^{\pm}$ *are all of dimension* 8, and S_3 permutes them. Thus it is immaterial which of them is identified with the vector or the spin representations. This is the famous *principle of triality*. There is an *octonionic model* for this case that makes explicit the principle of triality.[3,4]

To be useful in physics, one must discuss the entire theory of spinors over $\mathbf{R}$. Our point of view is to develop first the theory over $\mathbf{C}$ and then come down to $\mathbf{R}$ through standard devices that use data coming from complex conjugations.

Dirac's Equation of the Electron. The definition given above of the spin representations does not motivate them at all. Indeed, at the time of their discovery by Cartan, the spin representations were not called by that name; that came about only after Dirac's sensational discovery around 1930 of the spin representation and its relation to the Clifford algebra in dimension 4, on which he based the relativistic equation of the electron bearing his name. In Dirac's treatment the spin of the electron arose directly as a consequence of the structure of the Dirac equation and hence out of the structure of the representation theory of the Clifford algebra. This circumstance led to the general representations discovered by Cartan being named spin representations. The elements of the spaces on which the spin representations act were then called *spinors*. The fact that the spin representation cannot be obtained from tensors meant that the Dirac operator in quantum field theory must act on *spinor fields* rather than tensor fields. Since Dirac was concerned only with special relativity and so with *flat* Minkowski spacetime, there was no conceptual difficulty in defining the spinor fields there. But when one goes to *curved spacetime*, the spin modules of the orthogonal groups at each spacetime point form a structure that will exist in a global sense only when certain topological obstructions (cohomology classes) vanish. The structure is the so-called *spin structure* and the manifolds for which a spin structure exists are called *spin manifolds*. It is only on spin manifolds that one can formulate the global Dirac and Weyl equations.

Coming back to Dirac's discovery, his starting point was the Klein-Gordon equation

$$(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\varphi = -m^2\varphi \quad \left(\partial_\mu = \frac{\partial}{\partial x^\mu}\right)$$

where φ is the wave function of the particle (electron) and m is its mass. This equation is, of course, relativistically invariant. However, Dirac was dissatisfied with it, primarily because it was of the second order. He felt that the equation should be of the first order in time and hence, as all coordinates are on equal footing in special relativity, it should be of the first order in all coordinate variables.[5] Translation invariance meant that the differential operator should be of the form

$$D = \sum_\mu \gamma_\mu \partial_\mu$$

where the γ_μ are constants. To maintain relativistic invariance, Dirac postulated that

(5.1) $$D^2 = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 ,$$

and so his equation took the form

$$D\varphi = \pm im\varphi .$$

Here the factor i can also be understood from the principle that only the $i\partial_\mu$ are self-adjoint in quantum mechanics. Now a simple calculation shows that no *scalar* γ_μ can be found satisfying (5.1); the polynomial $X_0^2 - X_1^2 - X_2^2 - X_3^2$ is irreducible. Indeed, the γ_μ must satisfy the equations

(5.2) $$\gamma_\mu^2 = \varepsilon_\mu , \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0, \quad \mu \neq \nu,\ \varepsilon_0 = 1,\ \varepsilon_i = -1,\ i = 1,2,3 ,$$

and so the γ_μ cannot be scalars. But Dirac was not stopped by this difficulty and asked if he could find *matrices* γ_μ satisfying (5.2). He found the answer to be yes. In fact, he made the discovery that there is a solution to (5.2) where the γ_μ are 4×4 matrices, and that this solution is unique up to similarity in the sense that any other solution (γ'_μ) of degree 4 is of the form $(T\gamma_\mu T^{-1})$ where T is an invertible 4×4 matrix; even more, solutions occur only in degrees $4k$ for some integer $k \geq 1$ and are similar (in the above sense) to a direct sum of k copies of a solution in degree 4.

Because the γ_μ are 4×4 matrices, the wave function φ *cannot be a scalar anymore*; it has to have four components and Dirac realized that these extra components describe *some internal structure* of the electron. In this case he showed that they indeed encode the *spin* of the electron.

It is not immediately obvious that there is a natural action of the Lorentz group on the space of four-component functions on spacetime, with respect to which the Dirac operator is invariant. To see the existence of such an action, let $g = (\ell_{\mu\nu})$ be an element of the Lorentz group. Then it is immediate that with respect to the natural actions

$$D \circ g^{-1} = g^{-1} \circ D', \quad D' = \gamma'_\mu \partial_\mu , \quad \gamma'_\mu = \sum_\nu \ell_{\mu\nu}\gamma_\nu .$$

Since

$$D'^2 = (g \circ D \circ g^{-1})^2 = D^2$$

it follows that the ∂_μ's satisfy (5.2) and so

$$\gamma'_\mu = S(g)\gamma_\mu S(g)^{-1}$$

for all μ, $S(g)$ being an invertible 4×4 matrix determined uniquely up to a scalar multiple. The transformation property of D under the natural action of the Lorentz group can then be written, after changing g to g^{-1},

$$D \circ g = gS(g)^{-1}DS(g) \quad \text{or} \quad S(g)g^{-1}D = DS(g)g^{-1} .$$

The fact that $S(g)$ is unique up to a scalar means that

$$S : g \longmapsto S(g)$$

is a *projective* representation of the Lorentz group and can be viewed as an ordinary representation of the universal covering group of the Lorentz group, namely, $H = \mathrm{SL}(2, \mathbf{C})$. If we now define the action of H on the four-component functions is thus

$$\psi \longmapsto \psi^g := S(g)(\psi \circ g^{-1})$$

and the Dirac operator is invariant under this action:

$$D\psi^g = (D\psi)^g .$$

From the algebraic point of view one has to introduce the universal algebra C over $\mathbf{C}$ generated by the symbols γ_μ with relations (5.2) and study its representations. If we work over $\mathbf{C}$ we can forget the signs $\pm$ and take the relations between the γ_μ in the form

$$\gamma_\mu^2 = 1 , \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0, \quad \mu \neq \nu .$$

Dirac's result is then essentially that C has a unique irreducible representation, which is in dimension 4, and that any representation is a sum of copies of this one. Moreover, there is an action S of the group H on this representation space that is compatible with the action of the Lorentz group as automorphisms of C. S is the spin representation.

Clifford Algebras. The Clifford algebra, as the algebra over $\mathbf{R}$ with n generators

$$e_1, e_2, \ldots, e_n$$

and relations

$$e_r^2 = -1 , \quad e_re_s + e_se_r = 0 , \quad r \neq s ,$$

goes back to a paper of Clifford[6] in 1878 where it is viewed as a generalization of the quaternion algebra (for $n = 2$ it *is* the quaternion algebra). Their deeper significance became clear only after Dirac's discovery[7] of the spin representation, but only in dimension 4. In 1935, R. Brauer and H. Weyl wrote a seminal paper[8] in which they studied various questions concerning the spinors and spin representations over the real and complex field but in arbitrary dimensions and in the definite and Minkowski signatures. The geometric aspects of spinors were treated by Cartan in a book published in 1938.[9] The general algebraic study of spinors in *arbitrary* fields was carried out by C. Chevalley in his book.[10] The theory of spinors in arbitrary dimensions but for positive definite quadratic forms was developed in a famous paper of Atiyah, Bott, and Shapiro,[11] where they carried out many applications. In recent years, with the increasing interest of physicists in higher-dimensional spacetimes, spinors in arbitrary dimensions and *arbitrary signatures* have come to the foreground.

Our treatment leans heavily on that of Deligne.[3] One of its highlights is the study of the Clifford algebras and their representations from the point of view of the super category. This makes the entire theory extremely transparent.

5.2. Cartan's Theorem on Reflections in Orthogonal Groups

In this section we examine some elementary properties of orthogonal groups. For any finite-dimensional vector space V over a field k equipped with a symmetric nondegenerate bilinear form $(\cdot\,,\cdot)$, we write $O(V)$ for the orthogonal group of $(V, (\cdot))$ and $SO(V)$ for its normal subgroup of elements of determinant 1.

We would like to sketch a proof of Cartan's theorem on reflections and some consequences of it. We work over $k = \mathbf{R}$ or $\mathbf{C}$ and write $n = \dim(V)$.

A vector $v \in V$ is called *isotropic* if $(v, v) = 0$. For any nonisotropic vector $v \in V$ the *reflection* R_v is the orthogonal transformation of V that fixes every vector in the hyperplane orthogonal to v and takes v to $-v$. It can be computed to be given by

$$R_v x = x - 2\frac{(x, v)}{(v, v)} v\,, \qquad x \in V\,.$$

Note that $R_v^2 = 1$ and that the space of fixed points of R_v, being the hyperplane orthogonal to v, has dimension $n - 1$. Conversely, if $R \in O(V)$ is a *reflection*, i.e., such that $R^2 = 1$ and the subspace L of elements fixed by R has dimension $n - 1$, then $R = R_v$ for some nonisotropic v; we can then normalize v so that v is a unit vector, i.e., $(v, v) = 1$. To see this, let $x \in V$ be such that $v = Rx - x \neq 0$; then $Rv = -v$ while for $y \in L$ we have $(v, y) = (Rv, Ry) = -(v, y)$ so that $(v, y) = 0$, hence $v \perp L$; in particular, because $v \notin L$ we see that v is nonisotropic. Clearly $R = R_v$. A reflection has determinant -1. Cartan's theorem says that any element of $O(V)$ is a product of at most n reflections. The simplest example is when $V = k^2$ with the metric such that $(e_1, e_1) = (e_2, e_2) = 0$, $(e_1, e_2) = 1$. Then, for $v = e_1 + av_2$ where $a \neq 0$ the reflection R_v is given by

$$R_v = \begin{pmatrix} 0 & -a^{-1} \\ -a & 0 \end{pmatrix}$$

so that

$$T_c = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = R_v R_{v'}\,, \qquad v = e_1 + ae_2,\ v' = e_1 + ace_2\,.$$

However, in the general case T can be more complicated; for instance, it can be unipotent, and so one needs a more delicate argument. For the proof of Cartan's theorem, the following special case is essential. Here $V = k^4$ with basis

e_1, e_2, f_1, f_2 where

$$(e_i, e_j) = (f_i, f_j) = 0\,, \quad (e_i, f_j) = \delta_{ij}\,,$$

and

$$T = \begin{pmatrix} I_2 & J_2 \\ 0 & I_2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with I_2 as the unit 2×2 matrix. In this case let

$$R_1 = R_{e_2+f_2}\,, \quad R_2 = R_{e_2+cf_2}\,, \quad c \neq 0, 1\,.$$

A simple calculation shows that $T' = R_2 R_1 T$ has the following description, with $b = c(1 - c^{-1})$:

$$T' : e_1 \mapsto e_1\,, \quad e_2 - bf_1 \mapsto e_2 - bf_1\,, \quad e_1 + bf_2 \mapsto c(e_1 + bf_2)\,, \quad e_2 \mapsto c^{-1} e_2\,.$$

In other words, if L is the span of e_1 and $e_2 - bf_1$, we have $k^4 = L \oplus L^\perp$ and $L^\perp$ has the basis $\{e_1', e_2'\}$ where $e_1' = e_1 + bf_2$, $e_2' = b^{-1}e_2$, and T' is the identity on L and acts like $e_1' \mapsto ce_1'$, $e_2 \mapsto c^{-1}e_2'$ on $L^\perp$. Because $(e_1', e_1') = (e_2', e_2') = 0$ and $(e_1', e_2') = 1$, we find from the previous calculation that

$$T' = R_{e_1'+e_2'} R_{e_1'+ce_2'}\,.$$

This gives

$$T = R_{e_2+f_2} R_{e_2+cf_2} R_{e_1+bf_2+b^{-1}e_2} R_{e_1+bf_2+cb^{-1}e_2}\,,$$

a product of four reflections.

We now give Cartan's proof, which uses induction on n. If $T \in \mathrm{O}(V)$ fixes a nonisotropic vector, it leaves the orthogonal complement invariant and the result follows by induction; T is then a product of at most $n - 1$ reflections. Suppose that $x \in V$ is not isotropic and the vector $Tx - x$ is also not isotropic. Then the reflection R in the hyperplane orthogonal to $Tx - x$ will send x to Tx. This follows from the easily checked fact that the line segment joining x and Tx is bisected perpendicularly by the hyperplane orthogonal to $Tx - x$. So $RTx = x$ and since x is not isotropic, the argument just given applies and shows that RT is a product of at most $n - 1$ reflections, hence that R is a product of at most n reflections. However, it may happen that for all nonisotropic x, $Tx - x$ is isotropic. Then by continuity $Tx - x$ will be isotropic for all $x \in V$. We may also assume that T fixes no nonisotropic x. We shall now show that in this case $n = 4q$ and T is a direct sum of q transformations of the example in dimension 4 discussed above.

Let L be the image of V under $T - I$. Then L is an isotropic subspace of V and so $L \subset L^\perp$. We claim that $L = L^\perp$. If $x \in L^\perp$ and $y \in V$, then $Tx = x + \ell$ and $Ty = y + \ell'$ where $\ell, \ell' \in L$. Since $(Tx, Ty) = (x, y)$, we have $(x, \ell') + (y, \ell) + (\ell, \ell') = 0$. But $(x, \ell) = (\ell, \ell') = 0$ and so $(y, \ell) = 0$. Thus $\ell = 0$, showing that T is the identity on $L^\perp$. Since T cannot fix any nonisotropic vector, this means that $L^\perp$ is isotropic and so $L^\perp \subset L$, proving that $L = L^\perp$. Thus $n = 2p$ where p is the dimension of $L = L^\perp$. In this case we can find another isotropic subspace M such that $V = L \oplus M$ and $(\cdot, \cdot)$ is nonsingular on $L \times M$. This can seen by induction on p. Let $\{x_1, \ldots, x_p\}$ be a basis of L. The orthogonal complement M of $\{x_2, \ldots, x_p\}$ contains L and has dimension $p + 1$, and so has a vector $z \notin L$. Then $(z, x_1) \neq 0$, and we can replace z by $y_1 = z + tx_1$ for a suitable scalar t so that $(y_1, x_1) = 1$ and $(y_1, y_1) = 0$. If V' is the span of x_1, y_1, the form $(\cdot\,,\cdot)$ is nondegenerate on V' and so $V = V' \oplus V_1$ where $V_1 = V^{\perp}$. We have $\dim(V_1) = 2p-2$ and the span L_1 of $\{x_2, \dots, x_p\}$ is an isotropic subspace of V_1 of dimension $p-1$. So, by the induction hypothesis we can find an isotropic subspace M_1 of V_1 of dimension $p-1$ such that $(\cdot\,,\cdot)$ is nondegenerate on $L_1 \times M_1$. Then M, the span of y_1 and M_1, has the required properties. Since $Tm \equiv m \bmod L$ for $m \in M$, we see that with respect to the direct sum $V = L \oplus M$, T has the matrix

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}, \qquad B \in \mathrm{Hom}(M, L),$$

and the condition that $(Tx, Ty) = (x, y)$ for all $x, y \in V$ gives

$$(Bm, m') + (Bm', m) = 0, \qquad m, m' \in M.$$

We now claim that B is an isomorphism of M with L. Suppose that $Bm = 0$ for some nonzero $m \in M$. We choose $\ell \in L$ such that $(m, \ell) \neq 0$; then $m + \ell$ is not isotropic. Since $Bm = 0$ we have $T(m + \ell) = m + \ell$, a contradiction as T cannot fix any nonisotropic vector.

Thus B is an isomorphism of M with L. The nonsingularity of B implies that the skew-symmetric bilinear form

$$m, m' \longmapsto (Bm, m')$$

is nondegenerate and so we must have $p = 2q$ and there is a basis (m_i) of M such that $(Bm_i, m_j) = \delta_{j,q+i}$ $(1 \le i \le q)$. If (ℓ_i) is the dual basis in L, then the matrix of T in the basis (ℓ_i, m_j) is

$$\begin{pmatrix} I_{2q} & J_{2q} \\ 0 & I_{2q} \end{pmatrix}, \qquad J_{2q} = \begin{pmatrix} 0 & I_q \\ -I_q & o \end{pmatrix},$$

where I_r is the unit $r \times r$ matrix. Then $\dim(V) = 4q$ and T is a direct sum of q copies of the 4×4 matrix treated earlier as an example and the result for T follows immediately. This finishes the proof. We have thus proven the following:

THEOREM 5.2.1 *Let V be a vector space over $k = \mathbf{R}$ or $\mathbf{C}$ of dimension n equipped with $(\cdot\,,\cdot)$. Then any element of $\mathrm{O}(V)$ is a product of at most n reflections. An element of $\mathrm{O}(V)$ lies in $\mathrm{SO}(V)$ if and only if it is a product of an even number $2r \le n$ of reflections.*

Connected Components. We shall now determine the connected components of $\mathrm{O}(V)$. Since the determinant is ± 1 for elements of $\mathrm{O}(V)$ it is clear that the identity component is contained in $\mathrm{SO}(V)$. But $\mathrm{SO}(V)$ is not always connected. It is standard[12] that $\mathrm{SO}(V)$ is connected when V is complex or real and definite, and that in these cases the universal cover of $\mathrm{SO}(V)$ is a twofold cover; and further that if V is real and indefinite, $\mathrm{SO}(V)$ has two connected components. If V is Minkowskian the universal cover of $\mathrm{SO}(V)^0$ is again a double cover but in the other indefinite cases the fundamental group of $\mathrm{SO}(V)^0$ is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ so that the universal cover of $\mathrm{SO}(V)^0$ covers it four times. We want to obtain the result as a consequence of the above theorem of Cartan, as Cartan himself did.[9]

Let $V = \mathbf{R}^{p,q}$. We may assume that the quadratic form is

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 ,$$

$0 \le p \le q$, since $\mathrm{SO}(p, q) = \mathrm{SO}(q, p)$.

First assume that $p \ge 2$. Let $(e_i)_{1\le i\le p+q}$ be the standard basis for V. Let us call a nonisotropic vector u *timelike* if $(u, u) > 0$ and *spacelike* if $(u, u) < 0$. Clearly each of the sets of timelike and spacelike vectors is invariant under the orthogonal group. Let $V^{\pm}$ be the subspaces spanned by $(e_i)_{1\le i\le p}$ and $(e_i)_{p+1\le i\le p+q}$. The matrix of an element T of $\mathrm{SO}(V)$ is of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

corresponding to the direct sum $V = V^+ \oplus V^-$. We claim that $\det(A) \neq 0$. If not, there is a nonzero timelike vector u^+ such that Tu^+ is spacelike, a contradiction. So on any component of $\mathrm{SO}(V)$ the sign of $\det(T)$ is constant, and so we already have the parts $\mathrm{SO}(V)^{\pm}$ where this sign is greater than 0 or less than 0. Any element T of $\mathrm{SO}(V)$ can be written as $R_{v_1} \cdots R_{v_{2r}}$ where each v_j is either timelike or spacelike. But $R_v R_w = R_{R_v w} R_v$, and $R_v w$ is like w, and so we can arrange that in the product representation of T we have all the timelike and spacelike reflections together.

Any vector x with $(x, x) = 1$ can be written as $\cosh t\ u^+ + \sinh t\ u^-$ where $t \ge 0$ and $u^{\pm} \in V^{\pm}$ with $(u^{\pm}, u^{\pm}) = \pm 1$. It is clear that u^+ can be continuously joined to e_1, u^- similarly to e_{p+1}, and then changing t continuously to 0, we see that x can be continuously joined to e_1. Thus the timelike vectors form a connected domain. A similar argument shows that the spacelike vectors also form a connected domain and any spacelike vector can be joined by a continuous path to e_{p+1}. Since the map that takes a vector to the reflection in its orthogonal hyperplane is continuous, it follows that any element $T \in \mathrm{SO}(V)$ can be continuously joined to an element of the form $R^r_{e_1} R^r_{e_{p+1}}$ where r is 0 or 1. Clearly, $r = 0$ or 1 according as $T \in \mathrm{SO}(V)^{\pm}$ and the cases are distinguished by whether T is the product of an even or odd number, each of timelike and spacelike reflections. So we see that $\mathrm{SO}(V)^{\pm}$ are themselves connected and the identity component is $\mathrm{SO}(V)^+$, which is characterized as the set of T expressible as a product of an even number, each of timelike and spacelike reflections.

It remains to discuss the case when $p = 0, 1$. Assume that $q \ge 2$. The argument for the connectedness of the set of spacelike vectors remains valid, but for the timelike vectors (when $p = 1$) there are two connected components, depending on whether they can be connected to $\pm e_1$. For any timelike vector $x = \sum_i x_i e_i$ we have $x_1^2 - x_2^2 - \cdots - x_{q+1}^2 > 0$ and so $x_1^2 > 0$, so that the sign of x_1 is constant on any connected component. But $\pm e_1$ define the same reflection, and so the argument to determine the identity component of $\mathrm{SO}(V)$ remains valid. The cases for $q = 1$ are trivial.

If $k = \mathbf{C}$ and $V = \mathbf{C}^n$ with the quadratic form $z_1^2 + \cdots + z_n^2$ and standard basis (e_i), the set of nonisotropic vectors is connected; indeed, if u, v are two distinct nonisotropic vectors, $w = u + tv$ is nonisotropic for all but two values $t \in \mathbf{C}$, and so, such w form a connected set containing u and v. So any nonisotropic vector can be continuously joined to e_1, and the argument proceeds as before. We have thus obtained the following:

THEOREM 5.2.2 *The group* $\mathrm{SO}(n, \mathbf{C})$ *is connected. The group* $\mathrm{SO}(p, q)$ *is connected if and only if either* p *or* q *is* 0. *Otherwise it has two connected components, and the identity component consists of those elements that can be expressed as a product of an even number, each of the timelike and spacelike reflections.*

The case $p = 1$ deserves some additional remarks since it is the Minkowski signature and so plays an important role in physics. Denote the standard basis vectors as $e_0, e_1, \dots, e_q$ where $(e_0, e_0) = 1$ and $(e_j, e_j) = -1$ for $j = 1, \dots, q$. In this case the timelike vectors $x = x_0 e_0 + \sum_j x_j e_j$ are such that $x_0^2 > \sum_j x_j^2$ and hence the two components are those where $x_0 >$ or < 0. These are the forward and backward light cones. If x is a unit vector in the forward cone, we can use a rotation in the space V^- to move x to a vector of the form $x_0 e_0 + x_1 e_1$; then using hyperbolic rotations in the span of e_0, e_1, we can move it to e_0. Suppose now that x, x' are two unit vectors in the forward cone. We claim that $(x, x') > 1$ unless $x = x'$ (in which case $(x, x') = 1$). For this we may assume that $x = e_0$. Then $(x, x') = x_0' \geq 1$; if this is equal to 1, then $x_j' = 0$ for $j \geq 1$ and so $x' = e_0$. Thus

$$(*) \quad (x, x') > 1, \ = 1 \iff x = x', \quad (x, x) = (x', x') = 1, \ x_0, x_0' > 0.$$

We can now modify the argument of Theorem 5.2.1 to show that any $T \in \mathrm{O}(1, q)$ is a product of at most $n = q + 1$ spacelike reflections. This is by induction on q. Let $T \in \mathrm{O}(1, q)$ and suppose that x is a timelike unit vector. If $Tx = x$, then the orthogonal complement of x is a negative definite space of dimension $n - 1$, and since there are only spacelike reflections, we are through by Cartan's theorem. Otherwise $Tx = x'$ is a timelike vector distinct from x. Then

$$(x - x', x - x') = 2 - 2(x, x') < 0$$

by $(*)$ so that $x - x'$ is a spacelike vector. The reflection $R = R_{x-x'}$ is then spacelike and takes x to x'. Hence $T' = RT$ fixes x and we are in the previous case. Thus we have proven the following:

THEOREM 5.2.3 *If* $p = 1 \leq q$, *all elements of* $\mathrm{O}(1, q)$ *are products of at most* $n = q + 1$ *spacelike reflections, and they belong to* $\mathrm{SO}(1, q)^0$ *if and only if the number of reflections is even.*

5.3. Clifford Algebras and Their Representations

Tensors are objects functorially associated to a vector space. If V is a finite-dimensional vector space and

$$T^{r,s} = V^{*\otimes r} \otimes V^{\otimes s},$$

then the elements of $T^{r,s}$ are the tensors of rank (r, s). V is regarded as a module for $\mathrm{GL}(V)$, and then $T^{r,s}$ also becomes a module for $\mathrm{GL}(V)$. Spinors, on the other hand, are in a much more subtle relationship with the basic vector space. In the first place, the spinor space is attached only to a vector space with a metric. Let us define a *quadratic vector space* to be a pair (V, Q) where V is a finite-dimensional vector space over a field k of characteristic 0 and Q a nondegenerate quadratic form. Here a quadratic form is a function such that

$$Q(x) = \Phi(x, x)$$

where Φ is a symmetric bilinear form, with nondegeneracy of Q defined as the nondegeneracy of Φ. Thus

$$Q(x + y) = Q(x) + Q(y) + 2\Phi(x, y).$$

Notice that our convention, which is the usual one, differs from that of Deligne,[3] where he writes Φ for our 2Φ. A quadratic subspace of a quadratic vector space (V, Q) is a pair (W, Q_W) where W is a subspace of V and Q_W is the restriction of Q to W, with the assumption that Q_W is nondegenerate. For quadratic vector spaces (V, Q), (V', Q'), let us define the quadratic vector space $(V \oplus V', Q \oplus Q')$ by

$$(Q \oplus Q')(x + x') = Q(x) + Q(x'), \quad x \in V,\ x' \in V'.$$

Notice that V and V' are orthogonal in $V \oplus V'$. Thus for a quadratic subspace W of V, we have $V = W \oplus W^{\perp}$ as quadratic vector spaces. Given a quadratic vector space (V, Q) or V in brief, we have the orthogonal group $\mathrm{O}(V)$, the subgroup of $\mathrm{GL}(V)$ preserving Q, and its subgroup $\mathrm{SO}(V)$ of elements of determinant 1. If $k = \mathbf{C}$ and $\dim(V) \geq 3$, the group $\mathrm{SO}(V)$ is not simply connected, and $\mathrm{Spin}(V)$ is its universal cover, which is actually a double cover. The spinor spaces carry certain special irreducible representations of $\mathrm{Spin}(V)$. Thus, when the space V undergoes a transformation $\in \mathrm{SO}(V)$ and $g^{\sim}$ is an element of $\mathrm{Spin}(V)$ above g, the spinor space undergoes the transformation corresponding to $g^{\sim}$. The spinor space is, however, *not* functorially attached to V. Indeed, when (V, Q) varies, the spinor spaces do not vary in a natural manner unless additional assumptions are made (existence of spin structures). This is the principal difficulty in dealing with spinors globally on manifolds. However, we shall not treat global aspects of spinor fields on manifolds in this book.

The *Clifford algebra* $C(V, Q) = C(V)$ of the quadratic vector space (V, Q) is defined as the associative algebra generated by the vectors in V with the relations

$$v^2 = Q(v)1, \quad v \in V.$$

The definition clearly generalizes the Dirac definition (5.1) in dimension 4 and Clifford's in arbitrary dimension. The relations for the Clifford algebra are obviously equivalent to

$$xy + yx = 2\Phi(x, y)1, \quad x, y \in V.$$

Formally, let $T(V)$ be the tensor algebra over V, i.e.,

$$T(V) = \bigoplus_{r \geq 0} V^{\otimes r}$$

where $V^0 = k1$ and multiplication is $\otimes$. If

$$t_{x,y} = x \otimes y + y \otimes x - 2\Phi(x, y)1, \quad x, y \in V,$$

then

$$C(V) = T(V)/I$$

where I is the two-sided ideal generated by the elements $t_{x,y}$. If $(e_i)_{1\le i\le n}$ is a basis for V, then $C(V)$ is generated by the e_i and is the algebra with relations

$$e_ie_j + e_je_i = 2\Phi(e_i, e_j)\,, \quad i, j = 1, \dots, n.$$

The tensor algebra $T(V)$ is graded by $\mathbf{Z}$, but this grading does not descend to $C(V)$ because the generators $t_{x,y}$ are *not* homogeneous. However, if we consider the coarser $\mathbf{Z}_2$ grading of $T(V)$ where all elements spanned by tensors of even (odd) rank are regarded as even (odd), then the generators $t_{x,y}$ are even and so this grading descends to the Clifford algebra. Thus $C(V)$ is a *superalgebra*. The point of view of superalgebras may therefore be applied systematically to the Clifford algebras. Some of the more opaque features of classical treatments of Clifford algebras arise from an insistence on treating the Clifford algebra as an *ungraded* algebra. We shall see below that the natural map $V \longrightarrow C(V)$ is injective, and so we may (and shall) identify V with its image in $C(V)$: $V \subset C(V)$ and the elements of V are odd.

Since $C(V)$ is determined by Q, the subgroup of $\mathrm{GL}(V)$ preserving Q clearly acts on $C(V)$. This is the orthogonal group $\mathrm{O}(V)$ of the quadratic vector space V. For any element $g \in \mathrm{O}(V)$ the induced action on the tensor algebra T descends to an automorphism of $C(V)$.

The definition of the Clifford algebra is compatible with base change; if $k \subset k'$ and $V_{k'} := k' \otimes_k V$, then

$$C(V_{k'}) = C(V)_{k'} := k' \otimes_k C(V)\,.$$

Actually, the notions of quadratic vector spaces and Clifford algebras defined above may be extended to the case where k is any commutative ring with unit element in which 2 is invertible. The compatibility with base change remains valid in this general context. We shall, however, be concerned only with the case when k is a field of characteristic 0.

For later use we now introduce the *canonical* or *principal* antiautomorphism β of the *ungraded* Clifford algebra $C(V)$ that is the identity on V. Since β must take $x_1 \cdots x_r$ to $x_r \cdots x_1$ for $x_i \in V$, its uniqueness is clear. For the existence, observe that for any $r \ge 1$ there is a unique linear automorphism of $V^{\otimes r}$ that takes $x_1 \otimes \cdots \otimes x_r$ to $x_r \otimes \cdots \otimes x_1$ for $x_i \in V$; the direct sum of all these is the unique antiautomorphism of $T(V)$ that is the identity on V. Since the $t_{x,y}$ are fixed by β, it is clear that β descends to an antiautomorphism of the ungraded Clifford algebra $C(V)$.

By an ON basis for V we mean a basis (e_i) such that

$$\Phi(e_i, e_j) = \delta_{ij}\,.$$

If we only have the above for $i \ne j$, we speak of an orthogonal basis; in this case $Q(e_i) \ne 0$ and $e_ie_j + e_je_i = 2Q(e_i)\delta_{ij}$. For such a basis, if k is algebraically closed, there is always an ON basis. So in this case there is essentially only one Clifford algebra C_m for each dimension m. If k is not algebraically closed, there are many Clifford algebras. For instance, let $k = \mathbf{R}$. Then any quadratic vector space (V, Q) over $\mathbf{R}$ is isomorphic to $\mathbf{R}^{p,q}$ where p, q are integers ≥ 0, and $\mathbf{R}^{p,q}$

is the vector space $\mathbf{R}^{p+q}$ with the metric

$$Q(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 .$$

The numbers p, q are invariants of (V, Q), and we refer to either (p, q) or $p - q$ as the *signature* of V or Q. Thus, for $k = \mathbf{R}$, we have, as ungraded algebras,

$$C(\mathbf{R}^{0,1}) \simeq \mathbf{C}, \quad C(\mathbf{R}^{1,0}) \simeq \mathbf{R} \oplus \mathbf{R}, \quad C(\mathbf{R}^{0,2}) \simeq \mathbf{H}, \quad C(\mathbf{R}^{1,1}) \simeq M^2(\mathbf{R}),$$

where $\mathbf{H}$ is the algebra of quaternions and $M^2(\mathbf{R})$ is the 2×2 matrix algebra over $\mathbf{R}$.

Elementary Properties. Some of the elementary properties of Clifford algebras are as follows: For general k, $C(V)$ has dimension $2^{\dim(V)}$, and if $\dim(V) = n$, then the elements

$$1, e_I = e_{i_1} e_{i_2} \ldots e_{i_r}, \quad I = \{i_1, \ldots, i_r\},\ i_1 < \cdots < i_r,\ 1 \le r \le n,$$

form a basis for $C(V)$. If we change Q to $-Q$, we obtain $C(V)^{\mathrm{opp}}$, the algebra opposite to $C(V)$:

$$C(V, -Q) \simeq C(V)^{\mathrm{opp}}. \tag{5.3}$$

Notice here that we are speaking of opposite algebras in the super category. Let V, V' be quadratic vector spaces. We then have the important relation

$$C(V \oplus V') = C(V) \otimes C(V') \tag{5.4}$$

as superalgebras, *the tensor product being taken in the category of superalgebras.* We remark that this relation is *not true* if the tensor product algebra is the usual one in ungraded algebras; indeed, because V and V' are orthogonal, their elements *anticommute* in $C(V \oplus V')$, but in the ordinary tensor product they will have to commute. This is the first indication that it may be advantageous to treat the Clifford algebras as objects in the category of superalgebras.

PROOF OF (5.4): If A is an associative algebra with unit and (W, R) is a quadratic vector space, then in order that a linear map $L(W \longrightarrow A)$ extend to a map $C(W) \longrightarrow A$, it is necessary and sufficient that $L(w)^2 = R(w)1$ for all $w \in A$, and that for A a superalgebra, this is a map of superalgebras if $L(w)$ is odd for all $w \in W$. Let

$$(W, R) = (V, Q) \oplus (V', Q'), \quad A = C(V) \otimes C(V'),$$
$$L(v \oplus v') = v \otimes 1 + 1 \otimes v'.$$

Since v, v' are odd, $(1 \otimes v')(v \otimes 1) = -v \otimes v'$, and so we have

$$(v \otimes 1 + 1 \otimes v')^2 = R(v \oplus v')1$$

so that L extends to a map of $C(V \oplus V')$ into $C(V) \otimes C(V')$. To set up the inverse map, note that the inclusions $V, V' \subset V \oplus V'$ give even maps h, h' of $C(V), C(V') \longrightarrow C(V \oplus V')$, and hence a linear map $a \otimes a' \longmapsto h(a)h'(a')$ of $C(V) \otimes C(V')$ into $C(V \oplus V')$. Since h, h' preserve parity, this map will be a morphism of superalgebras if for $a, b \in C(V)$ and $a', b' \in C(V')$ we can show that

$$h(b)h'(a') = (-1)^{p(b)p(a')} h'(a')h(b).$$

This comes down to showing that for $v_i \in V$, $v'_j \in V'$ we have

$$v_1 \cdots v_r v'_1 \cdots v'_s = (-1)^{rs} v'_1 \cdots v'_s v_1 \cdots v_r$$

in $C(V \oplus V')$. This is obvious since, by definition, v_i and v'_j anticommute in $V \oplus V'$. It is trivial to check that the two maps thus constructed are inverses of each other; indeed, the compositions in either order are the identities at the level of the vectors and so are the identities everywhere. Thus (5.4) is proven. □

At this stage we can conclude that $C(V)$ has dimension 2^n where $n = \dim(V)$. In fact, if V has dimension 1 and v is nonzero in V with $Q(v) = a \neq 0$, then $C(V)$ is the span of 1 and v so that it has dimension 2; for arbitrary V of dimension n it follows from (4) that $C(V)$ has dimension 2^n. In particular, if $(e_i)_{1 \leq i \leq n}$ is a basis of V, then

$$1, e_I = e_{i_1} e_{i_2} \ldots e_{i_r}\,, \quad I = \{i_1, \ldots, i_r\},\ i_1 < \cdots < i_r,\ 1 \leq r \leq n\,,$$

form a basis for $C(V)$. This implies at once that the natural map $V \longrightarrow C(V)$ is injective, so that we shall assume from now on that $V \subset C(V)$.

PROOF OF (5.3): The identity map of V lifts to a morphism of $T(V)$ onto $C(V)^{\text{opp}}$ as superalgebras. We claim that this lift vanishes on the kernel of $T(V) \longrightarrow C(V^-)$ where we write V^- for $(V, -Q)$. It is enough to show that for $x \in V$, the image of $x \otimes x + Q(x)1$ in $C(V)^{\text{opp}}$ is 0. But this image is the element $-x^2 + Q(x)1$ in $C(V)$ and so is 0. Thus we have a surjective morphism $C(V^-) \longrightarrow C(V)^{\text{opp}}$. Since the dimensions are equal, this is an isomorphism. □

The Clifford Algebra and the Exterior Algebra. The Clifford algebra is filtered in a natural way because the tensor algebra that sits above it is filtered by the rank of tensors. Thus $C = C(V)$ acquires the filtration (C_r) where C_r is the span of elements of the form $v_1 \cdots v_s$ where $v_i \in V$ and $s \leq r$. Let C^{gr} be the associated graded algebra. Clearly, $C_1^{\text{gr}} = V$. If $v \in V$, then $v^2 \in C_0$, and so $v^2 = 0$ in C^{gr}. Hence we have a homomorphism of the exterior algebra $\Lambda(V)$ onto C^{gr} preserving degrees, which is an isomorphism because both spaces have dimension $2^{\dim(V)}$. Thus

$$C^{\text{gr}} \simeq \Lambda(V) \quad \text{(as graded algebras)}\,.$$

It is possible to construct a map going from the exterior algebra to the Clifford algebra inducing the above automorphism, namely, the so-called skew-symmetrizer map

$$\lambda : v_1 \wedge \cdots \wedge v_r \longmapsto \frac{1}{r!} \sum_{\sigma} \varepsilon(\sigma) v_{\sigma(1)} \cdots v_{\sigma(r)}$$

where the sum is over all permutations σ of $\{1, \ldots, r\}$, $\varepsilon(\sigma)$ is the sign of σ, and the elements on the right side are multiplied as elements of $C(V)$. Indeed, the right side above is skew-symmetric in the v_i, and so by the universality of the exterior power, the map λ is well-defined. If we choose a basis (e_i) of V such that the e_i

are mutually orthogonal, the elements $e_{i_1} \cdots e_{i_r}$ are clearly in the range of λ so that λ is surjective, showing that

$$\lambda : \Lambda(V) \simeq C(V)$$

is a linear isomorphism. If we follow λ by the map from C^r to C^{gr}, we obtain the isomorphism of $\Lambda(V)$ with C^{gr}, which is just the earlier isomorphism. The definition of λ makes it clear that it commutes with the action of $\mathrm{O}(V)$ on both sides.

Center and Supercenter. For any superalgebra A its *supercenter* $\mathrm{sctr}(V)$ is the subsuperalgebra whose homogeneous elements x are defined by

$$xy - (-1)^{p(x)p(y)}yx = 0\,, \quad y \in A\,.$$

This can be very different from the center $\mathrm{ctr}(V)$ of A regarded as an ungraded algebra. Notice that both $\mathrm{sctr}(V)$ and $\mathrm{ctr}(V)$ are themselves superalgebras.

PROPOSITION 5.3.1 *We have the following*:

(i) $\mathrm{sctr}(C(V)) = k1$.

(ii) $\mathrm{ctr}(C(V)) = k1$ *if* $\dim(V)$ *is even.*

(iii) *If* $\dim(V) = 2m+1$ *is odd, then* $\mathrm{ctr}(C(V))$ *is a superalgebra of dimension* $1|1$*; if* ε *is a nonzero odd element of it, then* $\varepsilon^2 = a \in k \setminus (0)$ *and* $\mathrm{ctr}(C(V)) = k[\varepsilon]$. *In particular, it is a superalgebra all of whose nonzero homogeneous elements are invertible and whose supercenter is* k. *If* $(e_i)_{0\le i\le 2m}$ *is an orthogonal basis for* V, *then we can take* $\varepsilon = e_0e_1\dots e_{2m}$. *If further* $e_i^2 = \pm 1$, *then* $\varepsilon^2 = (-1)^{m+q}1$ *where* q *is the number of* i*'s for which* $e_i^2 = -1$.

PROOF: Select an orthogonal basis $(e_i)_{1\le i\le n}$ for V. If $I \subset \{1,\dots,n\}$ is nonempty, then

$$e_I e_j = \alpha_{I,j} e_j e_I \quad \text{where } \alpha_{I,j} = \begin{cases} -(-1)^{|I|}, & j \in I, \\ (-1)^{|I|}, & j \notin I. \end{cases}$$

Let $x = \sum_I a_I e_I$ be a homogeneous element in the supercenter of $C(V)$ where the sum is over I with the parity of I being the same as $p(x)$. The above formulae and the relations $xe_j = (-1)^{p(x)}e_j x$ imply, since e_j is invertible,

$$(\alpha_{I,j} - (-1)^{p(x)})a_I = 0\,.$$

If we choose $j \in I$, then $\alpha_{I,j} = -(-1)^{p(x)}$, showing that $a_I = 0$. This proves (i). To prove (ii), let x above be in the center. We now have $xe_j = e_j x$ for all j. Then, as before,

$$(\alpha_{I,j} - 1)a_I = 0\,.$$

So $a_I = 0$ whenever we can find a j such that $\alpha_{I,j} = -1$. Thus $a_I = 0$ except when $\dim(V) = 2m+1$ is odd and $I = \{0,1,\dots,2m\}$. In this case $\varepsilon = e_0e_1\cdots e_{2m}$ commutes with all the e_j and so lies in $\mathrm{ctr}(V)$. Hence $\mathrm{ctr}(V) = k[\varepsilon]$. A simple calculation shows that

$$\varepsilon^2 = (-1)^m Q(e_0)\cdots Q(e_{2m})$$

from which the remaining assertions follow at once. □

REMARK. The center of $C(V)$ when V has odd dimension is an example of a super division algebra. A *super division algebra* is a superalgebra whose nonzero *homogeneous* elements are invertible. If $a \in k$ is nonzero, then $k[\varepsilon]$ with ε odd and $\varepsilon^2 = a1$ is a super division algebra since ε is invertible with inverse $a^{-1}\varepsilon$.

PROPOSITION 5.3.2 *Let* $\dim(V) = 2m + 1$ *be odd and let* $D = \mathrm{ctr}(V)$. *Then*

$$C(V) = C(V)^+ D \simeq C(V)^+ \otimes D$$

as superalgebras. Moreover, let $e_0 \in V$ *be such that* $Q(e_0) \neq 0$, $W = e_0^\perp$, *and* Q' *be the quadratic form* $-Q(e_0)Q_W$ *on* W *where* Q_W *is the restriction of* Q *to* W; *let* $W' = (W, Q')$. *Then*

$$C(V)^+ \simeq C(W')$$

as ungraded algebras.

PROOF: Let $(e_i)_{0 \le i \le 2m}$ be an orthogonal basis for V so that $e_1, \dots, e_{2m}$ is an orthogonal basis for W. Let $\varepsilon = e_0 \cdots e_{2m}$ so that $D = k[\varepsilon]$. In the proof r, s vary from 1 to $2m$. Write $f_r = e_0 e_r$. Then $f_r f_s = -Q(e_0)e_r e_s$ so that the f_r generate $C(V)^+$. If $\gamma_p \in C(V)^+$ is the product of the e_j $(j \neq p)$ in some order, $\gamma_p \varepsilon = c e_p$ where $c \neq 0$, and so D and $C(V)^+$ generate $C(V)$. By looking at dimensions we then have the first isomorphism. For the second, note that $f_r f_s + f_s f_r = 0$ when $r \neq s$ and $f_r^2 = -Q(e_0)Q(e_r)$, showing that the f_r generate the Clifford algebra over W'. □

Structure of Clifford Algebras over Algebraically Closed Fields. We shall now examine the structure of $C(V)$ and $C(V)^+$ when k is algebraically closed. Representations of $C(V)$ are morphisms into $\mathbf{End}(U)$ where U is a supervector space.

The Even-Dimensional Case. The basic result is the following:

THEOREM 5.3.3 *Let* k *be algebraically closed. If* $\dim(V) = 2m$ *is even,* $C(V)$ *is isomorphic to a full matrix superalgebra. More precisely,*

$$C(V) \simeq \mathbf{End}(S), \quad \dim(S) = 2^{m-1}|2^{m-1}.$$

This result is true even if k *is not algebraically closed provided* $(V, Q) \simeq (V_1, Q_1) \oplus (V_1, -Q_1)$.

This is a consequence of the following theorem.

THEOREM 5.3.4 *Suppose that* k *is arbitrary and* $V = U \oplus U^*$ *where* U *is a vector space with dual* U^*. *Let*

$$Q(u + u^*) = \langle u, u^* \rangle, \quad u \in U, \ u^* \in U^*.$$

Let $S = \Lambda U^*$ *be the exterior algebra over* U^*, *viewed as a superalgebra in the usual manner. Then* S *is a* $C(V)$*-module for the actions of* U *and* U^* *given by*

$$\mu(u^*) : \ell \longmapsto u^* \wedge \ell, \quad \partial(u) : \ell \longmapsto \partial(u)\ell, \quad \ell \in S,$$

where $\partial(u)$ is the odd derivation of S that is characterized by $\partial(u)(u^*) = \langle u, u^*\rangle$. Moreover, the map $C(V) \longrightarrow \mathbf{End}(S)$ defined by this representation is an isomorphism.

We shall first show that Theorem 5.3.4 $\Longrightarrow$ Theorem 5.3.3. If k is algebraically closed, we can find an ON basis $(e_j)_{1\le j\le 2m}$. If $f_r^{\pm} = 2^{-1/2}[e_r \pm i e_{m+r}]$ $(1 \le r \le m)$, then

$$(*) \qquad \Phi(f_r^{\pm}, f_s^{\pm}) = 0\,, \quad \Phi(f_r^{\pm}, f_s^{\mp}) = \delta_{rs}\,.$$

Let $U^{\pm}$ be the subspaces spanned by $(f_r^{\pm})$. We take $U = U^+$ and identify U^- with U^* in such a way that

$$\langle u^+, u^-\rangle = 2\Phi(u^+, u^-)\,, \quad u^{\pm} \in U^{\pm}\,.$$

Then

$$Q(u^+ + u^-) = \langle u^+, u^-\rangle$$

for $u^{\pm} \in U^{\pm}$, and we can apply Theorem 5.3.4. If k is not algebraically closed but $(V, Q) = (V_1, Q_1) \oplus (V_1, -Q_1)$, we can find a basis $(e_j)_{1\le j\le 2m}$ for V such that the e_j are mutually orthogonal, $(e_j)_{1\le j\le m}$ span $V_1 \oplus 0$ while $(e_{m+j})_{1\le j\le m}$ span $0 \oplus V_1$, and $Q(e_j) = -Q(e_{m+j}) = a_j \neq 0$. Let $f_r^+ = e_r + e_{m+r}$, $f_r^- = (2a_r)^{-1}(e_r - e_{m+r})$. Then the relations $(*)$ are again satisfied, and so the argument can be completed as before.

PROOF OF THEOREM 5.3.4: It is clear that $\mu(u^*)^2 = 0$. On the other hand, $\partial(u)^2$ is an even derivation that annihilates all u^* and so is 0 also. We regard S as $\mathbf{Z}_2$-graded in the obvious manner. It is a simple calculation that

$$\mu(u^*)\partial(u) + \partial(u)\mu(u^*) = \langle u, u^*\rangle 1\,, \quad u \in U,\ u^* \in U^*\,.$$

Indeed, for $g \in S$, by the derivation property, $\partial(u)\mu(u^*)g = \partial(u)(u^*g) = \langle u, u^*\rangle g - \mu(u^*)\partial(u)g$, which gives the above relation. This implies at once that

$$(\partial(u) + \mu(u^*))^2 = Q(u + u^*)1$$

showing that

$$r : u + u^* \longmapsto \partial(u) + \mu(u^*)$$

extends to a representation of $C(V)$ in S. Notice that the elements of V act as odd operators in S, and so r is a morphism of $C(V)$ into $\mathbf{End}(S)$.

We shall now prove that r is surjective as a morphism of ungraded algebras; this is enough to conclude that r is an isomorphism of superalgebras since $\dim(C(V)) = 2^{2\dim(U^*)} = \dim(\mathbf{End}(S))$ where all dimensions are of the ungraded vector spaces. Now, if A is an associative algebra of endomorphisms of a vector space acting irreducibly on it, and its commutant, namely, the algebra of endomorphisms commuting with A, is the algebra of scalars $k1$, then by Wedderburn's theorem, A is the algebra of all endomorphisms of the vector space in question. We shall now prove that r is irreducible and has scalar commutant. Let (u_i) be a basis of U and u_j^* the dual basis of U^*.

The proof of the irreducibility of r depends on the fact that if L is a nonzero subspace of S invariant under all $\partial(u)$, then $1 \in L$. If L is $k1$, this assertion in trivial; otherwise let $g \in L$ not be a scalar; then, replacing g by a suitable multiple of it, we can write

$$g = u_I^* + \sum_{J \neq I, |J| \leq |I|} a_J u_J^* , \quad I = \{i_1, \ldots, i_p\}, \ p \geq 1 .$$

Since

$$\partial(u_{i_p}) \cdots \partial(u_{i_1}) g = 1$$

we see that $1 \in L$. If now L is invariant under $C(V)$, applying the operators $\mu(u^*)$ to 1, we see that $L = S$. Thus S is irreducible. Let T be an endomorphism of S commuting with r. The proof that T is a scalar depends on the fact that the vector $1 \in S$, which is annihilated by all $\partial(u)$, is characterized (projectively) by this property. For this it suffices to show that if $g \in S$ has no constant term, then for some $u \in U$ we must have $\partial(u)g \neq 0$. If $g = \sum_{|I| \geq p} a_I u_I^*$ where $p \geq 1$ and some a_J with $|J| = p$ is nonzero, then $\partial(u_j)g \neq 0$ for $j \in J$. This said, since $\partial(u_i)T1 = T\partial(u_i)1 = 0$, we see that $T1 = c1$ for some $c \in k$. But then, because T commutes with all the $\mu(u^*)$, we have

$$Tu^* = T\mu(u^*)1 = \mu(u^*)T1 = cu^* , \quad u^* \in U^* .$$

Hence $T = cI$. This finishes the proof that r maps $C(V)$ onto $\mathbf{End}(S)$. □

REMARK 5.1. Write $V = U \oplus U^*$ as a direct sum of $V_i = U_i \oplus U_i^*$ $(i = 1, 2)$ where $\dim(U_i) \neq 0$. Then

$$C(V) \simeq C(V_1) \otimes C(V_2)$$

while an easy calculation shows that

$$r = r_1 \otimes r_2$$

where r_i is the representation of $C(V_i)$ defined above. Induction on m then reduces the surjectivity of r to the case when $\dim(U) = \dim(U^*) = 1$ where it is clear from an explicit calculation. The proof given here, although longer, reveals the structure of S in terms of the operators of multiplication and differentiation which are analogous to the creation and annihilation operators in Fock space. In fact, the analogy goes deeper and is discussed in the next remark.

REMARK 5.2. *The analogy with the Schrödinger representation.* There is an analogy of the Clifford algebra with the Heisenberg algebra that makes the representation r the *fermionic analogue to the Schrödinger representation.* If V is an even vector space with a symplectic form Φ, then the Heisenberg algebra $H(V)$ associated to (V, Φ) is the algebra generated by the commutation rules

$$\text{(H)} \qquad xy - yx = 2\Phi(x, y)1 , \quad x, y \in V .$$

For any symplectic Φ we can *always* write $V = U \oplus U^*$ with Φ vanishing on $U \times U$ and $U^* \times U^*$ and $2\Phi(u, u^*) = \langle u, u^* \rangle$. The algebraic representation of $H(V)$ is constructed on the *symmetric algebra* $\mathrm{Symm}(U^*)$ with u^* acting as the operator of multiplication by u^* and u acting as the (even) derivation $\partial(u)$. The splitting $V = U \oplus U^*$ is usually called a *polarization* of V. The commutation rule (H) is the bosonic analogue of the fermionic rule

$$\text{(C)} \qquad xy + yx = 2\Phi(x, y)1\,,$$

which defines the Clifford algebra. The analogy with the Clifford situation is now obvious. Unlike in the bosonic case, the polarization does not always exist in the fermionic case but will exist if k is algebraically closed. The vector 1 is called the *Clifford vacuum* by physicists. Notice that it is canonical *only after a polarization is chosen.* Indeed, there can be *no distinguished line in* S; otherwise S would be attached *functorially* to V and there would be no need to consider spin structures.

REMARK 5.3. For any field k the quadratic vector spaces of the form $(V_1, Q_1) \oplus (V_1, -Q_1)$ are called *hyperbolic*. When k is real, these are precisely the quadratic vector spaces $\mathbf{R}^{m,m}$ of signature 0.

From the fact that the Clifford algebra of an even-dimensional quadratic space is a full matrix superalgebra follows its simplicity. Recall the classical definition that an algebra is *simple* if it has no proper nonzero two-sided ideal. It is classical that full matrix algebras are simple. We have, from the theorems above, the following corollary.

COROLLARY 5.3.5 *For arbitrary* k, *if* V *is even-dimensional, then* $C(V)$ *is simple as an ungraded algebra.*

PROOF: $C(V)$ is simple if it stays simple when we pass to the algebraic closure $\bar{k}$ of k. So we may assume that k is algebraically closed. The result then follows from the fact that the ungraded Clifford algebra is a full matrix algebra. □

Modules for Clifford Algebras. Classically, the algebra $E(V)$ of all endomorphisms of a vector space V has the property that V is its only simple module and all its modules are direct sums of copies of V, so that any module is of the form $V \otimes W$ for W a vector space. We wish to extend this result to the superalgebra $\mathbf{End}(V)$ of any super vector space. In particular, such a result would give a description of all modules of a Clifford algebra $C(V)$ for V even dimensional and k algebraically closed.

We consider finite-dimensional modules of finite-dimensional superalgebras. Submodules are defined by invariant subsuper vector spaces. If A, B are superalgebras and V, W are modules for A, B, respectively, then $V \otimes W$ is a module for $A \otimes B$ by the action

$$a \otimes b : v \otimes w \longmapsto (-1)^{p(b)p(v)} av \otimes bw\,.$$

In particular, if $B = k$, $V \otimes W$ is a module for A where A acts only on the first factor. Imitating the classical case we shall say that a superalgebra A is *semisimple* if all its modules are completely reducible, i.e., direct sums of simple modules. Here, by a simple module for a superalgebra we mean an irreducible module, namely, one with no nontrivial proper submodule. If a module for A is a sum of simple modules, it is then a *direct sum* of simple modules; indeed, if $V = \sum_j V_j$ where the V_j are simple submodules, and (U_i) is a maximal subfamily of linearly independent members of the family (V_j), and if $U = \bigoplus U_i \neq V$, then for some j, we must have $V_j \not\subset U$, so that, by the simplicity of V_j, $V_j \cap U = 0$, contradicting the maximality of (U_i). In particular, a quotient of a direct sum of simple modules is a direct sum of simple modules. Now any module is a sum of cyclic modules generated by homogeneous elements, and a cyclic module is a quotient of the module defined by the left regular representation. Hence A is semisimple if and only if the left regular representation of A is completely reducible, and then any module is a direct sum of simple modules that occur in the decomposition of the left regular representation.

It is usual to call a supermodule M for a super algebra A semisimple if M is a direct sum of simple modules. Exactly as in the classical situation, M is semisimple if and only if it has the property that for any subsupermodule $N \subset M$, there is a subsupermodule N' complementary to N, i.e., $M = N \oplus N'$. In one direction, if M has this property and we start with a subsupermodule M_1 which is minimal, hence simple, and if M_1' is a complement, induction on the dimension of M shows that M_1', hence M itself, is a direct sum of simple modules. In the other direction let $M = \oplus_i M_i$ where the M_i are simple, and let N be a subsupermodule. Take a maximal subsupermodule N' such that $N \cap N' = 0$. If $N \oplus N' \neq M$, then some M_i is not contained in $N \oplus N'$ and so $M_i \cap (N \oplus N') = 0$. But then $N \cap (N' \oplus M_i) = 0$, contradicting the maximality of N'. Hence $M = N \oplus N'$. This result also shows that any submodule N of a semisimple module M is also semisimple; for, if we write $M = N \oplus N'$, then $N \simeq M/N'$ is semisimple by an earlier observation.

With an eye toward later use, let us discuss some basic facts relating semisimplicity and base change. The basic fact is that if A is a superalgebra, M is a module for A, and k'/k is a Galois extension (possibly of infinite degree), then M is semisimple for A if and only if $M' := k' \otimes_k M$ is semisimple for $A' := k' \otimes_k A$. This is proven exactly as in the classical case. In physics we need this only when $k = \mathbf{R}$ and $k' = \mathbf{C}$. For the sake of completeness we sketch the argument. Let $G = \mathrm{Gal}(k'/k)$. Then elements of G operate in the usual manner ($c \otimes m \mapsto c^g \otimes m$) on M' and the action preserves parity. To prove that the semisimplicity of M implies that of M', we may assume that M is simple. If $L' \subset M'$ is a simple submodule for A', then $\sum_{g \in G} L'^g$ is G-invariant and so is of the form $k' \otimes_k L$ where $L \subset M$ is a submodule. So $L = M$, showing that M' is semisimple, being a span of the simple modules L'^g. In the reverse direction it is a question of showing that if $L_1' \subset M'$ is a G-invariant submodule, there exists a G-invariant complementary submodule L_2'. It is enough to find an even map $f \in \mathbf{End}_{k'}(M')$ commuting with A and G such that

$$(*) \qquad f(M') \subset L_1', \quad f(\ell') = \ell', \quad \text{for all } \ell' \in L_1'.$$

We can then take L_2' to be the kernel of f.

By the semisimplicity of M' we can find even f_1 satisfying $(*)$ and commuting with A; indeed, if L_2'' is a complementary submodule to L_1', we can take f_1 to be the projection $M \longrightarrow L_1' \pmod{L_2''}$. Now f_1 is *defined over a finite Galois extension*

k''/k and so if $H = \mathrm{Gal}(k''/k)$ and

$$f = \frac{1}{|H|} \sum_{h \in H} h f_1 h^{-1} ,$$

then f commutes with A and H and satisfies $(*)$. But, if $g \in G$ and h is the restriction of g to k'', then $gfg^{-1} = hfh^{-1} = f$, and so we are done. In particular, applying this result to the left regular representation of A, we see that A is semisimple if and only if A' is semisimple.

It is also useful to make the following remark. Let A be a superalgebra and S a module for A. Suppose M is a direct sum of copies of S. Then $M \simeq S \otimes W$ where W is a purely even vector space. To see this, write $M = \bigoplus_{1 \le i \le r} M_i$ where $t_i : S \longrightarrow M_i$ is an isomorphism. Let W be a purely even vector space of dimension r with basis $(w_i)_{1 \le i \le r}$. Then the map

$$t : \sum_{1 \le i \le r} u_i \otimes w_i \longmapsto \sum_{1 \le i \le r} t_i(u_i)$$

is an isomorphism of $S \otimes W$ with M.

For any super vector space V, recall that ΠV is the super vector space with the same underlying vector space but with reversed parities, i.e., $(\Pi V)_0 = V_1$, $(\Pi V)_1 = V_0$. If V is a module for a superalgebra A, so is ΠV. If V is simple, so is ΠV. Notice that the identity map $V \longrightarrow \Pi V$ is *not* a morphism in the super category since it is *parity reversing*. One can also view ΠV as $V \otimes k^{0|1}$. Let

$$E(V) = \mathbf{End}(V)$$

for any super vector space V. If $\dim V_i > 0$ $(i = 0, 1)$, then $E(V)^+$, the even part of $E(V)$, is isomorphic to the algebra of all endomorphisms of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

and so is isomorphic to $E(V_0) \oplus E(V_1)$, and its center is isomorphic to $k \oplus k$. In particular, the center of $E(V)^+$ has two characters χ_i $(i = 0, 1)$ where the notation is such that χ_i is $(c_1, c_2) \longmapsto c_i$. So on V the center of $E(V)^+$ acts through (χ_1, χ_2) while on ΠV it acts through (χ_2, χ_1).

PROPOSITION 5.3.6 *For k arbitrary the superalgebra $E(V)$ has precisely two simple modules, namely, V and ΠV. Every module for $E(V)$ is a direct sum of copies of either V or ΠV. In particular, $E(V)$ is semisimple and any module for $E(V)$ is of the form $V \otimes W$ where W is a super vector space.*

PROOF: The ungraded algebra $E(V)$ is a full matrix algebra and it is classical that it is simple, V is its only simple module up to isomorphism, and any module is a direct sum of copies of V. The proposition extends these results to the super case where the same results are true except that we have to allow for parity reversal.

Let W be a simple module for $E(V)$. Since $E(V)$ is simple as an ungraded algebra, W is faithful, i.e., the kernel of $E(V)$ acting on W is 0. We first show that W is simple for $E(V)$ regarded as an ungraded algebra. To prove this, we need a preliminary observation. Let $I_j (j = 0, 1)$ be the element of $E(V)_0$ that is the identity on V_j and 0 on V_{1-j}. Then $I_j^2 = I_j$ and I_j lies in the center of $E(V)_0$. We claim that each I_j acts as 0 on one of W_0, W_1 and as the identity on the other. Fix i, j and suppose that $W_i = W_i(0) \oplus W_i(1)$ is the decomposition of W_i corresponding to the eigenvalues 0, 1 of I_j on W_i, with $W_i(0)$, $W_i(1)$ both being nonzero. Choose $w_r \in W_i(r)$, $w_r \neq 0$. Then $E(V)w_r$ is a graded nonzero submodule of W and so must be all of W. But then $E(V)_0 w_r = W_i$. On the other hand, because I_j lies in the center of $E(V)_0$, $E(V)_0$ leaves both $W_i(0)$ and $W_i(1)$ invariant and so $E(V)_0 w_r \subset W_i(r) \neq W_i$, a contradiction. Now $I_0 + I_1 = I$, the identity of $E(V)$, which acts as the identity on W, while neither I_0 nor I_1 can be zero on all of W since the action of $E(V)$ on W is faithful. Hence each I_j acts as 0 on one of W_0, W_1 and as the identity on the other.

This said, let U be a subspace stable under the ungraded $E(V)$. If $u = u_0 + u_1 \in U$ with $u_i \in W_i$, we have

$$I_0 u = \begin{cases} u_0 & \text{if } I_0 \text{ is 0 on } W_1 \\ u_1 & \text{if } I_0 \text{ is 0 on } W_1 . \end{cases}$$

Hence u_0 and u_1 both belong to U. Hence U has to be graded and so $U = 0$ or V. Hence we have an isomorphism $t(W \longrightarrow V)$ as ungraded modules for the ungraded $E(V)$. Write $t = t_0 + t_1$ where $p(t_i) = i$. Then $t_0 a + t_1 a = a t_0 + a t_1$ for all $a \in E(V)$. Because $p(at_0) = p(t_0 a) = p(a)$ and $p(at_1) = p(t_1 a) = 1 + p(a)$ we see that $at_0 = t_0 a$ and $at_1 = t_1 a$. If $t_0 \neq 0$, then t_0 is a nonzero element of $\mathrm{Hom}_{E(V)}(W, V)$ as supermodules and so, by the simplicity of V and W, we may conclude that t_0 is an isomorphism. Thus $W \simeq V$. If $t_1 \neq 0$, then $t_1 \in \mathrm{Hom}(W, \Pi V)$ and we argue as before that $W \simeq \Pi V$. We have thus proven that a simple $E(V)$-module is isomorphic to either V or ΠV.

It now remains to prove that an arbitrary module for $E(V)$ is a direct sum of simple modules. As we have already observed, it is enough to do this for the left regular representation. Now there is an isomorphism

$$V \otimes V^* \simeq E(V), \quad v \otimes v^* \longmapsto R_{v,v^*} : w \longmapsto v^*(w)v ,$$

of super vector spaces. If $L \in E(V)$, it is trivial to verify that $R_{Lv,v^*} = LR_{v,v^*}$, and so the above isomorphism takes $L \otimes 1$ to left multiplication by L in $E(V)$. Thus it is a question of decomposing $V \otimes V^*$ as an $E(V)$-module for the action $L \longmapsto L \otimes 1$. Clearly, $V \otimes V^* = \oplus_{e^*} V \otimes ke^*$ where e^* runs through a homogeneous basis for V^*. The map $v \longmapsto v \otimes e^*$ is an isomorphism of the action of $E(V)$ on V with the action of $E(V)$ on $V \otimes ke^*$. But this map is even for e^* even and odd for e^* odd. So the action of $E(V)$ on $V \otimes ke^*$ is isomorphic to V for even e^* and to ΠV for odd e^*. Hence the left regular representation of $E(V)$ is a direct sum of r copies of V and s copies of ΠV if $\dim(V) = r|s$. The direct sum of r copies of V is isomorphic to $V \otimes W_0$ where W_0 is purely even of dimension r. Since $\Pi V \simeq V \otimes k^{0|1}$ the direct sum of s copies of ΠV is isomorphic to $V \otimes W_1$ where W_1 is a purely odd vector space of dimension s. Hence the left regular representation is isomorphic to $V \otimes W$ where $W = W_0 \oplus W_1$. □

THEOREM 5.3.7 *Let V be an even-dimensional quadratic vector space. Then the Clifford algebra $C(V)$ is semisimple. Assume that either k is algebraically closed* *or k is arbitrary but V is hyperbolic. Then $C(V) \simeq$ **End**(S), $C(V)$ has exactly two simple modules S, ΠS, and any module for $C(V)$ is isomorphic to $S \otimes W$ where W is a super vector space.*

PROOF: By Theorem 5.3.3 we know that $C(V)$ is isomorphic to **End**(S). The result is now immediate from the proposition above. □

The Odd-Dimensional Case. We shall now extend the above results to the case when V has odd dimension. Let D be the super division algebra $k[\varepsilon]$ where ε is odd and $\varepsilon^2 = 1$. We first rewrite Proposition 5.3.2 as follows:

THEOREM 5.3.8 *Let* $\dim(V) = 2m + 1$ *and let k be algebraically closed. Then* $\mathrm{ctr}(C(V)) \simeq D$ *and, for some purely even vector space S_0 of dimension 2^m,*

$$C(V) \simeq C(V)^+ \otimes D\,, \quad C(V)^+ \simeq \mathrm{End}(S_0)\,, \quad \dim(S_0) = 2^m\,.$$

PROOF: If $(e_i)_{0 \le i \le 2m}$ is an ON basis for V and $\varepsilon = i^m e_0 e_1 \cdots e_{2m}$ where $i = (-1)^{1/2}$, then ε is odd, $\varepsilon^2 = 1$, and $\mathrm{ctr}(C(V)) = D = k[\varepsilon]$, by Proposition 5.3.1. The theorem is now immediate from Proposition 5.3.2 since $C(V)^+$ is isomorphic to the ungraded Clifford algebra in even-dimension $2m$, and so is a full matrix algebra in dimension 2^m. □

Let k be arbitrary and let U be an even vector space of dimension r. Write $E(U) = \mathrm{End}(U)$. Let A be the superalgebra $E(U) \otimes D$ so that the even part A^+ of A is isomorphic to $E(U)$. We construct a simple (super) module S for A as follows: $S = U \oplus U$ where $S_0 = U \oplus 0$ and $S_1 = 0 \oplus U$. $E(U)$ acts diagonally and ε goes to the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. It is obvious that S is simple. Notice that S is not simple for the ungraded algebra underlying A since the diagonal (as well as the antidiagonal) are stable under A. S can be written as $U \otimes k^{1|1}$ where A^+ acts on the first factor and D on the second with ε acting on $k^{1|1}$ by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The action of D on $k^{1|1}$ is also isomorphic to the left regular representation of D on itself.

PROPOSITION 5.3.9 *Let k be arbitrary. Then S is the unique simple module for $A = E(U) \otimes D$ where U is a purely even vector space over k. Any simple module for A is a direct sum of copies of S, and so is isomorphic to $S \otimes W$ where W is a purely even vector space.*

From this is we get the following theorem.

THEOREM 5.3.10 *If V is an odd-dimensional quadratic vector space, then $C(V)$ is semisimple. For k algebraically closed, $C(V) \simeq$* $\mathrm{End}(S_0) \otimes D$ *has a unique simple module $S = S_0 \otimes D$ up to isomorphism, and any module of $C(V)$ is isomorphic to $S \otimes W$ where W is a purely even vector space.*

PROOF: Theorem 5.3.10 follows from Proposition 5.3.9 and Theorem 5.3.8. It is therefore enough to prove Proposition 5.3.9. □

Let T be a simple module for A. Because $A^+ \simeq E(U)$, we have $T_0 \simeq aU$, $T_1 \simeq bU$ as A^+-modules for suitable integers $a, b \ge 0$. But the action of ε commutes with that of A^+ and $\varepsilon^2 = 1$, so that $\varepsilon(T_0 \longrightarrow T_1)$ is an isomorphism of A^+-modules. Hence we must have $a = b \ge 1$. But if R is a submodule of T_0,

$R \oplus \varepsilon R$ is stable for A and so it has to equal T. Thus $a = b = 1$, showing that we can take $T_0 = T_1 = U$ and ε as $(x, y) \longmapsto (y, x)$. But then $T = S$. To prove that any module for A is a direct sum of copies of S, it is enough (as we have seen already) to do this for the left regular representation. If $A^+ = \bigoplus_{1 \le j \le r} A_j$ where A_j as a left A^+-module is isomorphic to U, it is clear that $A_j \otimes D$ is isomorphic to $U \otimes k^{1|1} \simeq S$ as a module for A, and $A = \oplus_j (A_j \otimes D)$.

Representations of $C(V)^+$. We now obtain the representation theory of $C(V)^+$ over algebraically closed fields. Since this is an ungraded algebra, the theory is classical and not super.

THEOREM 5.3.11 *For any k, $C(V)^+$ is semisimple. Let k be algebraically closed. If $\dim(V) = 2m + 1$, $C(V)^+ \simeq \mathrm{End}(S_0)$ where S_0 is a purely even vector space of dimension 2^m, and so $C(V)^+$ has a unique simple module S_0. Let $\dim(V) = 2m$, let $C(V) \simeq \mathbf{End}(S)$ where $\dim(S) = 2^{m-1}|2^{m-1}$, and define $S^\pm$ to be the even and odd subspaces of S; then $C(V)^+ \simeq \mathrm{End}(S^+) \oplus \mathrm{End}(S^-)$. It has exactly two simple modules, namely, $S^\pm$, with $\mathrm{End}(S^\pm)$ acting as 0 on $S^\mp$, its center is isomorphic to $k \oplus k$, and every module is isomorphic to a direct sum of copies of $S^\pm$.*

The proof of the theorem is clear.

Center of the Even Part of the Clifford Algebra of an Even-Dimensional Quadratic Space. For later use we shall describe the center of $C(V)^+$ when V is of even dimension D and k arbitrary. Let $(e_i)_{1 \le i \le D}$ be an orthogonal basis. Let

$$e_{D+1} = e_1 \cdots e_D .$$

We have

$$e_{D+1} e_i = -e_i e_{D+1} .$$

Then

$$\mathrm{ctr}(C(V)^+) = k \oplus k e_{D+1} .$$

Moreover, if the e_i are orthonormal, then

$$e_{D+1}^2 = (-1)^{D/2} .$$

It is in fact enough to verify the description of the center over $\bar{k}$, and so we may assume that k is algebraically closed. We may then replace each e_i by a suitable multiple so that the basis becomes orthonormal. Since e_{D+1} anticommutes with all e_i, it commutes with all $e_i e_j$ and hence lies in the center of $C(V)^+$. If $a = \sum_{|I| \,\mathrm{even}} a_I e_I$ lies in the center of $C(V)^+$ and $0 < |I| < D$, then writing $I = \{i_1, \ldots, i_{2r}\}$, we use the fact that e_I anticommutes with $e_{i_1} e_s$ if $s \notin I$ to conclude that $a_I = 0$. Thus $a \in k1 \oplus e_{D+1}$.

5.4. Spin Groups and Spin Representations

In this section we shall define and study the spin groups and the spin representations associated to real and complex quadratic vector spaces V. We first treat the case when $k = \mathbf{C}$ and then the case when $k = \mathbf{R}$.

Summary. The spin group $\mathrm{Spin}(V)$ for a *complex* V is defined as the universal cover of $\mathrm{SO}(V)$ if $\dim(V) \geq 3$. Because the fundamental group of $\mathrm{SO}(V)$ is $\mathbf{Z}_2$ when $\dim(V) \geq 3$, it follows that in this case $\mathrm{Spin}(V)$ is a double cover of $\mathrm{SO}(V)$. If $\dim(V) = 1$, it is defined as $\mathbf{Z}_2$. For $\dim(V) = 2$, if we take a basis $\{x, y\}$ such that $\Phi(x,x) = \Phi(y,y) = 0$ and $\Phi(x,y) = 1$, then $\mathrm{SO}(V)$ is easily seen to be isomorphic to $\mathbf{C}^\times = \mathbf{C} \setminus (0)$ through the map

$$t \longmapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

The fundamental group of $\mathbf{C}^\times$ is $\mathbf{Z}$, and so $\mathrm{SO}(V)$ in this case has a unique double cover, which is defined as $\mathrm{Spin}(V)$. For any V we put $C = C(V)$ for the Clifford algebra of V and $C^+ = C(V)^+$ its even part. We shall obtain for all V a natural imbedding of $\mathrm{Spin}(V)$ inside C^+ as a complex algebraic group which lies as a double cover of $\mathrm{SO}(V)$; this double cover is unique inside C^+ if $\dim(V) \geq 3$. So modules for C^+ may be viewed by restriction as modules for $\mathrm{Spin}(V)$. The key property of the imbedding is that the restriction map gives a bijection between simple C^+-modules and *certain* irreducible $\mathrm{Spin}(V)$-modules. *These are precisely the spin and semi-spin representations.* Thus the spin modules are the irreducible modules for C^+, or, as we shall call them, *Clifford modules*, viewed as $\mathrm{Spin}(V)$-modules through the imbedding $\mathrm{Spin}(V) \hookrightarrow C(V)^+$. The algebra C^+ is semisimple, and so the restriction of any module for it to $\mathrm{Spin}(V)$ is a direct sum of spin modules. These are called *spinorial modules* of $\mathrm{Spin}(V)$.

Suppose now that V is a *real* quadratic vector space. If $V = \mathbf{R}^{p,q}$, we denote $\mathrm{SO}(V)$ by $\mathrm{SO}(p,q)$; this group does not change if p and q are interchanged, and so we may assume that $0 \leq p \leq q$. If $p = 0$ then $\mathrm{SO}(p,q)$ is connected; if $p \geq 1$, it has two connected components as we saw in Section 5.2. As usual, we denote the identity component of any topological group H by H^0. Let $V_{\mathbf{C}}$ be the complexification of V. Then the algebraic group $\mathrm{Spin}(V_{\mathbf{C}})$ is defined over $\mathbf{R}$, and so it makes sense to speak of the group of its real points. This is by definition $\mathrm{Spin}(V)$, and we have an exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V)^0 \longrightarrow 1\,.$$

If $\dim(V) = 1$, $\mathrm{Spin}(V) = \{\pm 1\}$. If V has signature $(1,1)$, then $\mathrm{Spin}(V)$ has two connected components. In all other cases it is connected and forms a double cover of $\mathrm{SO}(V)^0$. If $\dim(V) \geq 2$ is of signature $(p,q) \neq (1,1)$, then for $\min(p,q) \leq 1$, $\mathrm{SO}(p,q)$ has $\mathbf{Z}_2$ or $\mathbf{Z}$ as its fundamental group, and so has a unique double cover; $\mathrm{Spin}(V)$ is that double cover. If p, q are both ≥ 2, then $\mathrm{Spin}(p,q)$ is characterized as the unique double cover that induces a double cover of both $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$. Finally, if $\dim(V) \geq 3$, $\mathrm{Spin}(V)$ is the universal cover of $\mathrm{SO}(V)^0$ if and only if $\min(p,q) \leq 1$.

The relationship between the spin modules and modules for C^+ persists in the real case. The spinorial modules are the restriction to $\mathrm{Spin}(V)$ of C^+-modules. One can also describe them as modules of $\mathrm{Spin}(V)$ that are direct sums of the complex spin modules when we complexify.

Spin Groups in the Complex Case. Let V be a *complex* quadratic vector space. We write $C = C(V)$, $C^+ = C(V)^+$, the even part of $C(V)$. A motivation for expecting an imbedding of the spin group inside $C^\times$ may be given as follows: If $g \in \mathrm{O}(V)$, then g lifts to an automorphism of C that preserves parity. If V has even dimension, $C = \mathbf{End}(S)$, and so this automorphism is induced by an invertible homogeneous element $a(g)$ of $C = \mathbf{End}(S)$, uniquely determined up to a scalar multiple. It turns out that this element is even or odd according as $\det(g) = \pm 1$. Hence we have a projective representation of $\mathrm{SO}(V)$ that can be lifted to an ordinary representation of $\mathrm{Spin}(V)$ (at least when $\dim(V) \geq 3$) and hence to a map of $\mathrm{Spin}(V)$ into $C^{+\times}$. It turns out that this map is an imbedding, and further that such an imbedding can be constructed when the dimension of V is odd as well. Infinitesimally this means that there will be an imbedding of $\mathfrak{so}(V)$ inside C_L^+ where C_L^+ is the Lie algebra whose elements are those in C^+ with bracket $[a, b] = ab - ba$. We shall first construct this Lie algebra imbedding and then exponentiate it to get the imbedding $\mathrm{Spin}(V) \hookrightarrow C^{+\times}$.

To begin with we work over k, which can be either **R** or **C**. It is thus natural to introduce the *even Clifford group* Γ^+ defined by

$$\Gamma^+ = \{u \in C^{+\times} \mid uVu^{-1} \subset V\}$$

where $C^{+\times}$ is the group of invertible elements of C^+. Γ^+ is a closed (real or complex) Lie subgroup of $C^{+\times}$. For each $u \in \Gamma^+$ we have an action

$$\alpha(u) : v \longmapsto uvu^{-1}, \quad v \in V,$$

on V. Since

$$Q(uvu^{-1})1 = (uvu^{-1})^2 = uv^2u^{-1} = Q(v)1,$$

we have

$$\alpha : \Gamma^+ \longrightarrow \mathrm{O}(V)$$

with kernel as the centralizer in $C^{+\times}$ of C, i.e., $k^\times$.

If A is any finite-dimensional associative algebra over k, the Lie algebra of $A^\times$ is A_L where A_L is the Lie algebra whose underlying vector space is A with the bracket defined by $[a, b]_L = ab - ba$ $(a, b \in A)$. Moreover, the exponential map from A_L into $A^\times$ is given by the usual exponential series,

$$\exp(a) = e^a = \sum_{n \geq 0} \frac{a^n}{n!}, \quad a \in A.$$

Taking $A = C^+$, we see that the Lie algebra of $C^{+\times}$ is C_L^+. Thus, Lie (Γ^+), the Lie algebra of Γ^+, is given by

$$\mathrm{Lie}(\Gamma^+) = \{u \in C^+ \mid uv - vu \in V \text{ for all } v \in V\}.$$

For the map α from Γ^+ into $O(V)$, the differential $d\alpha$ is given by

$$d\alpha(u)(v) = uv - vu, \quad u \in \mathrm{Lie}(\Gamma^+),\ v \in V.$$

Clearly, $d\alpha$ maps $\mathrm{Lie}(\Gamma^+)$ into $\mathfrak{so}(V)$ with kernel as the centralizer in C^+ of C, i.e., k.

We now claim that $d\alpha$ is surjective. To prove this, it is convenient to recall that the orthogonal Lie algebra is the span of the momenta in its 2-planes. First let $k = \mathbf{C}$. Then there is an ON basis (e_i), the elements of the orthogonal Lie algebra are precisely the skew-symmetric matrices, and the matrices

$$M_{e_i,e_j} := E_{ij} - E_{ji}, \quad i < j,$$

where E_{ij} are the usual matrix units, form a basis for $\mathfrak{so}(V)$. The M_{e_i,e_j} are the infinitesimal generators of the group of rotations in the (e_i, e_j)-plane with the matrices

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Now a simple calculation shows that

$$M_{e_i,e_j} v = \Phi(e_j, v)e_i - \Phi(e_i, v)e_j .$$

So, if we define, for any two $x, y \in V$,

$$M_{x,y} v = \Phi(y, v)x - \Phi(x, v)y, \quad v \in V,$$

then $M_{x,y}$ is bilinear in x and y, and the $M_{x,y} \in \mathfrak{so}(V)$ for all $x, y \in V$ and span it. The definition of $M_{x,y}$ makes sense for $k = \mathbf{R}$ also, and it is clear (by going over to $\mathbf{C}$) that the $M_{x,y}$ span $\mathfrak{so}(V)$ in this case as well. Because the $M_{x,y}$ are bilinear and skew-symmetric in x and y, we see that there is a unique linear isomorphism of $\Lambda^2(V)$ with $\mathfrak{so}(V)$ that maps $x \wedge y$ to $M_{x,y}$:

$$\Lambda^2(V) \simeq \mathfrak{so}(V), \quad x \wedge y \longmapsto M_{x,y} .$$

For $x, y \in V$, a simple calculation shows that

$$d\alpha(xy)(v) = xyv - vxy = 2M_{x,y}v \in V, \quad v \in V .$$

Hence $xy \in \mathrm{Lie}(\Gamma^+)$ and $d\alpha(xy) = 2M_{x,y}$. The surjectivity of $d\alpha$ is now clear. Note that xy is the infinitesimal generator of the one-parameter group $\exp(txy)$, which must lie in Γ^+ since $xy \in \mathrm{Lie}(\Gamma^+)$. We have an exact sequence of Lie algebras

$$0 \longrightarrow k \longrightarrow \mathrm{Lie}(\Gamma^+) \xrightarrow{d\alpha} \mathfrak{so}(V) \longrightarrow 0 \tag{5.5}$$

where k is contained in the center of Lie (Γ^+). We now recall the following standard result from the theory of semisimple Lie algebras.

LEMMA 5.4.1 *Let $\mathfrak{g}$ be a Lie algebra over k, $\mathfrak{c}$ a subspace of the center of $\mathfrak{g}$ such that $\mathfrak{h} := \mathfrak{g}/\mathfrak{c}$ is semisimple. Then $\mathfrak{c}$ is precisely the center of $\mathfrak{g}$, $\mathfrak{g}_1 := [\mathfrak{g}, \mathfrak{g}]$ is a Lie ideal of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}_1$ is a direct product of Lie algebras. Moreover, $\mathfrak{g}_1$ is the unique Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{h}$ and $\mathfrak{g}_1 = [\mathfrak{g}_1, \mathfrak{g}_1]$. In particular, there is a unique Lie algebra injection γ of $\mathfrak{h}$ into $\mathfrak{g}$ inverting the map $\mathfrak{g} \longrightarrow \mathfrak{h}$, and its image is $\mathfrak{g}_1$.*

PROOF: Since the center of $\mathfrak{h}$ is 0, it is immediate that $\mathfrak{c}$ is precisely the center of $\mathfrak{g}$. For $X, Y \in \mathfrak{g}$, $[X, Y]$ depends only on the images of X, Y in $\mathfrak{h}$, and so we have an action of $\mathfrak{h}$ on $\mathfrak{g}$ that is trivial *precisely* on $\mathfrak{c}$. Because $\mathfrak{h}$ is semisimple, it follows that there is a *unique* subspace $\mathfrak{h}'$ of $\mathfrak{g}$ complementary to $\mathfrak{c}$ that is stable under $\mathfrak{h}$. Clearly $\mathfrak{h}'$ is a Lie ideal in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{h}'$ is a direct product; moreover, the natural map $\mathfrak{h}' \longrightarrow \mathfrak{h}$ is an isomorphism, so that, as $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$, we have

$$\mathfrak{h}' = [\mathfrak{h}', \mathfrak{h}'] = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_1 .$$

If $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{h}$, then $\mathfrak{a} \longrightarrow \mathfrak{h}$ is an isomorphism so that $\mathfrak{a}$ is stable under the action of $\mathfrak{h}$ and hence $\mathfrak{a} = \mathfrak{g}_1$. □

The Quadratic Subalgebra. We return to the exact sequence (5.5). Since $xy + yx$ is a scalar for $x, y \in V$, we have

$$d\alpha\left(\left(\frac{1}{4}\right)(xy - yx)\right) = d\alpha\left(\left(\frac{1}{2}\right)xy\right) = M_{x,y}, \quad x, y \in V .$$

Let us therefore define

$$C^2 = \text{linear span of } xy - yx , \quad x, y \in V, \ C^2 \subset \mathrm{Lie}(\Gamma^+) .$$

Then $d\alpha$ maps C^2 onto $\mathfrak{so}(V)$. Of course, $C^2 = 0$ if $\dim(V) = 1$. If x and y are orthogonal, then $xy - yx = 2xy$, from which it follows easily that C^2 is the span of the $e_r e_s$ $(r < s)$ for *any* orthogonal basis (e_i) of V (orthogonal bases always exist, but they may not be orthonormal). We claim that C^2 is a *Lie subalgebra* of C_L^+, which may be called the *quadratic subalgebra* of C_L^+. For this we may go over to **C** and assume that (e_r) is an ON basis of V. If $r < s$ and $t < u$, a simple calculation shows that

$$[e_r e_s, e_t e_u] = 2\delta_{st} e_r e_u - 2\delta_{rt} e_s e_u + 2\delta_{su} e_r e_t - 2\delta_{ru} e_t e_s .$$

The right side is easily seen to be in C^2; in fact, if $r = t$, $s = u$, or if r, s, t, u are distinct, the right side is 0, while in the case when $\{r, s\} \cap \{t, u\}$ consists of a single element, the right side is a linear combination of $e_p e_q$, $p \neq q$. Since $1, e_r e_s$ $(r < s)$ are linearly independent, it is clear that C^2 is linearly independent of $k1$ and $\dim(C^2) = \dim(\mathfrak{so}(V))$.

Since $d\alpha$ maps C^2 onto $\mathfrak{so}(V)$, it follows from the above discussion that

$$\mathrm{Lie}(\Gamma^+) = k \oplus C^2 , \quad d\alpha : C^2 \simeq \mathfrak{so}(V) ,$$

and the map

$$\gamma : M_{x,y} \longmapsto \left(\frac{1}{4}\right)(xy - yx) , \quad x, y \in V , \tag{5.6}$$

splits the exact sequence (5.5); i.e., it is a Lie algebra injection of $\mathfrak{so}(V)$ into $\mathrm{Lie}(\Gamma^+)$ such that

$$d\alpha \circ \gamma = \mathrm{id} \quad \text{on } \mathfrak{so}(V) .$$

We have

$$\gamma(\mathfrak{so}(V)) = C^2 .$$

The fact that C^2 is spanned by the $e_r e_s (r < s)$ shows that C^2 generates C^+ *as an associative algebra*. This is true for $k = \mathbf{R}, \mathbf{C}$.

THEOREM 5.4.2 *If* $\dim(V) \geq 3$, *then* $C^2 = [C^2, C^2]$ *is the unique subalgebra of* $\mathrm{Lie}(\Gamma^+)$ *isomorphic to* $\mathfrak{so}(V)$, *and* γ *the only Lie algebra map splitting* (5.5). *Moreover,* C^2 *generates* $C(V)^+$ *as an associative algebra. If further* $k = \mathbf{C}$ *and* G *is the complex analytic subgroup of* Γ^+ *determined by* C^2, *then* (G, α) *is a double cover of* $\mathrm{SO}(V)$ *and hence* $G \simeq \mathrm{Spin}(V)$. *In this case* G *is the unique connected subgroup of* Γ^+ *covering* $\mathrm{SO}(V)$ *and is the universal cover of* $\mathrm{SO}(V)$.

PROOF: If $\dim(V) \geq 3$, $\mathfrak{so}(V)$ is semisimple, and so it follows from the lemma that the exact sequence (5.5) splits *uniquely* and

$$\gamma(\mathfrak{so}(V)) = C^2 = [C^2, C^2].$$

Let $k = \mathbf{C}$. The fact that G is the unique connected subgroup of Γ^+ covering $\mathrm{SO}(V)$ follows from the corresponding uniqueness of C^2. It remains to show that G is a double cover. If $x, y \in V$ are orthonormal, we have $(xy)^2 = -1$ and $xy = (\frac{1}{2})(xy - yx)$ so that

$$a(t) := \exp\left\{\frac{t(xy - yx)}{2}\right\} = \exp(txy) = (\cos t)1 + (\sin t)xy$$

showing that the curve $t \longmapsto (\cos t)1 + (\sin t)xy$ lies in G. Taking $t = \pi$, we see that $-1 \in G$. Hence G is a nontrivial cover of $\mathrm{SO}(V)$. But the universal cover of $\mathrm{SO}(V)$ is its only nontrivial cover and so $G \simeq \mathrm{Spin}(V)$. This finishes the proof. □

Explicit Description of the Complex Spin Group. Let $k = \mathbf{C}$. We shall see now that we can do much better and obtain a very explicit description of G and also take care of the cases when $\dim(V) \leq 2$. This, however, requires some preparation. We introduce the *full Clifford group* Γ defined as follows:

$$\Gamma = \{u \in C^\times \cap (C^+ \cup C^-) \mid uVu^{-1} \subset V\}.$$

Clearly

$$\Gamma = (\Gamma \cap C^+) \cup (\Gamma \cap C^-), \qquad \Gamma \cap C^+ = \Gamma^+.$$

We now extend the action α of Γ^+ on V to an action α of Γ on V by

$$\alpha(u)(x) = (-1)^{p(u)} uxu^{-1}, \qquad u \in \Gamma,\ x \in V.$$

As in the case of Γ^+, it is checked that α is a homomorphism from Γ to $\mathrm{O}(V)$.

PROPOSITION 5.4.3 *We have an exact sequence*

$$1 \longrightarrow \mathbf{C}^\times 1 \longrightarrow \Gamma \stackrel{\alpha}{\rightarrow} \mathrm{O}(V) \longrightarrow 1.$$

Moreover, $\alpha^{-1}(\mathrm{SO}(V)) = \Gamma^+$ *and*

$$1 \longrightarrow \mathbf{C}^\times 1 \longrightarrow \Gamma^+ \stackrel{\alpha}{\rightarrow} \mathrm{SO}(V) \longrightarrow 1$$

is exact.

PROOF: If $v \in V$ and $Q(v) = 1$, we assert that $v \in \Gamma^-$ and $\alpha(v)$ is the reflection in the hyperplane orthogonal to v. In fact, $v^2 = 1$ so that $v^{-1} = v$, and, for $w \in V$, $\alpha(v)(w) = -vwv^{-1} = -vwv = w - 2\Phi(v, w)v$. By the theorem of E. Cartan (see Section 5.2), any element of $\mathrm{O}(V)$ is a product of reflections in hyperplanes orthogonal to unit vectors. Hence α maps Γ onto $O(V)$. If $\alpha(u) = 1$, then u lies in the *supercenter* of C and so is a scalar. This proves the first assertion. By Cartan's result, any element of $SO(V)$ is a product of an *even* number of reflections, and so, if G' is the group of all elements of the form $v_1 \cdots v_{2r}$ where the v_i are unit vectors, then $G' \subset \Gamma^+$ and α maps G' onto $SO(V)$. We first show that $\alpha(\Gamma^+) = SO(V)$. In fact, if the image of Γ^+ is more than $SO(V)$, it must be all of $O(V)$, and so for any unit vector $v \in V$, $\alpha(v)$ must also be of the form $\alpha(u)$ for some $u \in \Gamma^+$. Because the kernel of α is $\mathbf{C}^\times 1$, which is in C^+, it follows that $v = cu$ where c is a scalar and hence that $v \in \Gamma^+$, which is a contradiction. If $u \in \Gamma$ and $\alpha(u) \in SO(V)$, then there is $u' \in \Gamma^+$ such that $\alpha(u') = \alpha(u)$ and so $u = cu'$ where c is a scalar, showing that $u \in \Gamma^+$ already. This finishes the proof. □

Let us consider β, the unique antiautomorphism β of the ungraded Clifford algebra, which is the identity on V that we have called the principal or canonical antiautomorphism. Thus

$$\beta(x_1 \cdots x_r) = x_r \cdots x_1 , \quad x_i \in V . \tag{5.7}$$

Hence β preserves parity. We then have the following theorem, which gives the explicit description of $\text{Spin}(V)$ as embedded in $C^{+\times}$ for all dimensions:

THEOREM 5.4.4 *The map $x \longmapsto x\beta(x)$ is a homomorphism of Γ into $\mathbf{C}^\times 1$. Let G be the kernel of its restriction to Γ^+.*

(i) *If $\dim(V) = 1$, then $G = \{\pm 1\}$.*
(ii) *If $\dim(V) \geq 2$, then G is the analytic subgroup of $C^{+\times}$ defined by C^2 and (G, α) is a double cover of $SO(V)$. In particular,*

$$\text{Spin}(V) \simeq G = \{x \in C^{+\times} \mid xVx^{-1} \subset V,\ x\beta(x) = 1\} . \tag{5.8}$$

PROOF: Given $x \in \Gamma$ we can, by Cartan's theorem, find unit vectors $v_j \in V$ such that $\alpha(x) = \alpha(v_1) \cdots \alpha(v_r)$ and so $x = cv_1 \cdots v_r$ for a nonzero constant c. But then

$$x\beta(x) = c^2 v_1 \cdots v_r v_r \cdots v_1 = c^2$$

so that $x\beta(x) \in \mathbf{C}^\times 1$. If $x, y \in \Gamma$, then

$$x\beta(x)(y\beta(y)) = x(y\beta(y))\beta(x) = xy\beta(xy) .$$

Hence $x \longmapsto x\beta(x)$ is a homomorphism of Γ into $\mathbf{C}^\times 1$. Let G be the kernel of the restriction to Γ^+ of this homomorphism.

If $\dim(V) = 1$ and e is a basis of V, then C^+ is $\mathbf{C}$ so that $\Gamma^+ = \mathbf{C}^\times$. Hence $x\beta(x) = x^2$ for $x \in \mathbf{C}^\times$ and so $G = \{\pm 1\}$.

Let now $\dim(V) \geq 2$. If $g \in SO(V)$, we can find $u \in \Gamma^+$ such that $\alpha(u) = g$. Let $c \in \mathbf{C}^\times$ be such that $u\beta(u) = c^2 1$; then $v = c^{-1}u \in G$ and $\alpha(v) = \alpha(u) = g$. We thus see that α maps G onto $SO(V)$. If $u \in G$ and $\alpha(u) = 1$, then u is a scalar and so, as $u\beta(u) = u^2 = 1$, we see that $u = \pm 1$. Since $\pm 1 \in G$ it follows that α maps G onto $SO(V)$ with kernel $\{\pm 1\}$. We shall now prove that G is connected;

this will show that it is a double cover of $\mathrm{SO}(V)$. For this it is enough to show that $-1 \in G^0$. If $x, y \in V$ are orthogonal, we have, for all $t \in \mathbf{C}$,

$$\beta(\exp(txy)) = \sum_{n\geq 0} \frac{t^n}{n!}\beta((xy)^n) = \sum_{n\geq 0} \frac{t^n}{n!}(yx)^n = \exp(tyx)\,.$$

Hence, for all $t \in \mathbf{C}$, $\exp(txy) \in \Gamma^+$ and

$$\exp(txy)\beta(\exp(txy)) = \exp(txy)\exp(tyx) = \exp(txy)\exp(-txy) = 1\,.$$

Thus $\exp(txy)$ lies in G^0 for all $t \in \mathbf{C}$. If x, y are orthonormal, then $(xy)^2 = -1$ and so we have

$$\exp(txy) = (\cos t)1 + (\sin t)xy$$

as we have seen already. Therefore $-1 = \exp(\pi xy) \in G^0$. Hence G is a double cover of $\mathrm{SO}(V)$ and thus isomorphic to $\mathrm{Spin}(V)$. □

The fact that G is the analytic subgroup of Γ^+ defined by C^2 when $\dim(V) \geq 3$ already follows from Theorem 5.4.2. So we need only consider the case $\dim(V) = 2$. Let $x, y \in V$ be orthonormal. We know that $\exp(txy) \in G$ for all $t \in \mathbf{C}$. But

$$\exp(txy) = \exp\left(t\frac{(xy - yx0}{2}\right)$$

so that $xy - yx \in \mathrm{Lie}(G)$. Hence $\mathrm{Lie}(G) = \mathbf{C}xy = \mathbf{C}(xy - yx) = C^2$. Since it is a connected group of dimension 1, it follows that it is identical with the image of the one-parameter group $t \longmapsto \exp(txy)$.

We write $\mathrm{Spin}(V)$ for G.

PROPOSITION 5.4.5 *Let V be arbitrary. Then*

$$\mathrm{Spin}(V) = \{x = v_1 \cdots v_{2r},\ v_i \in V,\ Q(v_i) = 1\}\,. \tag{5.9}$$

PROOF: The right side of the formula above describes a group that is contained in $\mathrm{Spin}(V)$ and its image by α is the whole of $\mathrm{SO}(V)$ by Cartan's theorem. It contains -1 since $-1 = (-u)u$ where $u \in V$ with $Q(u) = 1$. So it is equal to $\mathrm{Spin}(V)$. □

Spin Groups for Real Orthogonal Groups. We now take up spin groups over the reals. Let V be a quadratic vector space over $\mathbf{R}$. Let $V_{\mathbf{C}}$ be its complexification. Then there is a unique conjugation $x \longmapsto x^{\mathrm{conj}}$ on the Clifford algebra $C(V_{\mathbf{C}})$ that extends the conjugation on $V_{\mathbf{C}}$, whose fixed points are the elements of $C(V)$. This conjugation commutes with β and so leaves $\mathrm{Spin}(V_{\mathbf{C}})$ invariant. The corresponding subgroup of $\mathrm{Spin}(V_{\mathbf{C}})$ of fixed points for the conjugation is a real algebraic Lie group, namely, the group of real points of $\mathrm{Spin}(V_{\mathbf{C}})$. It is by definition $\mathrm{Spin}(V)$:

$$\mathrm{Spin}(V) = \{x \in \mathrm{Spin}(V_{\mathbf{C}}), x = x^{\mathrm{conj}}\}\,. \tag{5.10}$$

Clearly, $-1 \in \mathrm{Spin}(V)$ always. If $\dim(V) = 1$, we have

$$\mathrm{Spin}(V) = \{\pm 1\}\,.$$

LEMMA 5.4.6 *Let* $\dim(V) \geq 2$ *and let* $x, y \in V$ *be mutually orthogonal and* $Q(x), Q(y) = \pm 1$. *Then* $e^{txy} \in \mathrm{Spin}(V)^0$ *for all real* t. *Let* J_{xy} *be the element of* $\mathrm{SO}(V)$ *that is* -1 *on the plane spanned by* x, y *and* $+1$ *on the orthogonal complement of this plane. Then*

$$e^{\pi xy} = -1\,, \quad Q(x)Q(y) > 0\,, \qquad \alpha(e^{(i\pi/2)xy}) = J_{xy}\,, \quad Q(x)Q(y) < 0\,.$$

In the second case, $e^{i\pi xy} = -1$.

PROOF: We have already seen that e^{txy} lies in $\mathrm{Spin}(V_{\mathbf{C}})$ for all complex t. Hence for t real it lies in $\mathrm{Spin}(V)$ and hence in $\mathrm{Spin}(V)^0$. Suppose that $Q(x)Q(y) > 0$. Then $(xy)^2 = -1$ and so

$$e^{txy} = (\cos t)1 + (\sin t)xy$$

for real t. Taking $t = \pi$ we get the first relation. Let now $Q(x)Q(y) < 0$. We have

$$\alpha(e^{itxy}) = e^{itd\alpha(xy)} = e^{2itM_{x,y}}\,.$$

Since $Q(x)Q(y) < 0$, the matrix of $M_{x,y}$ on the complex plane spanned by x and y is

$$\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

from which it follows that $\alpha(e^{itxy})$ is 1 on the complex plane orthogonal to x and y, while on the complex span of x and y it has the matrix

$$\cos 2t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pm i \sin 2t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Taking $t = \pi/2$ we get the second relation. Since $(xy)^2 = 1$ we have,

$$e^{txy} = (\cosh t)1 + (\sinh t)xy$$

for all complex t, so that $e^{i\pi xy} = -1$. □

THEOREM 5.4.7 *Let* V *be a real quadratic vector space, and let* $\mathrm{Spin}(V)$ *be the group of real points of* $\mathrm{Spin}(V_{\mathbf{C}})$. *Then*:

(i) *If* $\dim(V) = 1$, *then* $\mathrm{Spin}(V) = \{\pm 1\}$.

(ii) *If* $\dim(V) \geq 2$, $\mathrm{Spin}(V)$ *always maps onto* $\mathrm{SO}(V)^0$. *It is connected except when* $\dim(V) = 2$ *and* V *is indefinite. In this exceptional case*

$$(\mathrm{Spin}(V), \mathrm{SO}(V)^0, \alpha) \simeq (\mathbf{R}^\times, \mathbf{R}_+^\times, \sigma)\,, \quad \sigma(u) = u^2\,.$$

(iii) *In all other cases* $\mathrm{Spin}(V)$ *is connected and is a double cover of* $\mathrm{SO}(V)^0$. *If* $V = \mathbf{R}^{p,q}$, *then* $\mathrm{Spin}(p, q) := \mathrm{Spin}(V)$ *is characterized as the unique double cover of* $\mathrm{SO}(V)^0$ *when one of* $p, q \leq 1$, *and as the unique double cover that is nontrivial over both* $\mathrm{SO}(p)$ *and* $\mathrm{SO}(q)$, *when* $p, q \geq 2$. *In particular,* $\mathrm{Spin}(V)$ *is the universal cover of* $\mathrm{SO}(V)^0$ *if and only if* $\dim(V) \geq 3$ *and* $\min(p, q) = 0, 1$.

PROOF: We need only check (ii) and (iii). The Lie algebra map

$$d\alpha : \left(\frac{1}{4}\right)(xy - yx) \longmapsto M_{x,y}$$

maps Lie(Γ^+) onto $\mathfrak{so}(V)$. So, α maps $\mathrm{Spin}(V)^0$ onto $\mathrm{SO}(V)^0$, and Spin(V) into SO(V) with kernel $\{\pm 1\}$. Since the group SO(V) remains the same if we interchange p and q, we may suppose that $V = \mathbf{R}_{p,q}$ where $0 \le p \le q$ and $p + q \ge 2$.

First assume that $p = 0$. Then SO(V) is already connected. We can then find mutually orthogonal $x, y \in V$ with $Q(x) = Q(y) = -1$, and so, by the lemma above, $-1 \in \mathrm{Spin}(V)^0$. This proves that Spin($V$) is connected and is a double cover of SO(V).

Let $1 \le p \le q$. We shall first prove that Spin(V) maps into (hence onto) $\mathrm{SO}(V)^0$. Suppose that this is not true. Then the image of Spin(V) under α is the whole of SO(V). In particular, if $x, y \in V$ are mutually orthogonal and $Q(x) = 1$, $Q(y) = -1$, there is $u \in \mathrm{Spin}(V)$ such that $\alpha(u) = J_{xy}$. By the lemma above $\alpha(e^{(i\pi/2)xy}) = J_{xy}$ also, and so $u = \pm e^{(i\pi/2)xy}$. This means that $e^{(i\pi/2)xy} \in \mathrm{Spin}(V)$ and so must be equal to its conjugate. But its conjugate is $e^{(-i\pi/2)xy}$, which is its inverse, and so we must have $e^{i\pi xy} = 1$, contradicting the lemma.

Assume now that we are not in the exceptional case (ii). Then $q \ge 2$ and so we can find mutually orthogonal $x, y \in V$ such that $Q(x) = Q(y) = -1$. The argument for proving that Spin(V) is a double cover for $\mathrm{SO}(V)^0$ then proceeds as in the definite case.

Suppose now that we are in the exceptional case (ii). Then this last argument does not apply. In this case let $x, y \in V$ be mutually orthogonal and $Q(x) = 1$, $Q(y) = -1$. Then $(xy)^2 = 1$ and $\mathrm{Spin}(V_{\mathbf{C}})$ coincides with the image of the one-parameter group e^{txy} for $t \in \mathbf{C}$. But $e^{txy} = (\cosh t)1 + (\sinh t)xy$, and such an element lies in Spin(V) if and only if $\cosh t$, $\sinh t$ are both real. Thus

$$\mathrm{Spin}(V) = \{\pm a(t) \mid t \in \mathbf{R}\}, \quad a(t) = \cosh t\, 1 + \sinh t\, xy\,.$$

On the other hand,

$$\alpha(\pm a(t)) = e^{2tM_{x,y}} = (\cosh 2t)1 + (\sinh 2t)M_{x,y}$$

so that $\mathrm{SO}(V)^0$ is the group of all matrices of the form

$$m(t) = \begin{pmatrix} \cosh 2t & \sinh 2t \\ \sinh 2t & \cosh 2t \end{pmatrix}, \quad t \in \mathbf{R}\,.$$

This is isomorphic to $\mathbf{R}_+^\times$ through the map $m(t) \longmapsto e^{2t}$, while $\mathrm{Spin}(V) \simeq \mathbf{R}^\times$ through the map $\pm a(t) \longmapsto \pm e^t$. Assertion (ii) now follows at once.

It remains only to characterize the double cover when V is not exceptional. If $p = 0$, the fundamental group of $\mathrm{SO}(V)^0$ is $\mathbf{Z}$ when $q = 2$ and $\mathbf{Z}_2$ when $q \ge 3$; if $p = 1$, the fundamental group of $\mathrm{SO}(V)^0$ is $\mathbf{Z}_2$ for $q \ge 2$. Hence the double cover of $\mathrm{SO}(V)^0$ is unique in these cases without any further qualification. We shall now show that when $2 \le p \le q$, $\mathrm{Spin}(p, q)$ is the unique double cover of $S_0 = \mathrm{SO}(p, q)^0$ with the property described. If S is a double cover of S_0, the preimages L_p, L_q of SO(p), SO(q) are compact, and for L_r $(r = p, q)$ there are only two possibilities: either (i) it is connected and a double cover of SO(r) or (ii) it has two connected components and $L_r^0 \simeq \mathrm{SO}(r)$. We must show that L_p, L_q have property (i) and $\mathrm{Spin}(p,q)$ is the unique double cover possessing this property for both $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$.

To this end we need a little preparation. Let $\mathfrak{g}$ be a real semisimple Lie algebra and let G_0 be a connected real Lie group with finite center (for instance, a matrix group) and Lie algebra $\mathfrak{g}$. Let us consider the category $\mathcal{G}$ of pairs (G,π) where G is a connected, real, semisimple Lie group and π a *finite* covering map $G \longrightarrow G_0$; G then has finite center as well. Morphisms $f : (G_1,\pi_1) \longrightarrow (G_2,\pi_2)$ are finite covering maps compatible with the π_i $(i=1,2)$. We generally suppress the maps π in the discussion below. If G_i $(i=1,2)$ are two objects in $\mathcal{G}$ there is a third group G that covers both G_i finitely, for instance, the fiber product $G_1 \times_{G_0} G_2$. Any G in $\mathcal{G}$ has maximal compact subgroups; these are all connected and mutually conjugate, and all of them contain the center of G. If $f : G_1 \longrightarrow G_2$ and K_i is a maximal compact of G_i, then $f(K_1)$ (resp., $f^{-1}(K_2)$) is a maximal compact of G_2 (resp., G_1). Fix a maximal compact K_0 of G_0. Then for each G in $\mathcal{G}$, the preimage K of K_0 is a maximal compact of G and a map $G_1 \longrightarrow G_2$ gives a map $K_1 \longrightarrow K_2$ with the kernel being the same. Suppose now that G, G_i $(i=1,2)$ are in $\mathcal{G}$ and $G \longrightarrow G_i$ with kernel F_i $(i=1,2)$. It follows from our remarks above that to prove that there is a map $G_1 \longrightarrow G_2$, it is enough to prove that there is a map $K_1 \longrightarrow K_2$. For the existence of a map $K_1 \longrightarrow K_2$, it is clearly necessary and sufficient that $F_1 \subset F_2$.

In our case $G_0 = \mathrm{SO}(p,q)^0$, $K_0 = SO(p) \times \mathrm{SO}(q)$. Then $\mathrm{Spin}(p,q)$ is in the category $\mathcal{G}$ and $K_{p,q}$, the preimage of K_0, is a maximal compact of it. Since both p and q are ≥ 2, it follows from the lemma that -1 lies in the connected component of the preimages of both $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$. So if K_r is the preimage of $\mathrm{SO}(r)$ $(r=p,q)$, then $K_r \longrightarrow \mathrm{SO}(r)$ is a double cover. Let G_1 be a double cover of G_0 with preimages L_p, L_q, $L_{p,q}$ of $\mathrm{SO}(p)$, $\mathrm{SO}(q)$, K_0 with the property that L_r is connected and $L_r \longrightarrow \mathrm{SO}(r)$ is a double cover. It is enough to show that there is a map $G_1 \longrightarrow \mathrm{Spin}(p,q)$ above G_0 since such a map is bijective because G is also a double cover of G_0. By our remarks above this comes down to showing that there is a map $L_{p,q} \longrightarrow K_{p,q}$ above K_0. Since the fundamental group of $\mathrm{SO}(r)$ for $r \geq 2$ is $\mathbf{Z}$ for $r=2$ and $\mathbf{Z}_2$ for $r \geq 3$, $\mathrm{SO}(r)$ has a unique double cover, and so we have isomorphisms $L_r \simeq K_r$ above $\mathrm{SO}(r)$ for $r=p,q$.

The Lie algebra of K_0 is the direct product of the Lie algebras of $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$. This implies that L_p, L_q, as well as K_p, K_q, commute with each other and $L_{p,q} = L_pL_q$, $K_{p,q} = K_pK_q$. Let $M_{p,q} = \mathrm{Spin}(p) \times \mathrm{Spin}(q)$. Then we have unique maps $M_{p,q} \longrightarrow L_{p,q}, K_{p,q}$ with $\mathrm{Spin}(r) \simeq L_r, K_r$ $(r=p,q)$. To show that we have an isomorphism $L_{p,q} \simeq K_{p,q}$, it is enough to show that the kernels of $M_{p,q} \longrightarrow L_{p,q}, K_{p,q}$ are the same. The kernel of $M_{p,q} \longrightarrow K_0$ is $\mathbf{Z}_2 \times \mathbf{Z}_2$. Since $\mathrm{Spin}(r) \simeq L_r, K_r$, it follows that the kernels of $M_{p,q} \longrightarrow L_{p,q}, K_{p,q}$, which are both nontrivial, have the property that their intersections with $\mathrm{Spin}(p) \times 1$ and $1 \times \mathrm{Spin}(q)$ are trivial. But $\mathbf{Z}_2 \times \mathbf{Z}_2$ has only one nontrivial subgroup that has trivial intersection with both of its factors, namely, the diagonal. The uniqueness of this subgroup gives the map $L_{p,q} \simeq K_{p,q}$ that we want. This finishes the proof. □

REMARK. The above discussion also gives a description of $K_{p,q}$, the maximal compact of Spin(p, q). Let us write ε_r for the nontrivial element in the kernel of $K_r \longrightarrow \mathrm{SO}(r)$ $(r = p, q)$. Then

$$K_{p,q} = (K_p \times K_q)/Z\,, \quad Z = \{(1, 1), (\varepsilon_p, \varepsilon_q)\}\,.$$

Thus a map of $K_p \times K_q$ factors through to $K_{p,q}$ if and only if it maps ε_p and ε_q to the same element.

We shall now obtain the analogue of Proposition 5.4.5 in the real case.

PROPOSITION 5.4.8 *For $p, q \geq 0$ we have*

$$\text{(5.11)} \quad \mathrm{Spin}(p, q) = \{v_1 \cdots v_{2a} w_1 \cdots w_{2b} \mid v_i, w_j \in V,\ Q(v_i) = 1,\ Q(w_j) = -1\}\,.$$

PROOF: By the results of Section 5.2, we know that the elements of $\mathrm{SO}(p, q)^0$ are exactly the products of an even number of spacelike reflections and an even number of timelike reflections; here a reflection in a hyperplane orthogonal to a vector $v \in V$ with $Q(v) = \pm 1$ is spacelike or timelike according as $Q(v) = +1$ or -1. It is then clear that the right side of (5.11) is a group that is mapped by α *onto* $\mathrm{SO}(p, q)^0$. Because it contains -1, the result follows at once. □

5.5. Spin Representations as Clifford Modules

We are now in a position to begin the study of the spin representations. Their definition given at the beginning in terms of the Dynkin diagrams does not furnish us with the tools to study them in depth. It turns out that if we identify $\mathfrak{so}(V)$ with C^2 via the isomorphism

$$\gamma : \mathfrak{so}(V) \stackrel{\sim}{\longrightarrow} C^2$$

defined earlier, the irreducible spin modules are precisely those that are restrictions to $\mathfrak{so}(V)$ of uniquely determined irreducible modules for $C(V)^+$. Thus the spin modules can be completely identified with modules for $C(V)^+$.

In view of this we begin by considering the following situation: A is a finite-dimensional associative algebra over the field k, which is either $\mathbf{R}$ or $\mathbf{C}$. Let $A^\times$ be the group of invertible elements of A. Then $A^\times$ is a Lie group over k and its Lie algebra is A_L, which is A with the bracket $[a, b] = ab - ba$. The exponential map $A_L \to A^\times$ is the usual one:

$$\exp(a) = e^a = \sum_{n \geq 0} \frac{a^n}{n!}\,.$$

Let $\mathfrak{g} \subset A_L$ be a Lie algebra and G the corresponding analytic subgroup of $A^\times$. *We assume that A is generated as an associative algebra by the elements of $\mathfrak{g}$.* The exponential map $\mathfrak{g} \longrightarrow G$ is the restriction of the exponential map from A_L to $A^\times$. A finite-dimensional representation ρ of $\mathfrak{g}$ is said to be of A-type if there is a representation μ of A such that ρ is the restriction of μ to $\mathfrak{g}$. Since $\mathfrak{g} \subset A$ and generates A as an associative algebra, we have a surjective map $\mathcal{U}(\mathfrak{g}) \longrightarrow A$, where $\mathcal{U}(\mathfrak{g}) \supset \mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$, which is the identity on $\mathfrak{g}$.

So the representations of $\mathfrak{g}$ of A-type, which are just the A-modules, are precisely those whose extensions to $\mathcal{U}(\mathfrak{g})$ factor through the map $\mathcal{U}(\mathfrak{g}) \longrightarrow A$. Clearly the extension μ to A of a representation ρ of $\mathfrak{g}$ of A-type is unique.

We now wish to extend the notion of A-type to representations of G. A representation r of G is said to be of A-type if $\rho = dr$ is a representation of $\mathfrak{g}$ of A-type.

We now have the following important but completely elementary result:

PROPOSITION 5.5.1 *Let $\mathfrak{g}$ generate A as an associative algebra. If ρ is a representation of $\mathfrak{g}$ of A-type and μ is its extension to A, then ρ integrates to the representation $r = \mu|_G$. In particular, if r is a representation of G, it is of A-type if and only if it is the restriction to G of a representation μ of A; μ is then unique and is the extension to A of dr. Finally, the categories of A-modules and of A-type modules of G or $\mathfrak{g}$ are identical.*

PROOF: For the first assertion, let $r' = \mu|_G$. Then for $a \in \mathfrak{g}$ we have

$$(dr')(a) = \left(\frac{d}{dt}\right)_{t=0} r'(e^{ta}) = \left(\frac{d}{dt}\right)_{t=0} \mu(e^{ta}) = \left(\frac{d}{dt}\right)_{t=0} e^{t\mu(a)} = \mu(a)\,.$$

Hence $dr' = \rho$. For the second assertion, let r be a representation of G of A-type, $\rho = dr$, and μ the extension of ρ to A. By the first assertion we have $r = \mu|_G$. Conversely, let μ be a representation of A and $r = \mu|_G$, $\rho = \mu|_{\mathfrak{g}}$. Once again, by the first assertion ρ integrates to r, showing that $\rho = dr$. If μ, μ' are two representations of A that have the same restriction r on G, then $\mu|_{\mathfrak{g}} = \mu'|_{\mathfrak{g}} = dr$ and so $\mu = \mu'$. The last statement is trivial.

The imbedding

$$\gamma : \mathfrak{so}(V) \longrightarrow C_L^+\,, \quad M_{x,y} \longmapsto \left(\frac{1}{4}\right)(xy - yx)$$

has the property that its image generates C^+ as an associative algebra. Hence the conditions of the above proposition are satisfied with $G = \mathrm{Spin}(V)$, $\mathfrak{g} = \mathfrak{so}(V)$ (identified with its image under γ), and $A = C^+$. By a *Clifford module* we mean any module for $\mathfrak{so}(V)$ or $\mathrm{Spin}(V)$, which is the restriction to $\mathrm{Spin}(V)$ or $\mathfrak{so}(V)$ of a module for C^+. Every Clifford module is a direct sum of irreducible ones. If the ground field k is $\mathbf{C}$, we know all these: if $\dim(V)$ is odd, there is exactly one irreducible module S for $C(V)^+$; if $\dim(V)$ is even, there are two irreducible $C(V)^+$-modules $S^\pm$. Their dimensions are given by

$$\dim(S) = 2^{\frac{D-1}{2}}\,, \quad \dim(S^\pm) = 2^{\frac{D}{2}-1} \quad D = \dim(V)\,.$$

If $k = \mathbf{R}$, the classification of the irreducible $C(V)^+$ is more complicated and will be taken up in the next chapter.

Identification of the Clifford Modules with the Spin Modules. We shall now take $k = \mathbf{C}$ and show that $S^\pm$ and S are the spin modules. In the discussion below we shall have to use the structure theory of the orthogonal algebras. For details of this theory, see chapter 4 in Varadarajan (1984).[2]

$\dim(V) = 2m$. We take a basis $(e_i)_{1\le i\le 2m}$ for V such that the matrix of the quadratic form is

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Thus

$$\Phi(e_r, e_{m+r}) = 1\,, \quad 1 \le r \le m\,,$$

and all other scalar products between the e's are zero. In what follows we use $r, s, r', \ldots$ as indices varying between 1 and m. The matrices of $\mathfrak{so}(V)$ are those of the form

$$\begin{pmatrix} A & B \\ C & -A^{\mathsf{T}} \end{pmatrix}, \quad B^{\mathsf{T}} = -B,\ C^{\mathsf{T}} = -C\,,$$

where A, B, C, D are $m \times m$ matrices. In the usual structure theory of this classical algebra, the Cartan subalgebra is the set of diagonal matrices $\simeq \mathbf{C}^m$ via

$$(a_1, \ldots, a_m) \longmapsto \operatorname{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_m)\,.$$

We write E'_{ij} for the usual matrix units of the $m \times m$ matrix algebra and define

$$E_{ij} = \begin{pmatrix} E'_{ij} & 0 \\ 0 & -E'_{ji} \end{pmatrix}, \quad F_{pq} = \begin{pmatrix} 0 & E'_{pq} - E'_{qp} \\ 0 & 0 \end{pmatrix}, \quad G_{pq} = \begin{pmatrix} 0 & 0 \\ E'_{pq} - E'_{qp} & 0 \end{pmatrix}.$$

Then the E_{ij}, F_{pq}, G_{pq} are the root vectors with corresponding roots $a_i - a_j$, $a_p + a_q$, $-(a_p + a_q)$. For the positive system of root vectors we choose

$$E_{ij},\ i < j\,, \quad F_{pq},\ p < q\,.$$

Writing $M_{t,u}$ for M_{e_t,e_u}, it is easy to check that

$$M_{r,m+s} = E_{rs}\,, \quad M_{r,s} = F_{rs}\,, \quad M_{m+r,m+s} = G_{rs}\,, \quad M_{r,m+r} = E_{rr}\,.$$

Thus the positive root vectors are

$$M_{r,m+s},\ r < s\,, \quad M_{r,s},\ r < s\,.$$

The simple roots are

$$\alpha_1 = a_1 - a_2, \ldots, \alpha_{m-1} = a_{m-1} - a_m, \alpha_m = a_{m-1} + a_m\,,$$

with the corresponding normalized coroots

$$H_r = E_{rr} - E_{r+1,r+1}\ (1 \le r \le m-1)\,, H_m = E_{m-1,m-1} + E_{mm}\,.$$

The Dynkin diagram is

$$\circ - \circ - \circ \quad \cdots \quad \circ - \circ < {}^{\circ}_{\circ} \quad (m \text{ vertices})$$

The linear functions corresponding to the fundamental weights at the right extreme nodes of the Dynkin diagram are

$$\delta^{\pm} := \left(\frac{1}{2}\right)(a_1 + a_2 + \cdots + a_{m-1} \pm a_m)\,.$$

Since the $\pm a_i$ are the weights of the defining representation in $\mathbf{C}^{2m}$, the weights of the tensor representations are those of the form

$$k_1 a_1 + k_2 a_2 + \cdots + k_m a_m$$

where the k_i are *integers*, and so it is clear that the irreducible representations with highest weights $\delta^{\pm}$ cannot occur in the tensors; this was Cartan's observation. We shall now show that the representations with highest weights $\delta^{\pm}$ are none other than the C^+-modules $S^{\pm}$ viewed as modules for $\mathfrak{so}(V)$ through the injection $\mathfrak{so}(V) \hookrightarrow C_L^+$.

The representation of the full algebra C as described in Theorem 5.3.4 acts on ΛU^* where U^* is the span of the e_{m+s}. The duality between U, the span of the e_r, and U^* is given by $\langle e_r, e_{m+s}\rangle = 2\delta_{rs}$. The action of C^+ is through even elements and so preserves the even and odd parts of ΛU^*. We shall show that these are separately irreducible and are equivalent to the representations with highest weights $\delta^{\pm}$. To decompose ΛU^* we find all the vectors that are killed by the positive root vectors. It will turn out that these are the vectors in the span of 1 and e_{2m}. So 1 generates the even part and e_{2m} the odd part; the respective weights are δ^+, δ^-, and so the claim would be proven.

The action of C on ΛU^* is as follows:

$$e_r : u^* \longmapsto \partial(e_r)(u^*)\,, \quad e_{m+r} : u^* \longmapsto e_{m+r} \wedge u^*\,.$$

The injection γ takes $M_{x,y}$ to $(\frac{1}{4})(xy - yx)$ and so γ is given as follows:

$$M_{r,s} \longmapsto \left(\frac{1}{2}\right)e_r e_s\,, \quad M_{r,m+s} \longmapsto \left(\frac{1}{2}\right)e_r e_{m+s}\,, \quad r < s\,,$$

$$M_{r,m+r} \longmapsto \left(\frac{1}{2}\right)(e_r e_{m+r} - 1)\,.$$

We thus have

$$\gamma(M_{r,m+r})1 = \frac{1}{2}\,, \quad \gamma(M_{r,m+r})e_{2m} = \left(\left(\frac{1}{2}\right) - \delta_{rm}\right)e_{2m}\,.$$

Let us now determine all vectors v killed by

$$\gamma(M_{r,m+s})\,, \quad \gamma(M_{r,s})\,, \quad r < s\,.$$

Because $\operatorname{diag}(a_1, \ldots, a_m, -a_1, \ldots, -a_m) = \sum_r a_r M_{r,m+r}$, we see that 1 has weight δ^+ while e_{2m} has weight δ^-. Since 1 is obviously killed by the positive root vectors, we may suppose that v has no constant term and has the form

$$v = \sum_{|I|\geq 1} c_I e_{m+I}\,.$$

We know that v is killed by all $\partial(e_{j_1})\partial(e_{j_2})$ $(1 \leq j_1 < j_2 \leq m)$. If we apply $\partial(e_{j_1})\partial(e_{j_2})$ to a term e_{m+I} with $|I| \geq 2$, we get $e_{m+I'}$ if I contains $\{j_1, j_2\}$ where $I' = I \setminus \{j_1, j_2\}$, or 0 otherwise, from which it is clear that $c_I = 0$. So

$$v = \sum_j c_j e_{m+j}\,.$$

Since $\gamma(M_{r,m+s})v = 0$ for $r < s$, we conclude that $c_r = 0$ for $r < m$.

$\dim(V) = 2m + 1$. We take a basis $(e_t)_{0\leq t\leq 2m}$ with the e_t $(1 \leq t \leq 2m)$ as above and e_0 a vector of norm 1 orthogonal to them. The positive system of roots now consists of

$$a_r - a_s \ (r < s)\,, \qquad -(a_r + a_s)\ (r < s)\,, \qquad -a_r\,,$$

with corresponding root vectors

$$M_{r,s}\ (r < s)\,, \qquad M_{m+r,s}\,, \qquad M_{0,r}\,.$$

The diagram is

$$\circ - \circ - \circ \quad \cdots \quad \circ - \circ \Rightarrow \circ \quad (m \text{ vertices})$$

and

$$\delta = \frac{1}{2}(a_1 + \cdots + a_m)$$

is the fundamental weight corresponding to the right extreme node of the diagram.

Let $f_t = ie_0e_t$. Then $f_s f_t = e_s e_t$ and so C^+ is generated by the (f_t) with the same relations as the e_t. This gives the fact already established that C^+ is the full ungraded Clifford algebra in dimension $2m$ and so is a full matrix algebra. It has thus a unique simple module S. We wish to identify it with the irreducible module with highest weight δ corresponding to the right extreme node of the diagram of $\mathfrak{so}(V)$. We take the module S for C^+ to be $\Lambda(F)$ where F is the span of the f_{m+s}, with f_r acting as $\partial(f_r)$ and f_{m+r} acting as multiplication by f_{m+r}. Then, as $e_t e_u = f_t f_u$, γ is given as follows:

$$M_{r,s} \mapsto \frac{1}{2} f_r f_s\,, \quad M_{m+r,s} \mapsto \frac{1}{2} f_{m+r} f_s\,, \quad M_{r,m+r} \mapsto \frac{1}{2} f_r f_{m+r} - \frac{1}{2}\,,$$

$$M_{0,s} \mapsto \frac{-i}{2} f_s\,, \quad M_{0,m+s} \mapsto \frac{-i}{2} f_{m+s}\,.$$

It is easy to show, as in the previous example, that 1 is of weight δ and is killed by

$$\gamma(M_{0,r})\,, \quad \gamma(M_{r,s})\ (r < s)\,, \quad \gamma(M_{m+r,s})\,,$$

so that it generates the simple module of highest weight δ. To prove that this is all of S, it is enough to show that the only vectors killed by all the positive root vectors are the multiples of 1. Now if v is such a vector, the argument of the previous case shows that $v = a1 + bf_{2m}$. But then $\partial(f_m)v = b = 0$. This finishes the proof. □

Let V be a complex quadratic vector space of dimension D. Then for D odd the spin module has dimension $2^{(D-1)/2}$, while for D even the semispin modules have dimension $2^{D/2-1}$. Combining both, we see that

(5.12) $$\text{dimension of the spin module(s)} = 2^{[\frac{D+1}{2}]-1}\,, \quad D \geq 1\,,$$

in all cases where $[x]$ is the largest integer $\leq x$.

REMARK. The identification of the spin modules with Clifford modules has a very important consequence. If V is a quadratic space and W a quadratic subspace, it is obvious that the restriction of a $C(V)^+$-module to $C(W)^+$ splits as a direct sum of simple modules, and so the restriction of a spinorial module for Spin(V) to Spin(W) is spinorial. There are many situations like this occurring in physics, and one can explicitly write down some of these "branching rules."[3]

Centers of the Complex and Real Spin Groups. We shall now determine the centers of the spin groups, both in the complex and real cases. Let us first consider the case of a complex quadratic space V of dimension $D \geq 3$. If D is odd, $\mathrm{SO}(V)$ has trivial center and so is the adjoint group, and its fundamental group is $\mathbf{Z}_2$. As $\mathrm{Spin}(V)$ is the universal cover of $\mathrm{SO}(V)$, its center is $\mathbf{Z}_2$.

In even dimensions the determination of the center of $\mathrm{Spin}(V)$ is more delicate. If V above has even dimension $D = 2m$, the center of $\mathrm{SO}(V)$ is $\mathbf{Z}_2$, consisting of $\pm I$, I being the identity endomorphism of V. Its preimage in $\mathrm{Spin}(V)$, say Z, is the center of $\mathrm{Spin}(V)$ and is a group with four elements; hence, it is either $\mathbf{Z}_4$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. We shall now determine in terms of D when these two possibilities occur. For this we need to use the obvious fact that the center of $\mathrm{Spin}(V)$ is the intersection of $\mathrm{Spin}(V)$ with the center of $C(V)^+$. We have already determined the center of $C(V)^+$. If $(e_i)_{1\leq i\leq D}$ is an ON basis and $e_{D+1} = e_1 \cdots e_D$, then the center of $C(V)^+$ is spanned by 1 and e_{D+1}. Now $e_{D+1}^2 = (-1)^m$, e_{D+1} anticommutes with all e_i, and $\beta(e_{D+1}) = (-1)^m e_{D+1}$, so that $x = a + be_{D+1}$ lies in the spin group if and only if $xVx^{-1} \subset V$ and $x\beta(x) = 1$. The second condition reduces to $a^2 + b^2 = 1$, $ab(1 + (-1)^m) = 0$, while the first condition, on using the fact that $x^{-1} = \beta(x)$, reduces to $ab(1 - (-1)^m) = 0$. Hence we must have $ab = 0$, $a^2 + b^2 = 1$, showing that

$$\text{center } (\mathrm{Spin}(V)) = \{\pm 1, \pm e_{D+1}\}\,.$$

If m is even, $e_{D+1}^2 = 1$ and so the center is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. For m odd we have $e_{D=1}^2 = -1$, and so the center is $\mathbf{Z}_4$ generated by $\pm e_{D+1}$. Thus,

$$\text{center } (\mathrm{Spin}(V)) \simeq \begin{cases} \mathbf{Z}_2 & \text{if } D = 2k+1 \\ \mathbf{Z}_4 & \text{if } D = 4k+2 \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } D = 4k. \end{cases}$$

Suppose now that V is a *real* quadratic vector space of D. If D is odd it is immediate that the center of $\mathrm{Spin}(V)$ is $\{\pm 1\} \simeq \mathbf{Z}_2$. Now let D be even and let $V = \mathbf{R}^{a,b}$ where $a \leq b$ and $a + b = D$. If a, b are both odd, $-I \notin \mathrm{SO}(a) \times \mathrm{SO}(b)$, and so the center of $\mathrm{SO}(V)^0$ is trivial. This means that the center of $\mathrm{Spin}(V)$ is $\{\pm 1\} \simeq \mathbf{Z}_2$. Suppose that both a and b are even. Then $-I \in \mathrm{SO}(a) \times \mathrm{SO}(b)$, and so the center of $\mathrm{Spin}(V)^0$ consists of $\pm I$. Hence the center of $\mathrm{Spin}(V)$ has four elements and so coincides with Z, the center of $\mathrm{Spin}(V_{\mathbf{C}})$. Thus we have the following:

$$\text{center of } \mathrm{Spin}(\mathbf{R}^{a,b}) \simeq \begin{cases} \mathbf{Z}_2 & \text{if } D = 2k+1 \text{ or } D = 2k,\ a, b \text{ odd} \\ \mathbf{Z}_4 & \text{if } D = 2k,\ a, b \text{ even.} \end{cases}$$

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CHAPTER 6

Fine Structure of Spin Modules

6.1. Introduction

The goal of this chapter is to discuss some specialized questions concerning the spin representations that are of importance for applications to physics. Among them are the following:

(1) for a *real* quadratic vector space V, to get a description of the *real* irreducible representations of $\mathrm{Spin}(V)$,
(2) for any irreducible spin module S (real or complex), to discuss the existence, uniqueness, and symmetry properties of $\mathrm{Spin}(V)$-invariant forms on $S \times S$, and
(3) the same question as (2) but for $\mathrm{Spin}(V)$-invariant morphisms of $S \times S$ into $\Lambda^r(V)$.

Our principal tool will be the result established in Chapter 5 that the spin modules are precisely the Clifford modules, i.e., modules for $C(V)^+$.

For applications to physics, the theory of spin modules over **C** is not enough; one needs the theory over **R**. Representation theory over **R** is a little more subtle than the usual theory over **C** because Schur's lemma takes a more complicated form. If V is a real vector space and $A \subset \mathrm{End}_{\mathbf{R}}(V)$ is an algebra acting irreducibly on V (we say that V is a *simple* module), the commutant A' of A, namely, the algebra of elements of $\mathrm{End}_{\mathbf{R}}(V)$ commuting with A, is a *division algebra*. Indeed, if $R \in A'$, the kernel and image of R are submodules, and so each is either 0 or V. So, if $R \neq 0$, then both are 0 and so R is bijective, hence invertible, and $R^{-1} \in A'$. Now **R**, **C**, **H** are all division algebras over **R**, **H** being the algebra of quaternions. Examples can be given to show that all three arise as commutants of simple modules of **R**-algebras. For instance, if A denotes any one of these, it is a simple module for the left regular representation, and its commutant is isomorphic to $A^{\mathrm{opp}} \simeq A$. A classical theorem of Frobenius asserts that these are the only (associative) division algebras over **R**. So simple modules for a real algebra may be classified into three types according to the division algebra arising as the commutants in their simple modules.

The first main goal of this chapter is to determine the types of the simple modules for the even parts of the Clifford algebras of real quadratic vector spaces. The main result is that the types are governed by the *signature of the quadratic space* mod 8. This is the first of two beautiful periodicity theorems that dominate the theory of spinors.

It is not difficult to see that the types depend on the signature. Indeed, if we replace V by $V \oplus W$ where W is hyperbolic, then $C(V \oplus W) \simeq C(V) \otimes C(W)$, and $C(W)$ is a full endomorphism superalgebra of a super vector space U. One can show that the simple modules for $C(V)$ and $C(V \oplus W)$ are S and $S \otimes U$ and the commutants are the same. Hence the types for $C(V)$ and $C(V \oplus W)$ are the same. Since two spaces V_1, V_2 have the same signature if and only if we can write $V_i = V \oplus W_i$ for $i = 1, 2$ where the W_i are hyperbolic, it is immediate that the types of $C(V_1)$ and $C(V_2)$ are the same. A little more work is needed to come down to the even parts. However, one needs a much closer look to see that there is a periodicity mod 8 here.

The treatment of the reality question will be completed in Section 6.4, and it will turn out that the main results depend only on the *signature* of the quadratic vector space mod 8. In Sections 6.5 and 6.6 we study the second and third questions mentioned above. The main goal here is to prove that over **C** the results are governed by periodicity mod 8 of the *dimension* of V. This is still true over **R**, but one has to add additional detail due to the variation of the types of the irreducible modules. Thus in the real case the answers to questions 2 and 3 depend on both the signature and dimension mod 8. In the physics literature the results are described in terms of the *Regge Clock*,[1] which has period 8 and whose two hands correspond to signature and dimension. In Section 6.7 we discuss the special case of Minkowski signature. Finally, in Section 6.8 we study the image of the real spin group inside the complex space of spinors, which is useful in understanding a variety of physical situations.[2] Our treatment of all these questions owes very much to Deligne's work.[3] Only in Section 6.8 do we go beyond what is in Deligne's notes.

It is thus clear that basic to everything is a study of the *real* Clifford modules for the Clifford algebra of a real quadratic vector space. This is a special case of the following general question: if A is an associative superalgebra over **R** and $A_{\mathbf{C}}$ is its complexification, how to determine the types of the real simple (irreducible) supermodules of A if one has complete knowledge of the simple modules over **C** of $A_{\mathbf{C}}$? For the purely even case when $A_{\mathbf{C}}$ is a full matrix algebra, this is quite classical and is just the theory of the Brauer group. So what we have to do is to work out a super version of the theory of the Brauer group. Notice, however, that the algebraic question is most naturally studied over any field k of characteristic 0 with an algebraic closure $\overline{k} \supset k$. This is what we shall do, postponing to the very end the special case when $k = \mathbf{R}, \overline{k} = \mathbf{C}$. The reader who is interested in the final results when $k = \mathbf{R}$ may go directly to Section 6.4 where we sketch a derivation of the main results on the structure of real Clifford algebras by a direct method that in essence goes back to Atiyah-Bott-Shapiro.[4]

6.2. The Central Simple Superalgebras

The Brauer Group of a Field. We begin with a discussion of the types of simple modules of an associative algebra over k. All algebras are finite dimensional and have units, and all modules are finite dimensional. The field k need not be algebraically closed. If A is an associative algebra over k and M is a module for A, we write A_M for the image of A in $\mathrm{End}_k(M)$. If M is simple, then the commutant $D = A'_M$ of A in M is a division algebra, as we have seen above. However, unlike the case when k is algebraically closed, this division algebra need not be k. The classical theorem of Wedderburn asserts that A_M is the commutant of D, i.e.,

$$A_M = \mathrm{End}_D(M)\,.$$

We can reformulate this as follows: The definition $m \cdot d = dm$ $(m \in M, d \in D)$ converts M into a right vector space over D^{opp}, the division algebra opposite to D. Let $(m_i)_{1\le i\le r}$ be a D^{opp}-basis for M. If we write, for any $a \in A_M$, $am_j = \sum_i m_i \cdot a_{ij}$, then the map

$$a \longmapsto (a_{ij})$$

is an isomorphism of A_M with the algebra $M^r(D^{\mathrm{opp}})$ of all matrices with entries from D^{opp}:

$$A_M \simeq M^r(D^{\mathrm{opp}}) \simeq M^r(k) \otimes D^{\mathrm{opp}}\,.$$

Here, for any field k', $M^r(k')$ is the full matrix algebra over k'.

The classical theory of the Brauer group is well-known, and we shall now give a quick summary of its basic results. We shall not prove these here, but we shall prove their super versions later on. Given an associative algebra A over k and a field $k' \supset k$, we define

$$A_{k'} = k' \otimes_k A\,.$$

We shall say that A is *central simple* (CS) if $A_{\overline{k}}$ is isomorphic to a full matrix algebra:

$$A \text{ CS} \Longleftrightarrow A_{\overline{k}} \simeq M^r(\overline{k})\,.$$

Since

$$M^r(k') \otimes M^s(k') \simeq M^{rs}(k')\,,$$

it follows that if A, B are CS algebras so is $A \otimes B$. Since

$$M^r(k')^{\mathrm{opp}} \simeq M^r(k')\,,$$

it follows that for A a CS algebra, A^{opp} is also a CS algebra. The basic facts about CS algebras are summarized in the following proposition. Recall that for an algebra A over k and a module M for it, M is called *semisimple* if it is a direct sum of simple modules. M is semisimple if and only if $\overline{M} := M \otimes_k \overline{k}$ is semisimple for $A_{\overline{k}} = \overline{k} \otimes_k A$. A itself is called *semisimple* if all its modules are semisimple. This will be the case if A, viewed as a module for itself by left action, is semisimple. Also, we have an action of $A \otimes A^{\mathrm{opp}}$ on A given by the morphism t from $A \otimes A^{\mathrm{opp}}$ into $\mathrm{End}_k(A)$ defined as follows:

$$t(a \otimes b) : x \longmapsto axb\,, \quad a, x \in A,\ b \in A^{\mathrm{opp}}\,.$$

PROPOSITION 6.2.1 *The following are equivalent*:

(i) A *is* CS.
(ii) $t : A \otimes A^{\mathrm{opp}} \simeq \mathrm{End}_k(A)$.
(iii) $\mathrm{ctr}(A) = k$ *and* A *is semisimple.*
(iv) $A = M^r(k) \otimes K$ *where* K *is a division algebra with* $\mathrm{ctr}(K) = k$.
(v) $\mathrm{ctr}(A) = k$ *and* A *has no proper nonzero two-sided ideal.*

In this case A has a unique simple module with commutant D and $A \simeq M^r(k) \otimes D^{\mathrm{opp}}$. Moreover, if M is any module for A and B is the commutant of A in M, then the natural map $A \longrightarrow \mathrm{End}_B(M)$ is an isomorphism,

$$A \simeq \mathrm{End}_B(M)\,.$$

Finally, in (iv), *K^{opp} is the commutant of A in its simple modules.*

An algebra A over k is *central* if its center is k, and *simple* if it has no (proper) nonzero two-sided ideal. Thus CS is the same as central and simple. Two central simple algebras over k are *similar* if the division algebras that are the commutants of their simple modules are isomorphic. This is the same as saying that they are both of the form $M^r(k) \otimes K$ for the same central division algebra K but possibly different r. Similarity is a coarser notion of equivalence than isomorphism since A and $M^r(k) \otimes A$ are always similar. Write $[A]$ for the similarity class of A. Since $M^r(k)$ has zero divisors as soon as $r > 1$, $M^r(k) \otimes K$ and K cannot both be division algebras unless $r = 1$, and so it follows that for central *division algebras* similarity and isomorphism coincide. Thus each similarity class contains a unique *isomorphism class* of central division algebras. On the set of similarity classes we now define a multiplication, the so-called *Brauer multiplication*, by the rule

$$[A]\cdot[B] = [A \otimes B]\,.$$

Since

$$(M^r(k) \otimes A) \otimes (M^s(k) \otimes B) = M^{rs}(k) \otimes (A \otimes B)\,,$$

it follows that Brauer multiplication is well-defined. In particular, if E, F are two central division algebras, there is a central division algebra G such that $E \otimes F$ is the full matrix algebra over G, and

$$[E]\cdot[F] = [G]\,.$$

The relations

$$[M^r(k) \otimes A] = [A]\,, \quad A \otimes B \simeq B \otimes A\,, \quad A \otimes A^{\mathrm{opp}} \simeq M^r(k)\,, \quad r = \dim(A)\,,$$

show that Brauer multiplication converts the set of similarity classes into a *commutative group* with $[k]$ as its identity element and $[A^{\mathrm{opp}}]$ as the inverse of $[A]$. This group is called the *Brauer group of the field k* and is denoted by $\mathrm{Br}(k)$. If k is algebraically closed, we have $\mathrm{Br}(k) = 1$ since every CS algebra over k is a full matrix algebra. For $k = \mathbf{R}$ we have

$$\mathrm{Br}(\mathbf{R}) = \mathbf{Z}_2\,.$$

In fact, $\mathbf{R}$ and $\mathbf{H}$ are the only central division algebras over $\mathbf{R}$ (note that $\mathbf{C}$ as an $\mathbf{R}$-algebra is not central), and $\mathbf{H}$ is therefore isomorphic to its opposite. Hence the square of the class of $\mathbf{H}$ is 1. For our purposes we need a super version of Brauer's theory because the Clifford algebras are CS only in the super category. However, the entire discussion above may be extended to the super case and will lead to a treatment of the Clifford modules from the perspective of the theory of the super Brauer group. The theory of the super Brauer group was first worked out by C. T. C. Wall.[5] Our discussion follows Deligne's.[3]

Central Simple (CS) Superalgebras over a Field. Recall from the remark following Proposition 5.3.1 the notion of a super division algebra, namely, a superalgebra whose nonzero homogeneous elements are invertible. For any field k and any $u \in k^\times$ let $D = D_{k,u}$ be the super division algebra $k[\varepsilon]$ where ε is odd and $\varepsilon^2 = u$. It is obvious that the isomorphism class of $D_{k,u}$ depends only on the image of u in $k^\times/k^{\times 2}$. Clearly

$$D_{k,u}^{\mathrm{opp}} = D_{k,-u}\,.$$

In particular, if $D_k := D_{k,1}$, then $D^{\mathrm{opp}} = k[\varepsilon^0]$ where ε^0 is odd and ${\varepsilon^0}^2 = -1$. If k is algebraically closed, D_k is the only super division algebra apart from k. To see this, let B be a super division algebra over k algebraically closed. If u is an odd nonzero element, it is invertible and so multiplication by u is an isomorphism of B_1 with B_0. But B_0 is an ordinary division algebra over k and so is k itself, so that $\dim(B_1) = 1$. Since u^2 is nonzero and even, we have $u^2 = a1$, and so replacing u by $\varepsilon = a^{-1/2}u$, we see that $B = D_k$. If there is no ambiguity about k, we write D for D_k. Because of this result we have

$$D \simeq D^{\mathrm{opp}} \quad (k \text{ algebraically closed })\,.$$

In imitation of the classical case and guided by the Clifford algebras, we define a superalgebra A over k to be *central simple* (CS) if

$$\text{(CS)} \qquad A_{\bar{k}} \simeq M^{r|s}(\bar{k}) \quad \text{or} \quad A_{\bar{k}} \simeq M^n(\bar{k}) \otimes D_{\bar{k}}\,.$$

From our results on Clifford algebras, namely, Theorems 5.3.3 and 5.3.8, we see that the Clifford algebra $C(V)$ of a quadratic vector space over k is always central simple in the super category. We shall prove presently the super version of Proposition 6.2.1 that will allow us to define the notions of similarity for CS superalgebras and of Brauer multiplication between them, and prove that this converts the set of similarity classes of CS superalgebras over a field into a commutative group, the super Brauer group of the field.

We begin with some preparatory lemmas. If A, B are superalgebras over k and V, W are modules for A, B, respectively, recall that $V \otimes W$ is a module for $A \otimes B$ if we define

$$(a \otimes b)(v \otimes w) = (-1)^{p(b)p(v)} av \otimes bw\,.$$

Let A be a subsuperalgebra of $\mathbf{End}_k(V)$. The *supercommutant* A' of A is the superalgebra whose homogeneous elements x are defined by

$$ax = (-1)^{p(a)p(x)} xa\,, \quad a \in A\,.$$

We must distinguish this from the superalgebra, denoted by A'_u, which is the ordinary commutant of the ungraded algebra A_u underlying A, namely, consisting of elements $x \in \mathrm{End}_k(V)$ such that $ax = xv$ for all $a \in A$. Note, however, that A' and A'_u have the same even part. If A is a superalgebra and V a supermodule for A, we write A_V for the image of A in $\mathbf{End}_k(V)$.

LEMMA 6.2.2 *We have*

$$(A \otimes B)'_{V \otimes W} = A'_V \otimes B'_W\,.$$

Furthermore,

$$\mathrm{sctr}(A \otimes B) = \mathrm{sctr}(A) \otimes \mathrm{sctr}(B)\,.$$

PROOF: It is an easy check that $A' \otimes B' \subset (A \otimes B)'$. We shall now prove the reverse inclusion. First we shall show that

$$(*) \qquad (A \otimes 1)' = A' \otimes \mathbf{End}_k(W)\,.$$

Let $c = \sum_j a_j \otimes b_j \in (A \otimes 1)'$ where the b_j are linearly independent in $\mathbf{End}_k(W)$. Then $c(a \otimes 1) = (-1)^{p(c)p(a)}(a \otimes 1)c$ for a in A. Writing this out and observing that $p(c) = p(a_j) + p(b_j)$ for all j, we get

$$\sum_j (-1)^{p(a)p(b_j)}\big[a_j a - (-1)^{p(a)p(a_j)} a a_j\big] \otimes b_j = 0\,.$$

The linear independence of the b_j implies that $a_j \in A'$ for all j, proving $(*)$. If now $c \in (A \otimes B)'$, we can write $c = \sum_j a_j \otimes b_j'$ where the a_j are in A' and linearly independent. Proceeding as before but this time writing out the condition that $c \in (1 \otimes B)'$, we get $b_j \in B'$ for all j. Hence $c \in A' \otimes B'$. The second assertion is proven in a similar fashion. □

Our next result is the Wedderburn theorem in the super context.

LEMMA 6.2.3 *Let A be a superalgebra and V a semisimple module for A. Then, with primes denoting commutants,*

$$A_V = A_V''\,.$$

PROOF: We may assume that $A = A_V$. Let v_j $(1 \le j \le N)$ be homogeneous nonzero elements in V. It is enough to prove that if $L \in A''$, then there is $a \in A$ such that $av_j = Lv_j$ for all j. Consider first the case when $N = 1$. Since V is a direct sum of simple subsupermodules, it follows as in the classical case that any subsupermodule W has a complementary supermodule and hence there is a projection $V \longrightarrow W$, necessarily even, that lies in A'. Applying this to the submodule $W = Av_1$, we see that there is a projection $P(V \longrightarrow Av_1)$ that lies in A'. By assumption L commutes with P and so L leaves Av_1 invariant, i.e., $Lv_1 \in Av_1$. This proves the assertion for $N = 1$. Let now $N > 1$. Consider $V^N = V \otimes U$ where U is a super vector space with homogeneous basis $(e_j)_{1 \le j \le N}$ where e_j has the same parity as v_j. Then V^N, being the direct sum of the $V \otimes ke_j$, is itself semisimple. By Lemma 6.2.2, $(A \otimes 1)'' = A'' \otimes 1$. Let $v = \sum_j v_j \otimes e_j$. Then v is even and by what has been proven above, given $L \in A''$, we can find $a \in A$ such that $(L \otimes 1)v = (a \otimes 1)v$, i.e.,

$$\sum Lv_j \otimes e_j = \sum av_j \otimes e_j\,.$$

This implies that $Lv_j = av_j$ for all j, finishing the proof. □

LEMMA 6.2.4 *If A is a superalgebra and M a simple supermodule for A, then the supercommutant of A_M is a super division algebra. If B is a super division algebra over k that is not purely even, and V is a super vector space, then*

$$\mathbf{End}_k(V) \otimes B \simeq \mathrm{End}_k(V') \otimes B$$

where V' is the ungraded vector space V and $\mathrm{End}_k(V')$ *is the purely even algebra of all endomorphisms of V'. In particular,*

$$M^{r|s}(k) \otimes B \simeq M^{r+s}(k) \otimes B \,.$$

PROOF: Let L be a homogeneous element of A'_M. Then the kernel and image of L are subsupermodules and the argument is then the same as in the classical Schur lemma. For the second assertion we begin by regarding B as a module for B^{opp} by

$$b \cdot b' = (-1)^{p(b)p(b')} b'b \quad (b \in B^{\mathrm{opp}}, b' \in B)\,.$$

Clearly, the commutant of this module is B acting by left multiplication on itself. By Lemma 6.2.2 we therefore have, on $V \otimes B$,

$$(1 \otimes B^{\mathrm{opp}})' = \mathbf{End}_k(V) \otimes B\,.$$

Choose now $\eta \neq 0$ in B_1. Let v_i form a basis of V with v_i $(i \leq r)$ even and v_i $(i > r)$ odd. Then, the *even* elements

$$v_1 \otimes 1, \dots, v_r \otimes 1,\ v_{r+1} \otimes \eta, \dots, v_{r+s} \otimes \eta$$

form a B^{opp}-basis of $V \otimes B$. This implies that

$$V \otimes B \simeq V' \otimes B$$

as B^{opp}-modules. Since the commutant of $1 \otimes B^{\mathrm{opp}}$ in $V' \otimes B$ is $\mathrm{End}_k(V') \otimes B$, the result follows. ☐

It follows easily from the definition that if A, B are CS superalgebras, then so are $A \otimes B$ and A^{opp}. To see this, write $D_k = D_{k,1}$. We then have the following:

$$M^{r|s}(k) \otimes M^{p|q}(k) \simeq M^{rp+sq|rq+sp}(k)$$
$$M^{r|s}(k) \otimes (M^n(k) \otimes D_k) \simeq M^{nr|ns}(k) \otimes D_k \simeq M^{n(r+s)}(k) \otimes D_k$$
$$(M^m(k) \otimes D_k) \otimes (M^n(k) \otimes D_k)^{\mathrm{opp}} \simeq M^{mn}(k) \otimes M^{1|1}(k) \simeq M^{mn|mn}(k)\,.$$

Taking $\bar{k}$ instead of k and remembering that $D^{\mathrm{opp}} \simeq D$, we see that $A \otimes B$ is CS if A, B are CS. In the second relation we are using Lemma 6.2.4. The verification of the third comes down to seeing that

$$D_k \otimes D_k^{\mathrm{opp}} \simeq M^{1|1}\,.$$

This last relation is proven as follows: For any superalgebra A, we have an action $t = t_A$ of $A \otimes A^{\mathrm{opp}}$ on A given by

$$t(a \otimes b)(x) = (-1)^{p(b)p(x)} axb\,, \quad a, x \in A,\ b \in A^{\mathrm{opp}}\,.$$

Thus

$$t : A \otimes A^{\mathrm{opp}} \longrightarrow \mathbf{End}_k(A)$$

is a morphism of superalgebras. In the special case when $A = D_k$ we can compute t explicitly and verify that it is an isomorphism. In the basis $1, \varepsilon$ for D_k we have, for the action of t,

$$t(1\otimes 1)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad t(1\otimes\varepsilon)=\begin{pmatrix}0&-1\\1&0\end{pmatrix},$$
$$t(\varepsilon\otimes 1)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad t(\varepsilon\otimes\varepsilon)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

So the result is true in this case. To see that the opposite of a CS superalgebra is also CS, we first prove that

$$\mathbf{End}_k(V)^{\text{opp}}\simeq\mathbf{End}_k(V).$$

Let V^* be the dual of V and for $T\in\mathbf{End}_k(V)$ let us define $T^*\in\mathbf{End}_k(V^*)$ by

$$(T^*v^*)(v)=(-1)^{p(T^*)p(v^*)}v^*(Tv).$$

It is then easily checked that $p(T^*)=p(T)$ and

$$(T_1T_2)^*=(-1)^{p(T_1)p(T_2)}T_2^*T_1^*,$$

which proves that the map $T\longmapsto T^*$ is an isomorphism of $\mathbf{End}_k(V)$ with $\mathbf{End}_k(V^*)^{\text{opp}}$. However, we have, noncanonically, $V\simeq V^*$, and so

$$\mathbf{End}_k(V)\simeq\mathbf{End}_k(V^*)^{\text{opp}}\simeq\mathbf{End}_k(V)^{\text{opp}}.$$

Next, as $D\simeq D^{\text{opp}}$ for k algebraically closed, we have

$$(M^n(k)\otimes D)^{\text{opp}}\simeq M^n(k)\otimes D,$$

where we are using the easily proven fact that $(A\otimes B)^{\text{opp}}\simeq A^{\text{opp}}\otimes B^{\text{opp}}$.

We shall now prove the super version of Proposition 6.2.1. Recall from Section 5.3 that for a superalgebra, the complete reducibility of all its modules is equivalent to the complete reducibility of the left regular representation, and that we have called such superalgebras *semisimple*. Let us write $M^{r|s}=M^{r|s}(\bar{k})$, $D=D_{\bar{k},1}$.

THEOREM 6.2.5 *Let k be of characteristic* 0. *The following are equivalent*:

(i) *A is* CS.
(ii) $t: A\otimes A^{\text{opp}}\longrightarrow\mathbf{End}_k(A)$ *is an isomorphism.*
(iii) $\operatorname{sctr}(A)=k$ *and the ungraded algebra A is semisimple.*
(iv) $\operatorname{sctr}(A)=k$ *and the super algebra A is semisimple.*
(v) $\operatorname{sctr}(A)=k$ *and A has no proper nonzero two-sided homogeneous ideal.*
(vi) $A=M^{r|s}(k)\otimes K$ *where K is a super division algebra with* $\operatorname{sctr}(K)=k$.
(vii) $\operatorname{sctr}(A)=k$ *and A has a faithful semisimple representation.*

PROOF: (i) $\Longrightarrow$ (ii). Since the map t is already well-defined, the question of its being an isomorphism can be settled by working over $\bar{k}$. Hence we may assume that k is already algebraically closed. We consider two cases.

Case 1. $A\simeq M^{r|s}$. Let E_{ij} be the matrix units with respect to the usual homogeneous basis of $k^{r|s}$. Then

$$t(E_{ij}\otimes E_{q\ell}): E_{mn}\longmapsto(-1)^{[p(q)+p(\ell)][p(m)+p(n)]}\delta_{jm}\delta_{nq}E_{i\ell},$$

and so $t(E_{ij}\otimes E_{q\ell})$ takes E_{jq} to $\pm E_{i\ell}$ and E_{mn} to 0 if $(m,n)\neq(j,q)$. This proves that the image of t is all of $\mathbf{End}_{\bar{k}}(A)$. Computing dimensions, we see that t is an isomorphism.

Case 2. $A \simeq \mathrm{End}_k(V) \otimes D$ where V is a purely even vector space. We have already verified that t is an isomorphism when $V = k$, i.e., $A = D$. If we write $t_{A\otimes B}, t_A, t_B$ for the maps associated to $A\otimes B$, A, B, then a simple calculation shows (after the identifications $(A \otimes B) \otimes (A \otimes B)^{\mathrm{opp}} \simeq (A \otimes A^{\mathrm{opp}}) \otimes (B \otimes B^{\mathrm{opp}})$ and $t_{A\otimes B} \simeq t_A \otimes t_B$) that

$$t_{A\otimes B} = t_A \otimes t_B \,.$$

Hence the result for $A = \mathrm{End}_k(V) \otimes D$ follows from those for End_k and D.

(ii) $\Longrightarrow$ (iv). Let $x \in \mathrm{sctr}(A)$. Then $xa = (-1)^{p(x)p(a)}xa$ for all $a \in A$. We now assert that $x \otimes 1$ is in the supercenter of $A \otimes A^{\mathrm{opp}}$. In fact,

$$\begin{aligned}(x \otimes 1)(a \otimes b) &= xa \otimes b = (-1)^{p(x)p(a)}ax \otimes b \\ &= (-1)^{p(x)p(a\otimes b)}(a \otimes b)(x \otimes 1)\,,\end{aligned}$$

proving our claim. By (ii), the supercenter of $A \otimes A^{\mathrm{opp}}$ is k and so $x \otimes 1 \in k$, showing that $x \in k$. We shall now show that the left regular representation of the superalgebra A is completely reducible. Let L be a (graded) subspace of A stable and irreducible under left translations. Then, under our assumption (ii), the spaces $t(a\otimes b)[L] = Lb$ span A as b varies among the homogeneous elements of A. This means that the spaces Lb span A. Right multiplication by b is a map of L with Lb commuting with the left action, and so Lb is a quotient of L or ΠL according as b is even or odd, thus irreducible as a supermodule for the left regular representation. Thus A is the sum of simple subsupermodules for the left action and hence A is semisimple.

(iv) $\Longrightarrow$ (ii). We begin by remarking that if L and M are simple nonzero subsupermodules of A under the left action, then $M = Lb$ for some homogeneous $b \in A$ if and only if either $M \simeq L$ or $M \simeq \Pi L$. Indeed, if $M = Lb$, then right multiplication by b is a nonzero element of $\mathrm{Hom}_A(L, M)$ if b is even and $\mathrm{Hom}_A(\Pi L, M)$ if M is odd, and hence is an isomorphism. For the reverse result, write $A = L \oplus L_1 \cdots$ where $L, L_1, \ldots$ are simple subsupermodules of A for the left action. Let $T(L \longrightarrow M)$ be a homogeneous linear isomorphism $L \simeq M$ as A-modules. Define T as 0 on $L_1, L_2, \ldots$. Then T is homogeneous and commutes with left action. If $T1 = b$, then b is homogeneous and $Ta = ab$ for $a \in L$. Hence $M = Lb$ as we wished to show.

This said, let us write $A = \oplus A_i$ where the A_i are simple subsupermodules of A for the left action. We write $i \sim j$ if A_i is isomorphic under left action to A_j or ΠA_j. This is the same as saying, in view of our remark above, that for some homogeneous b, $A_j = A_i b$; $\sim$ is an equivalence relation. Let $I, J, \ldots$ be the equivalence classes and $A_I = \bigoplus_{i\in I} A_i$. Each A_I is graded, and A_I does not change if we start with another A_j with $i \sim j$. Moreover, A_I is invariant under left as well as right multiplication by elements of A and so invariant under the action of $A \otimes A^{\mathrm{opp}}$. We now claim that each A_I is *irreducible* as a supermodule under the action of $A \otimes A^{\mathrm{opp}}$. To show this, it is enough to prove that if M is a graded subspace of A_I stable and irreducible under the left action, then the subspaces Mb for homogeneous b span A_I. Now A_I is a sum of submodules all equivalent to A_i or ΠA_i for some $i \in I$, and so M has to be equivalent to A_i or ΠA_i also. So, by the remark made at the outset, $A_i = Mb_0$ for some homogeneous b_0; but then because the $A_i b$ span A_I, it is clear that the Mb span A_I. Thus A_I is a simple module for $A \otimes A^{\mathrm{opp}}$. Since $A = \sum_I A_I$, it follows that the action of $A \otimes A^{\mathrm{opp}}$ on A is semisimple. So Lemma 6.2.3 is applicable to the image R of $A \otimes A^{\mathrm{opp}}$ in $\mathbf{End}_k(A)$. Let $T \in R'$ and $T1 = \ell$. Then, for all $x \in A$,

$$Tx = Tt(x \otimes 1)1 = t(x \otimes 1)\ell = x\ell\,.$$

The condition on T is that

$$(*) \qquad t(a \otimes b)T = (-1)^{p(T)p(t(a\otimes b))}Tt(a \otimes b)$$

for all $a, b \in A$. Since $t(a \otimes b)(x) = \pm axb$, it follows that $p(t(a \otimes b)) = p(a \otimes b) = p(a) + p(b)$. Moreover, because $T1 = \ell$, we have $p(T) = p(\ell)$. Hence applying both sides of $(*)$ to 1, we get

$$(-1)^{p(b)p(\ell)}a\ell b = (-1)^{p(\ell)[p(a)+p(b)]}T(ab)\,.$$

Taking $a = 1$ we see that $Tb = \ell b$ so that the above equation becomes

$$(-1)^{p(b)p(\ell)}a\ell b = (-1)^{p(\ell)[p(a)+p(b)]}\ell ab\,.$$

Taking $b = 1$ we get $a\ell = (-1)^{p(a)p(\ell)}\ell a$, showing that ℓ lies in the supercenter of A. So $\ell \in k$, showing that T is a scalar. But then $R'' = \mathbf{End}_k(A)$ so that the map $t : A \otimes A^{\mathrm{opp}} \longrightarrow \mathbf{End}_k(A)$ is surjective. By counting dimensions we see that this must be an isomorphism. Thus we have (ii).

(iv) $\Longrightarrow$ (v). It is enough to prove that (v) follows from (ii). But, under (ii), A, as a module for $A \otimes A^{\mathrm{opp}}$, is simple. Since two-sided homogeneous ideals are stable under t, we get (v).

(v) $\Longrightarrow$ (vi). By (v) we know that any nonzero morphism of A into a superalgebra is faithful. Take a simple module M for A. Its supercommutant is a super division algebra D, and by Lemma 6.2.3 we have $A_M = \mathbf{End}_D(M)$. The map $A \longrightarrow A_M$ is faithful and so

$$A \simeq \mathbf{End}_D(M)\,.$$

This implies that

$$A \simeq M^{r|s}(k) \otimes K\,, \qquad K = D^{\mathrm{opp}}\,.$$

Since the supercenter of a product is the product of supercenters, we see that the supercenter of K must reduce to k. Thus we have (vi).

(vi) $\Longrightarrow$ (i). It is enough to prove that if K is a super division algebra whose supercenter is k, then K is CS. Now the left action of K on itself is simple and so semisimple. Thus K is semisimple. We now pass to the algebraic closure $\overline{k}$ of k. Then $\overline{K} = K_{\overline{k}}$ is semisimple and has supercenter $\overline{k}$. Thus $\overline{K}$ satisfies (iv), and hence (v) so that any nonzero morphism of $\overline{K}$ is faithful. Let M be a simple module for $\overline{K}$ and E the super commutant in M. Then, with $F = E^{\mathrm{opp}}$, $\overline{K} \simeq M^{r|s} \otimes F$. For F there are only two possibilities: $F = \overline{k}$, D. In the first case $\overline{K} \simeq M^{r|s}$, while in the second case $\overline{K} \simeq M^{r+s} \otimes D$ by Lemma 6.2.4. Hence $\overline{K}$ is CS.

(iii) $\Longleftrightarrow$ (i). It is enough to prove (i) $\Longrightarrow$ (iii) when k is algebraically closed. It is only a question of the semisimplicity of A as an ungraded algebra. If $A = M^{r|s}$, then the ungraded A is M^{r+s} and so the result is clear. If $A = M^n \otimes D$, then it is a question of proving that the ungraded D is semisimple. But as an ungraded algebra, $D \simeq k[u]$ where $u^2 = 1$ and so $D \simeq k \oplus k$, hence semisimple.

For the converse, let us suppose (iii) is true. Let us write A_u for A regarded as an ungraded algebra. We first argue as in the proof of (iv) $\Longrightarrow$ (ii) above that A_u is semisimple as a module for $A_u \otimes A_u^{\rm opp}$. Take now a filtration of homogeneous left ideals

$$A_0 = A \supset A_1 \supset \cdots \supset A_r \supset A_{r+1} = 0$$

where each $M_i := A_i / A_{i+1}$ is a simple supermodule. Let R be the set of elements that map to the zero endomorphism in each M_i. Then R is a homogeneous two-sided ideal. If $x \in R$, then $x A_i \subset A_{i+1}$ for all i, and so $x^r = 0$. Now, the ungraded algebra A_u is semisimple by assumption. Hence, because R is stable under $A_u \otimes A_u^{\rm opp}$, we can find a two-sided ideal R' such that $A = R \oplus R'$. Since $RR' \subset R \cap R' = 0$, we have $RR' = R'R = 0$. Write $1 = u + u'$ where $u \in R$, $u' \in R'$. Then $uu' = u'u = 0$ and so $1 = (u + u')^r = u^r + u'^r = u'^r$, showing that $1 \in R'$. Hence $R = R1 = 0$. This means that A_u, and hence A, acts faithfully in $\bigoplus_i M_i$.

The kernel of A in M_i and M_j are the same if either $M_i \simeq M_j$ or $M_i \simeq \Pi M_j$. Hence, by omitting some of the M_i we may select a subfamily M_i $(1 \le i \le s)$ such that for $i \neq j$ we have $M_i \not\simeq M_j, \Pi M_j$, and that A acts faithfully on $M = \bigoplus_{1 \le i \le s} M_i$. We may thus suppose that $A = A_M$. Let $P_i(M \longrightarrow M_i)$ be the corresponding (even) projections. If A'_u is the ordinary commutant of A_u, it is clear that $P_i \in A'_u$ for all i. We claim that $P_i \in (A'_u)'_u$ for all i. Let $S \in A'_u$ be homogeneous. Then $S[M_i]$ is a supermodule for A that is a quotient of M_i or ΠM_i, and so is either 0 or equivalent to M_i or ΠM_i. Hence it cannot be equivalent to any M_j for $j \neq i$, and so $S[M_i] \subset M_i$ for all i. So S commutes with P_i for all i. Thus $P_i \in (A'_u)'_u$ for all i. But because A_u is semisimple, we have $A_u = (A'_u)'_u$ and so $P_i \in A$ for all i. Hence $P_i \in A \cap A' = \mathrm{sctr}(A) = k$ for all i. Thus there is only one index i and $P_i = 1$ so that M is simple. But then $A = \mathrm{End}_K(M)$ by Lemma 6.2.3 where K is the supercommutant of A in M. Hence $A = M^{r|s} \otimes B$ where $B = K^{\rm opp}$. B is a super division algebra with supercenter k, and so we have (vi). But, then because (vi) implies (i), we are done.

(vii) $\Longleftrightarrow$ (i). The argument in the preceding implication actually proves that (vii) implies (i). The reverse is trivial since the left action of A on itself is semisimple and faithful if A is CS.

This completes the proof of the entire theorem. □

THEOREM 6.2.6 *Let k be arbitrary and A a* CS *superalgebra over k. Let M be any module for A, and let B be the commutant of A in M. Then the natural map $A \longrightarrow \mathrm{End}_B(M)$ is an isomorphism*:

$$A \simeq \mathrm{End}_B(M)\,.$$

Moreover, the supercommutants in the simple modules for A are all isomorphic. If B is such a commutant, then B is a super division algebra with supercenter k, and $A \simeq M^{r|s}(k) \otimes B^{\rm opp}$. Finally, if $A = M^{r|s}(k) \otimes K$ where K is a super division algebra with supercenter k, $K^{\rm opp}$ is the commutant of A in its simple modules.

PROOF: The first assertion is immediate from Lemma 6.2.3 since A acts faithfully on M and is semisimple by Theorem 6.2.5. To prove the second assertion, let M, N be two simple modules for A. Let $\overline{M}, \overline{N}$ be their extensions to $\overline{k}$ as modules for $\overline{A} := A_{\overline{k}}$. We consider two cases.

Case 1. $\overline{A} \simeq \mathbf{End}(V)$ where V is a super vector space over $\overline{k}$. Then $\overline{M} \simeq V \otimes R$, $\overline{N} \simeq V \otimes S$ where R, S are super vector spaces. Unless one of R, S is purely even and the other purely odd, we have $\mathrm{Hom}_{\overline{A}}(\overline{M}, \overline{N}) \neq 0$. Hence $\mathrm{Hom}_A(M, N) \neq 0$, and so because M and N are simple, we must have $M \simeq N$. In the exceptional case we replace N by ΠN to conclude as before that $M \simeq \Pi N$. So to complete the proof we must check that the commutants of A in M and ΠM are isomorphic. But parity reversal does not change the action of A and hence does not change the supercommutant.

Case 2. $A_{\overline{k}} \simeq \mathrm{End}_{\overline{k}}(V) \otimes D$ where V is a purely even vector space. In this case we have seen that there is a unique simple module and so the argument proceeds as before.

For the last assertion let $A = M^{r|s}(k) \otimes K$ where K is a super division algebra with k as supercenter. Let $M = k^{r|s} \otimes K$ be viewed as a module for A in the obvious manner, K acting on K by left multiplication. It is easy to check that this is a simple module. The commutant is $1 \otimes K^{\mathrm{opp}} \simeq K^{\mathrm{opp}}$ as we wanted to show. □

6.3. The Super Brauer Group of a Field

Let k be arbitrary. We have seen that if A is a CS superalgebra, then A is isomorphic to $M^{r|s}(k) \otimes B$ where B is a CS super division algebra, i.e., a super division algebra with supercenter k. B is also characterized by the property that B^{opp} is the super commutant of A in its simple modules. Two CS superalgebras A_1, A_2 are said to be *similar* if their associated division algebras are isomorphic, i.e., if $A_i \simeq M^{r_i|s_i}(k) \otimes D$ where D is a central super division algebra. Similarity is an equivalence relation that is coarser than isomorphism, and the similarity class of A is denoted by $[A]$. We define Brauer multiplication of the similarity classes as before by

$$[A] \cdot [B] = [A \otimes B].$$

It is obvious that this depends only on the classes and not on the representative superalgebras in the class. This is a commutative product and has $[k]$ as the unit element. The relation

$$A \otimes A^{\mathrm{opp}} \simeq \mathbf{End}_k(A)$$

shows that $[A^{\mathrm{opp}}]$ is the inverse of $[A]$. Thus the similarity classes form a commutative group. This is the *super Brauer group* of k, denoted by $\mathrm{sBr}(k)$. Our goal in this section is to get information about the structure of $\mathrm{sBr}(k)$ and the subset of classes of the Clifford algebras inside it.

First of all we have, for k algebraically closed,

$$\mathrm{sBr}(k) = \{[k], [D]\} = \mathbf{Z}_2.$$

In fact, this is clear from the fact that

$$(M^n \otimes D) \otimes (M^n \otimes D) \simeq M^{n^2} \otimes (D \otimes D^{\mathrm{opp}}) \simeq M^{n^2} \otimes M^{1|1} \simeq M^{n^2|n^2}$$

so that $[M^n \otimes D]^2 = 1$. For arbitrary k, going from k to $\bar{k}$ gives a homomorphism

$$\mathrm{sBr}(k) \longrightarrow \mathbf{Z}_2 .$$

This is surjective because $[D_k]$ goes over to $[D]$. The kernel of this map is the subgroup H of $\mathrm{sBr}(k)$ of those similarity classes of CS superalgebras that become isomorphic to $M^{r|s}$ over $\bar{k}$. For example, the Clifford algebras of even-dimensional quadratic vector spaces belong to H. In what follows when we write $A \in H$ we really mean $[A] \in H$. Our aim is to examine H.

Fix $A \in H$. Then, $\overline{A} = A_{\bar{k}} \simeq \mathbf{End}_{\bar{k}}(S)$ and so, over $\bar{k}$, $\overline{A}$ has two simple supermodules, namely S and ΠS. Let $\dim(S) = r|s$ and let

$$I(A) = \{S, \Pi S\} .$$

Changing S to ΠS, we may assume that $r > 0$. We may view elements of $I(A)$ as representations of A over $\bar{k}$. Let L denote one of these and let $\sigma \in G_k := \mathrm{Gal}(\bar{k}/k)$. In S (or ΠS) we take a homogeneous basis and view L as a morphism of A into $M^{r|s}(\bar{k})$. Then $a \longmapsto L(a)^\sigma$ is again a representation of A in $\bar{k}$, and its equivalence class does not depend on the choice of the basis used to define L^σ. L^σ is clearly simple and so is isomorphic to either S or ΠS. Hence G_k acts on $I(A)$, and so we have a map α_A from G_k to $\mathbf{Z}_2$ identified with the group of permutations of $I(A)$. If A is purely even, i.e., $s = 0$, then it is clear that $S^\sigma \simeq S$ for any $\sigma \in G_k$. So $\alpha_A(\sigma)$ acts as the identity on $I(A)$ for all σ for such A. Suppose now that A is not purely even so that $r > 0, s > 0$. Let Z^+ be the center of A^+ and $\overline{Z}^+$ its extension to $\bar{k}$, the center of $\overline{A}^+$. Then $\overline{Z}^+$ is canonically isomorphic, over $\bar{k}$, to $\bar{k} \oplus \bar{k}$, and has two characters χ_1, χ_2 where the notation is chosen so that $\overline{Z}^+$ acts on S by $\chi_1 \oplus \chi_2$; then it acts on ΠS by $\chi_2 \oplus \chi_1$. So in this case we can identify $I(A)$ with $\{\chi_1, \chi_2\}$ so that $S \mapsto \Pi S$ corresponds to $(\chi_1, \chi_2) \mapsto (\chi_2, \chi_1)$. Now G_k acts on $\overline{Z}^+$ and hence on $\{\chi_1, \chi_2\}$, and this action corresponds to the action on $I(A)$. In other words, if we write, for any $\bar{k}$-valued character χ of Z^+, χ^σ for the character

$$\chi^\sigma(z) = \chi(z)^\sigma , \quad z \in Z^+ ,$$

then σ fixes the elements of $I(A)$ or interchanges them according as

$$(\chi_1^\sigma, \chi_2^\sigma) = (\chi_1, \chi_2) \quad \text{or} \quad (\chi_1^\sigma, \chi_2^\sigma) = (\chi_2, \chi_1) .$$

PROPOSITION 6.3.1 *The map $A \longmapsto \alpha_A$ is a homomorphism of H into the group* $\mathrm{Hom}(G_k, \mathbf{Z}_2)$. *It is surjective and its kernel K is* $\mathrm{Br}(k)$. *In particular, we have an exact sequence*

$$1 \longrightarrow \mathrm{Br}(k) \longrightarrow H \longrightarrow k^\times/(k^\times)^2 \longrightarrow 1 .$$

PROOF: For any simple module S of a superalgebra A, the identity map is an odd bijection interchanging S with ΠS, while for arbitrary linear homogeneous maps we have $p(x \otimes y) = p(x) + p(y)$. So, if $A_1, A_2 \in H$ and $\{S_i, \Pi S_i\}$ are the simple modules for A_i, then $A_1 \otimes A_2 \in H$ and its simple modules are $S_1 \otimes S_2 \simeq \Pi S_1 \otimes \Pi S_2$, $\Pi(S_1 \otimes S_2) \simeq S_1 \otimes \Pi S_2 \simeq \Pi S_1 \otimes S_2$. This shows that $\alpha_{A_1 \otimes A_2}(\sigma) = \alpha_{A_1}(\sigma)\alpha_{A_2}(\sigma)$ for all $\sigma \in G_k$.

To prove the surjectivity of $A \longmapsto \alpha_A$, let $f \in \mathrm{Hom}(G_k, \mathbf{Z}_2)$. We may assume that f is not trivial. The kernel of f is then a subgroup of G_k of index 2 and so determines a quadratic extension $k' = k(\sqrt{a})$ of k for some $a \in k^\times \setminus k^{\times 2}$. We must find $A \in H$ such that the corresponding α_A is just f, i.e., $S^\sigma \simeq S$ if and only if σ fixes $b = \sqrt{a}$. Let $V = k \oplus k$ with the quadratic form $Q = x^2 - ay^2$. If f_1, f_2 is the standard basis for V, then $Q(f_1) = 1$, $Q(f_2) = -a$ while $\Phi(f_1, f_2) = 0$. Let $e_1 = bf_1 + f_2, e_2 = (1/4a)(bf_1 - f_2)$. Then, writing Q, Φ for the extensions of Q, Φ to $V' = k' \otimes_k V$, and remembering that $Q(x) = \Phi(x, x)$, we have $Q(e_1) = Q(e_2) = 0$ and $\Phi(e_1, e_2) = \frac{1}{2}$. The simple module S for $C(V')$ then has the basis $\{1, e_2\}$ with

$$e_1 \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since $2bf_1 = e_1 + 4ae_2, 2f_2 = e_1 - 4ae_2$, we have

$$f_1 \mapsto \begin{pmatrix} 0 & \frac{1}{2b} \\ \frac{2a}{b} & 0 \end{pmatrix}, \quad f_2 \longmapsto \begin{pmatrix} 0 & \frac{1}{2} \\ -2a & 0 \end{pmatrix}.$$

The algebra $C(V')^+ = k'[f_1 f_2]$ is already abelian and so coincides with $k' \otimes_k Z^+$. In the module S we have

$$f_1 f_2 \longmapsto \begin{pmatrix} -\frac{a}{b} & 0 \\ 0 & \frac{a}{b} \end{pmatrix}.$$

If now σ is the nontrivial element of $\mathrm{Gal}(k'/k)$, then σ changes b to $-b$, so that in S^σ we have

$$f_1 f_2 \longmapsto \begin{pmatrix} \frac{a}{b} & 0 \\ 0 & -\frac{a}{b} \end{pmatrix}.$$

Thus

$$S^\sigma \simeq \Pi S\,,$$

which is exactly what we wanted to show.

It remains to determine the kernel K of the homomorphism $A \longmapsto \alpha_A$. Certainly A is in K if it is purely even. Suppose that A is not purely even and $\overline{A}$ isomorphic to $M^{r|s}$ with $r > 0, s > 0$. Using the characters of $\overline{Z}^+$ to differentiate between S and ΠS, we see that for α_A to be the identity element of $\mathrm{Hom}(G_k, \mathbf{Z}_2)$, it is necessary and sufficient that $\chi_i^\sigma = \chi_i$ on Z^+, i.e., the χ_i take their values in k. So they are k-valued characters of Z^+. It is then obvious that the map $(\chi_1, \chi_2) : Z^+ \longrightarrow k \oplus k$ is an isomorphism. Conversely, if $Z^+ \simeq k \oplus k$, it is obvious that α_A is the identity. So we obtain the result that A lies in K if and only if either A is purely even or the center of its even part is isomorphic over k to $k \oplus k$.

We shall now prove that K is isomorphic to $\mathrm{Br}(k)$. For A in K let D be a super division algebra with supercenter k such that $[A] = [D]$. Then D^+, which is a division algebra over k, cannot contain a subalgebra isomorphic to $k \oplus k$, and so D must be purely even. For any purely even division algebra D with center k, the algebra $A = M^{r|s}(k) \otimes D$ is, for $s = 0$, purely even and is a classical central simple algebra in the similarity class of the central division algebra D, while for $s > 0$,

$$A^+ \simeq (M^{r|s}(k))^+ \otimes D \simeq (M^r(k) \otimes D) \oplus (M^s(k) \otimes D)\,,$$

and so its center is $\simeq k \oplus k$. Thus the elements of K are precisely the classical similarity classes of purely even division algebras with center k with multiplication as Brauer multiplication. So the kernel is isomorphic to $\mathrm{Br}(k)$.

To complete the proof it only remains to identify $\mathrm{Hom}(G_k, \mathbf{Z}_2)$ with $k^\times/(k^\times)^2$. The nontrivial elements in $\mathrm{Hom}(G_k, \mathbf{Z}_2)$ are in canonical bijection with the subgroups of G_k of index 2, and these in turn are in canonical bijection with the quadratic extensions of k, and so, by standard results in Galois theory, in correspondence with $k^\times/(k^\times)^2$. We need only verify that this correspondence is a group map. Given $a \in k^\times$, make a fixed choice of $\sqrt{a}$ in $\overline{k}$ and write $b = \sqrt{a}$. For $\sigma \in G_k$, b/b^σ is independent of the choice of the square root of a, and so it depends only on a. Let $\chi_a(\sigma) = b/b^\sigma$. Then, as $b^\sigma = \pm b$ it follows that χ_a takes values in $\mathbf{Z}_2$. Moreover, the map $a \longmapsto \chi_a$ is a group homomorphism and $\chi_a = 1$ if and only if $a \in (k^\times)^2$. Thus we have the group isomorphism

$$\frac{k^\times}{(k^\times)^2} \simeq \mathrm{Hom}(G_k, \mathbf{Z}_2)\,.$$

This finishes the proof. □

We suggested at the beginning of this chapter that when $k = \mathbf{R}$ the type of Clifford modules of a real quadratic vector space V depends only on the signature. For arbitrary k there is a similar result that relates the super Brauer group with the Witt group of the field. Recall that $W(k)$, *the Witt group of* k, is the group F/R where F is the free additive abelian group generated by the isomorphism classes of quadratic vector spaces over k and R is the subgroup generated by the relations

$$[V \oplus V_h] - [V] = 0$$

where V_h is hyperbolic, i.e., of the form $(V_1, Q_1) \oplus (V_1, -Q_1)$. If L is an abelian group and $V \longmapsto f(V)$ a map of quadratic spaces into L, it will define a morphism of $W(k)$ into L if and only if

$$f(V \oplus V_h) = f(V)\,.$$

We write $[V]_W$ for the Witt class of V. As an example let us calculate the Witt group of $\mathbf{R}$. Any real quadratic space V of signature (p, q) is isomorphic to $\mathbf{R}^{p,q}$; we write $\mathrm{sign}(V) = p - q$. It is obvious that in $W(\mathbf{R})$,

$$[\mathbf{R}^{0,1}]_W = -[\mathbf{R}^{1,0}]_W\,, \quad [\mathbf{R}^{p,q}]_W = (p-q)[\mathbf{R}^{1,0}]_W\,.$$

Clearly, $\mathrm{sign}(V_h) = 0$ and so $\mathrm{sign}(V \oplus V_h) = \mathrm{sign}(V)$. Thus sign induces a morphism s from $W(\mathbf{R})$ into $\mathbf{Z}$. We claim that this is an isomorphism. To see this, let t be the morphism from $\mathbf{Z}$ to $W(\mathbf{R})$ that takes 1 to $[\mathbf{R}^{1,0}]_W$. Clearly, $st(1) = 1$ and so st is the identity. Also, $s([\mathbf{R}^{p,q}]_W) = p - q$ so that

$$ts([\mathbf{R}^{p,q}]_W) = t(p-q) = (p-q)t(1) = (p-q)[\mathbf{R}^{1,0}]_W = [\mathbf{R}^{p,q}]_W$$

by what we saw above. So ts is also the identity. Thus

$$W(\mathbf{R}) \simeq \mathbf{Z}\,.$$

Now we have a map

$$V \longmapsto [C(V)]$$

from quadratic spaces into the super Brauer group of k, and we have already seen that $C(V_h)$ is a full matrix superalgebra over k. Hence $[C(V_h)]$ is the identity and so

$$[C(V \oplus V_h)] = [C(V) \otimes C(V_h)] = [C(V)] .$$

Thus by what we said above we have a map

$$f : W(k) \longrightarrow \mathrm{sBr}(k)$$

such that for any quadratic vector space V,

$$f([V]_W) = [C(V)] .$$

In summary we have the following theorem:

THEOREM 6.3.2 *For any field k of characteristic 0 let* $\mathrm{sBr}(k)$ *be the super Brauer group of k. We then have the exact sequences*

$$1 \longrightarrow H \longrightarrow \mathrm{sBr}(k) \longrightarrow \mathbf{Z}_2 \longrightarrow 1 ,$$

$$1 \longrightarrow \mathrm{Br}(k) \longrightarrow H \longrightarrow k^\times/(k^\times)^2 \longrightarrow 1 .$$

Moreover, if $W(k)$ is the Witt group of k, there is a map

$$f : W(k) \longrightarrow \mathrm{sBr}(k)$$

such that for any quadratic vector space V,

$$f([V]_W) = [C(V)]$$

where $[V]_W$ is the class of V in $W(k)$ and $[C(V)]$ is the class of $C(V)$ in $\mathrm{sBr}(k)$.

Representation Theory of the Even Parts of CS Superalgebras. For applications we need the representation theory of the algebra C^+ where C is a Clifford algebra. More generally, let us examine the representation theory of algebras A^+ where A is a CS superalgebra over k. If A is purely even, there is nothing more to do as we are already in the theory of the classical Brauer group. Thus all simple modules of A over k have commutants D^{opp} where D is the (purely even) central division algebra in the similarity class of A. So we may assume that A is not purely even. Then we have the following proposition.

THEOREM 6.3.3 *Let A be a* CS *superalgebra that is not purely even and write $A = M^{r|s}(k) \otimes B$ where B is the central super division algebra in the similarity class of A. Then we have the following*:

(i) *If B is purely even, $A^+ \simeq M^r(B) \oplus M^s(B)$ where $M^p(B) = M^p(k) \otimes B$.*

(ii) *If B is not purely even, then*

$$A \simeq M^{r+s}(B) , \quad A^+ \simeq M^{r+s}(B^+) .$$

In particular, A^+ is always semisimple as a classical algebra, and the types of its simple modules depend only on the class of A in $\mathrm{sBr}(k)$. *In case* (i) *A has two simple modules, both with commutants B^{opp}, while in case* (ii) *A has a unique simple module with commutant $B^{+\mathrm{opp}}$.*

PROOF: If B is not purely even, Lemma 6.2.4 shows that $A \simeq M^{r+s}(B)$. The assertions follow from the classical theory of simple algebras. □

REMARK. It must be noted that when B is not purely even, B^+ *need not be central.*

6.4. Real Clifford Modules

We now specialize to the case $k = \mathbf{R}$ (finally!). It will turn out that the information on $\mathrm{sBr}(k)$ available from Section 6.3 is enough to show that $\mathrm{sBr}(\mathbf{R})$ is $\mathbf{Z}_8 = \mathbf{Z}/8\mathbf{Z}$, the cyclic group of order 8, as well as to determine the map $V \longrightarrow [C(V)]$.

Let us now take $k = \mathbf{R}$. Then $\mathbf{R}^\times/\mathbf{R}^{\times 2} = \mathbf{Z}_2$ while $\mathrm{Br}(\mathbf{R}) = \mathbf{Z}_2$ also. Hence, by Proposition 6.3.1,

$$|\mathrm{sBr}(\mathbf{R})| = 8 .$$

On the other hand, since $W(\mathbf{R}) \simeq \mathbf{Z}$, there is a homomorphism f of $\mathbf{Z}$ into $\mathrm{sBr}(\mathbf{R})$ such that if V is a real quadratic space, then

$$[C(V)] = f(\mathrm{sign}(V))$$

where $\mathrm{sign}(V)$ is the signature of V. Since $\mathrm{sBr}(\mathbf{R})$ is of order 8, it follows that the class of $C(V)$ depends only on the signature mod 8. We thus have periodicity mod 8.

It remains to determine $\mathrm{sBr}(\mathbf{R})$ and the map $V \longrightarrow [C(V)]$. We shall show that $\mathrm{sBr}(\mathbf{R})$ is actually cyclic; i.e., it is equal to $\mathbf{Z}_8 = \mathbf{Z}/8\mathbf{Z}$, and that f is the natural map $\mathbf{Z} \longrightarrow \mathbf{Z}_8$. Let $\mathbf{R}[\varepsilon]$ be the super division algebra over $\mathbf{R}$ with ε odd and $\varepsilon^2 = 1$; $\mathbf{R}[\varepsilon] = D_{\mathbf{R},1}$ in earlier notation. We shall in fact show that $\mathbf{R}[\varepsilon]$ has order 8. If V is a real quadratic space of dimension 1 containing a unit vector, $C(V)$ is the algebra $\mathbf{R}[\varepsilon]$ where ε is odd and $\varepsilon^2 = 1$. Its opposite is $\mathbf{R}[\varepsilon^0]$ where ε^0 is odd and ${\varepsilon^0}^2 = -1$,

$$\mathbf{R}[\varepsilon]^{\mathrm{opp}} = \mathbf{R}[\varepsilon^0] .$$

Both $\mathbf{R}[\varepsilon]$ and $\mathbf{R}[\varepsilon^0]$ are central super division algebras and so, because the order of $\mathrm{sBr}(\mathbf{R})$ is 8, their orders can only be 2, 4, or 8. We wish to exclude the possibilities that the orders are 2 and 4. We consider only $\mathbf{R}[\varepsilon]$. Write $A = \mathbf{R}[\varepsilon]$.

By direct computation, we see that $A \otimes A$ is the algebra $\mathbf{R}[\varepsilon_1, \varepsilon_2]$ where the ε_i are odd, $\varepsilon_i^2 = 1$, and $\varepsilon_1\varepsilon_2 = -\varepsilon_2\varepsilon_1$. We claim that this is a central super division algebra. It is easy to check that the supercenter of this algebra is just $\mathbf{R}$. We claim that it is a super division algebra. The even part is $\mathbf{R}[\varepsilon_1\varepsilon_2]$, and because $(\varepsilon_1\varepsilon_2)^2 = -1$, it is immediate that it is $\simeq \mathbf{C}$, hence a division algebra. On the other hand $(u\varepsilon_1 + v\varepsilon_2)^2 = u^2 + v^2$, and so $u\varepsilon_1 + v\varepsilon_2$ is invertible as soon as $(u, v) \neq (0, 0)$. Thus $\mathbf{R}[\varepsilon_1, \varepsilon_2]$ is a central super division algebra. We claim that its square, namely the class of $[A]^4$, is nontrivial. First of all, if $[A]^4$ were trivial we should have $[A]^2 = [A^{\mathrm{opp}}]^2$, which would imply that $\mathbf{R}[\varepsilon_1, \varepsilon_2] \simeq \mathbf{R}[{\varepsilon_1}^0, {\varepsilon_2}^0]$. Then we should be able to find $a, b \in \mathbf{R}$ such that $(a\varepsilon_1 + b\varepsilon_2)^2 = a^2 + b^2 = -1$, which is impossible. So $[A]^4 \neq 1$. Hence $[A]$ must be of order 8, proving that $\mathrm{sBr}(\mathbf{R})$ is cyclic of order 8 and is generated by $\mathbf{R}[\varepsilon]$.

The central super division algebras corresponding to the powers $[\mathbf{R}[\varepsilon]]^m (0 \leq m \leq 7)$ are thus the representative elements of $\mathrm{sBr}(\mathbf{R})$. These can now be written down. For $m = 2$ it is $\mathbf{R}[\varepsilon_1, \varepsilon_2]$. Now, $[\mathbf{R}[\varepsilon]]^2$ becomes isomorphic over $\mathbf{C}$ to

$D \otimes D \simeq M^{1|1}$. If we go back to the discussion in Proposition 6.3.1, we see that $[A]^2 \in H$ and $[A]^4 \in \mathrm{Br}(\mathbf{R})$; since $[A]^4$ is a nontrivial element of $\mathrm{Br}(\mathbf{R})$, the corresponding division algebra must be purely even and isomorphic to $\mathbf{H}$. Thus for $m = 4$ it is purely even and $\mathbf{H}$. For $m = 6$ it is the opposite of the case $m = 2$ and so is $\mathbf{R}[\varepsilon_1^0, \varepsilon_2^0]$. We now consider the values $m = 3, 5, 7$. But $[A]^7 = [A^{\mathrm{opp}}]$ and $[A]^5 = [A^{\mathrm{opp}}]^3$. Now $[A]^5 = [A]^4 \cdot [A]$ and so $\mathbf{H} \otimes \mathbf{R}[\varepsilon]$ is in the class $[A]^5$, $\mathbf{H}$ being viewed as purely even. It is immediate that $\mathbf{H} \otimes \mathbf{R}[\varepsilon] = \mathbf{H} \oplus \mathbf{H}\varepsilon$ is already a super division algebra and so is the one defining the class $[A]^5$. Consequently, $[A]^3$ corresponds to the super division algebra $\mathbf{H} \otimes \mathbf{R}[\varepsilon^0]$. If $V = \mathbf{R}^{p,q}$ and (e_i) is the standard basis,

$$C(V) \simeq \mathbf{R}[\varepsilon]^{\otimes p} \otimes \mathbf{R}[\varepsilon^0]^{\otimes q} \simeq \mathbf{R}[\varepsilon]^{p-q}\,.$$

We have thus obtained the following result.

THEOREM 6.4.1 *The group* $\mathrm{sBr}(\mathbf{R})$ *is cyclic of order* 8 *and is generated by* $[\mathbf{R}[\varepsilon]]$. *If* V *is a real quadratic space, then* $[C(V)] = [\mathbf{R}[\varepsilon]]^{\mathrm{sign}(V)}$ *where* $\mathrm{sign}(V)$ *is the signature of* V. *The central super division algebras* $D(m)$ *in the classes* $[\mathbf{R}[\varepsilon]]^m (0 \leq m \leq 7)$ *are given as follows*:

m	$D(m)$	m	$D(m)$
0	$\mathbf{R}$	4	$\mathbf{H}$
1	$\mathbf{R}[\varepsilon]$	5	$\mathbf{H} \otimes \mathbf{R}[\varepsilon]$
2	$\mathbf{R}[\varepsilon_1, \varepsilon_2]$	6	$\mathbf{R}[\varepsilon_1^0, \varepsilon_2^0]$
3	$\mathbf{H} \otimes \mathbf{R}[\varepsilon^0]$	7	$\mathbf{R}[\varepsilon^0]$

In the above table $\mathbf{R}[\varepsilon_1, \varepsilon_2]$ *is the (super division) algebra generated over* $\mathbf{R}$ *by* $\varepsilon_1, \varepsilon_2$ *with* $\varepsilon_j^2 = 1$ $(j = 1, 2)$, $\varepsilon_1\varepsilon_2 = -\varepsilon_2\varepsilon_1$, *while* $\mathbf{R}[\varepsilon_1^0, \varepsilon_2^0]$ *is the (super division) algebra generated over* $\mathbf{R}$ *by* $\varepsilon_1^0, \varepsilon_2^0$ *with* ${\varepsilon_j^0}^2 = -1$ $(j = 1, 2)$, $\varepsilon_1^0\varepsilon_2^0 = -\varepsilon_2^0\varepsilon_1^0$.

Reality of Clifford Modules. We are now in a position to describe the Clifford modules over $\mathbf{R}$, namely, the types of the simple modules for $C(V)^+$ where V is a real quadratic vector space. Here we have to go from $C(V)$ to $C(V)^+$, and we use Theorem 6.3.3 for this purpose. We must remember during the following discussion that the dimension and signature of a real quadratic vector space are of the same parity. The only purely even central super division algebras over $\mathbf{R}$ are $\mathbf{R}$ and $\mathbf{H}$. If the class of $C(V)$ corresponds to $\mathbf{R}$ (resp., $\mathbf{H}$), then Theorem 6.3.3 shows that $C(V)^+$ has two simple modules with commutant $\mathbf{R}$ (resp., $\mathbf{H}$). From Theorem 6.4.1 we see that this happens if and only if $\mathrm{sign}(V) \equiv 0 \pmod 8$ (resp., $\equiv 4 \pmod 8$) and the corresponding commutant is $\mathbf{R}$ (resp., $\mathbf{H}$). For the remaining values of the signature, the class of $C(V)$ is not purely even. For the values (mod 8) 1, 3, 5, 7 of the signature of V, the super commutants of the simple modules are $\mathbf{R}[\varepsilon^0]$, $\mathbf{H} \otimes \mathbf{R}[\varepsilon]$, $\mathbf{H} \otimes \mathbf{R}[\varepsilon^0]$, and $\mathbf{R}[\varepsilon]$, respectively, and for these values C^+ has a unique simple module with commutant $\mathbf{R}$, $\mathbf{H}$, $\mathbf{H}$, $\mathbf{R}$, respectively. For the values 2, 6 (mod 8) of the signature of V, $C(V)^+$ has a unique simple module with commutant $\mathbf{C}$. Hence we have proven the following theorem:

THEOREM 6.4.2 *Let V be a real quadratic vector space and let $s = \mathrm{sign}(V)$ be its signature. Then $C(V)^+$ is semisimple and the commutants of the simple modules of $C(V)^+$, which are also the commutants of the simple spin modules of* $\mathrm{Spin}(V)$, *are given as follows*:

$s \pmod 8$	0	1,7	2,6	3,5	4
Commutant	**R**, **R**	**R**	**C**	**H**	**H**, **H**

REMARK. One may ask how much of this theory can be obtained by arguments of a general nature. Let us first consider the case when $\dim(V)$ is odd. Then $C(V)^+_{\mathbf{C}}$ is a full matrix algebra. So we are led to the following general situation: We have a real algebra A with complexification $A_{\mathbf{C}}$ that is a full matrix algebra. So $A_{\mathbf{C}}$ has a unique simple module S and we wish to determine the types of simple modules of A over **R**. The answer is that A also has a *unique* simple module over **R**, but this may be either of real type or quaternionic type. To see this we first make the simple remark that if M, N are two real modules for a real algebra and $M_{\mathbf{C}}$, $N_{\mathbf{C}}$ are their complexifications, then

$$\mathrm{Hom}_{A_{\mathbf{C}}}(M_{\mathbf{C}}, N_{\mathbf{C}}) \neq 0 \Longrightarrow \mathrm{Hom}_A(M, N) \neq 0\,.$$

Indeed, there is a natural conjugation in the complex Hom space ($\overline{f}(m) = \overline{f(\overline{m})}$), and the real Hom space consists precisely of those elements of the complex Hom space fixed by it, so that the real Hom spans the complex Hom over **C**. This proves the above implication. This said, let $S_{\mathbf{R}}$ be a real simple module for A and $S_{\mathbf{C}}$ its complexification. If $S_{\mathbf{R}}$ is of type **R**, then $S_{\mathbf{C}}$ is simple and so $\simeq S$. If S' is another simple real module of type **R**, its complexification $S'_{\mathbf{C}}$ is also $\simeq S$, and so by the remark above, $\mathrm{Hom}(S_{\mathbf{R}}, S') \neq 0$, showing that $S' \simeq S_{\mathbf{R}}$. If S' were to be of type **H**, its commutant would be of dimension 4 and so $S'_{\mathbf{C}} = 2S$; but then $2S$ would have two real forms, namely, $2S_{\mathbf{R}}$, S', hence $\mathrm{Hom}(S', 2S_{\mathbf{R}}) \neq 0$, a contradiction. If S' is of type **C** its commutant is of dimension 2 and so the same is true for $S'_{\mathbf{C}}$; but the commutant in aS is of dimension a^2, so that this case does not arise. Thus A also has a unique simple module, but it may be either of type **R** or type **H**. Now, for a Clifford algebra C over **R** of odd dimension, $C^+_{\mathbf{C}}$ is a full matrix algebra and so the above situation applies. The conclusion is that there is a unique simple spin module over **R** that may be of type **R** or **H**.

In the case when V has even dimension $2m$, the argument is similar but slightly more involved because the even part of the Clifford algebra now has two simple modules over the complexes, say $S^{\pm}$. In fact, if

$$S : C(V)_{\mathbf{C}} \simeq \mathbf{End}(\mathbf{C}^{2^{m-1}|2^{m-1}})\,,$$

then

$$S(a) = \begin{pmatrix} S^+(a) & 0 \\ 0 & S^-(a) \end{pmatrix}, \quad a \in C(V)^+_{\mathbf{C}}\,,$$

and $S^{\pm}$ are the two simple modules for $C(V)^+_{\mathbf{C}}$. However, these two are exchanged by inner automorphisms of the Clifford algebra that are induced by real invertible odd elements. Let g be a real invertible odd element of $C(V)$. Then

$$S(g) = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix},$$

and we find

$$S(gag^{-1}) = \begin{pmatrix} \alpha S^-(a)\alpha^{-1} & 0 \\ 0 & \beta S^+(a)\beta^{-1} \end{pmatrix}, \quad a \in C(V)^+_{\mathbf{C}},$$

so that

$$S^{+g} \simeq S^-, \quad S^{-g} \simeq S^+, \quad S^{\pm g}(a) = S^{\pm}(gag^{-1}),\ a \in C(V)^+_{\mathbf{C}}.$$

If now g is real, i.e., $g \in C(V)$, then the inner automorphism by g preserves $C(V)^+$ and exchanges $S^{\pm}$. Such g exist: if $u \in V$ has unit norm, then $u^2 = 1$ so that u is real, odd, and invertible ($u^{-1} = u$). The situation here is therefore of a real algebra A with complexification $A_{\mathbf{C}}$ that is semisimple and has two simple modules $S^{\pm}$ that are exchanged by an automorphism of A. In this case A has either two or one simple modules: if it has two, both are of the same type, which is either **R** or **H**. If it has just one, it is of type **C**.

To prove this, we remark first that if S' is a simple module for A, $S'_{\mathbf{C}}$ is $S^{\pm}$, $S^+ \oplus S^-$, $2S^{\pm}$ according as S' is of type **R**, **C**, **H**. This statement is obvious for the real type. If the type is **C**, the commutant has dimension 2; the complexification is $mS^+ \oplus nS^-$, whose commutant has dimension $m^2 + n^2$ and this is 2 only when $m = n = 1$. If S' is of type **H**, the commutant is of dimension 4 and $m^2 + n^2 = 4$ only for $m = 2$, $n = 0$, or $m = 0$, $n = 2$. This said, assume first that $S_{\mathbf{R}}$ is a simple module for A of type **R**. Then its complexification is either S^+ or S^-. Using the automorphism g we obtain a second simple module of type **R** whose complexification is the other of $S^{\pm}$. So we have simple modules $S^{\pm}_{\mathbf{R}}$ of type **R** with complexifications $S^{\pm}$. There will be no other simple modules of type **R**, and in fact, no others of other types either. For if S' is simple of type **C**, its complexification is $S^+ \oplus S^-$, which has two real forms, namely $S^+_{\mathbf{R}} \oplus S^-_{\mathbf{R}}$, as well as S', which is impossible by our remark. If S' is quaternionic, the same argument applies to $2S^+ \oplus 2S^-$.

If A has a simple module of complex type, it has to be unique since its complexification is uniquely determined since $S^+ \oplus S^-$, and by the above argument A cannot have any simple module of type **R**. But A cannot have a simple module of type **H** either. For if S' were to be one such, then the complexification of S' would be $2S^{\pm}$, and the argument using the odd automorphism g would imply that A has two simple modules $S^{\pm}_{\mathbf{H}}$ with complexifications $2S^{\pm}$; but then $2S^+ \oplus 2S^-$ would have two real forms, $S^+_{\mathbf{H}} \oplus S^-_{\mathbf{H}}$ and $2S'$, which is impossible.

Finally, if $S_{\mathbf{R}}$ is of type **H**, then what we have seen above implies that A has two simple modules of type **H** and no others.

However, these general arguments cannot decide when the various alternatives occur nor will they show that these possibilities are governed by the value of the signature mod 8. These can be done only by a much more detailed analysis.

Method of Atiyah-Bott-Shapiro. Atiyah, Bott, and Shapiro worked with the definite case, and among many other things, they determined the structure of the Clifford algebras and their even parts over the reals.[4] Now all signatures are obtained from the definite quadratic spaces by adding hyperbolic components. In fact,

$$\mathbf{R}^{p,q} = \begin{cases} \mathbf{R}^{p,p} \oplus \mathbf{R}^{0,q-p} & 0 \le p \le q \\ \mathbf{R}^{q,q} \oplus \mathbf{R}^{p-q,0} & 0 \le q \le p, \end{cases} \qquad [\mathbf{R}^{m,0}] = -[\mathbf{R}^{0,m}] .$$

It is therefore enough to determine the types of the Clifford algebras where the quadratic form is *positive* or *negative definite*. This is what Atiyah et al. did.[4] We shall present a variant of their argument in what follows. The argument is in two steps. We first take care of the definite case, and then reduce the general signature (p, q) to the signature $(0, q')$.

We first consider only negative definite quadratic vector spaces. Since we are ultimately interested only in $C(V)^+$, it is enough to work with ungraded algebras and ungraded tensor products. We write C_m for the ungraded Clifford algebra of the real quadratic vector space $\mathbf{R}_{0,m}$. It is thus generated by $(e_j)_{1\le j\le m}$ with relations

$$e_j^2 = -1, \quad 1 \le j \le m, \quad e_r e_s + e_s e_r = 0, \quad r \ne s .$$

Let us write M^r for the matrix algebra $M^r(\mathbf{R})$. The algebra generated by e_1, e_2 with the relations

$$e_1^2 = e_2^2 = 1, \quad e_1 e_2 + e_2 e_1 = 0,$$

is clearly isomorphic to M^2 by

$$e_1 \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

On the other hand, if $F_\pm = \mathbf{R}[e]$ where $e^2 = \pm 1$, then

$$F_+ \simeq \mathbf{R} \oplus \mathbf{R}, \quad a + be \longmapsto (a+b, a-b),$$
$$F_- \simeq \mathbf{C}, \qquad a + be \longmapsto a + ib .$$

Hence for any algebra A, we have

$$A \otimes F_+ = A[e] = A \oplus A .$$

Finally, we have the obvious isomorphisms of Clifford algebras

$$C_1 \simeq \mathbf{C}, \quad C_2 \simeq \mathbf{H}.$$

In what follows we write $\mathbf{C}$ for the complex numbers viewed as an $\mathbf{R}$-algebra.

We consider first the case when $m = 2n$ is even. Then we know that the center of C_{2n} is $\mathbf{R}$. Let

$$f_1 = e_1 \cdots e_{2n-2} e_{2n-1}, \quad f_2 = e_1 \cdots e_{2n-2} e_{2n} .$$

It is then immediate that the f_i commute with the e_j $(1 \le j \le 2n-2)$, while

$$f_1^2 = f_2^2 = (-1)^n, \quad f_1 f_2 + f_2 f_1 = 0 .$$

Hence the algebra A_n generated by f_1, f_2 is isomorphic to C_2 if n is odd and to M^2 if n is even. Moreover,

$$C_{2n} = C_{2n-2} \otimes A_n .$$

We therefore have

$$C_{4n+2} = C_{4n} \otimes \mathbf{H}, \quad C_{4n} = C_{4n-2} \otimes M^2 .$$

Using the fact that $\mathbf{H} \otimes \mathbf{H} = \mathbf{H} \otimes \mathbf{H}^0 = M^4$, we obtain

$$C_{4n} = C_{4n-2} \otimes M^2 = C_{4n-4} \otimes M^2 \otimes \mathbf{H} = C_{4n-6} \otimes M^4 \otimes \mathbf{H} = C_{4n-8} \otimes M^{16},$$
$$C_{4n+2} = C_{4n} \otimes \mathbf{H} = C_{4n-8} \otimes M^{16} \otimes \mathbf{H} = C_{4n-6} \otimes M^{16} .$$

Thus we have the periodicity

$$C_{2n+8} = C_{2n} \otimes M^{16} .$$

Moreover,

$$C_2 = \mathbf{H}, \quad C_4 = \mathbf{H} \otimes M^2 , \quad C_6 = \mathbf{H} \otimes M^2 \otimes \mathbf{H} = M^8 , \quad C_8 = M^8 \otimes M^2 = M^{16} .$$

We thus obtain the following:

$$C_2 = \mathbf{H}, \quad C_4 = M^2 \otimes \mathbf{H}, \quad C_6 = M^8 , \quad C_8 = M^{16} , \quad C_{2n+8} = C_{2n} \otimes M^{16} .$$

We take up next the C_m with odd $m = 2n + 1$. Take the basis as e_j $(0 \le j \le 2n + 1)$, and let

$$\gamma = e_0 e_1 \cdots e_{2n} .$$

Then by Proposition 5.3.1 γ commutes with all the e_j $(1 \le j \le 2n)$ and

$$\gamma^2 = (-1)^{n+1} .$$

Moreover,

$$C^+_{2n+1} \simeq C_{2n}, \quad C_{2n+1} \simeq C_{2n} \otimes \mathbf{R}[\gamma] ,$$

by Proposition 5.3.2. Hence we have

$$C_{4n+1} = C_{4n} \otimes \mathbf{C}, \quad C_{4n+3} = C_{4n+2} \otimes F = C_{4n+2} \oplus C_{4n+2} .$$

Thus, writing A for $\mathbf{C}$ or F_+, we have

$$C_{2n+9} = C_{2n+8} \otimes A = C_{2n} \otimes A \otimes M^{16} = C_{2n+1} \otimes M^{16}$$

since $2n + 9$ and $2n + 1$ have the same residue mod 4. So there is again periodicity mod 8. Now $C_5 = \mathbf{H} \otimes \mathbf{C} \otimes M^2$ while $\mathbf{H} \otimes \mathbf{C}$, viewed as a complex algebra, is just $M^2(\mathbf{C})$, so that $C_5 = M^2 \otimes \mathbf{C}$. Hence we have the following:

$$C_1 = \mathbf{C}, \quad C_3 = \mathbf{H} \oplus \mathbf{H}, \quad C_5 = M^4 \otimes \mathbf{C}, \quad C_7 = M^8 \oplus M^8 ,$$
$$\mathbf{C}_{2n+9} = C_{2n+1} \otimes M^{16} .$$

Combining the odd and even cases, we finally have the following table:

$$\begin{array}{lll} C_1 = \mathbf{C}, & C_2 = \mathbf{H}, & \\ C_3 = \mathbf{H} \oplus \mathbf{H}, & C_4 = M^2 \otimes \mathbf{H}, & C_{m+8} = C_m \otimes M^{16}, \\ C_5 = M^4 \otimes \mathbf{C}, & C_6 = M^8 , & \\ C_7 = M^8 \oplus M^8 , & C_8 = M^{16} . & \end{array}$$

It only remains to determine the structure of the even parts. We have

$$C^+_{n+1} = C_n$$

since the e_0e_j $(1 \le j \le n)$ generate C_{n+1}^+ and they also generate C_n. Also,

$$C_1^+ = \mathbf{R}.$$

Hence we have the following:

$$\begin{array}{lll} C_1^+ = \mathbf{R}, & C_2^+ = \mathbf{C}, & \\ C_3^+ = \mathbf{H}, & C_4^+ = \mathbf{H} \oplus \mathbf{H}, & C_{m+8}^+ = C_m^+ \otimes M^{16}, \\ C_5^+ = M^2 \otimes \mathbf{H}, & C_6^+ = M^4 \otimes \mathbf{C}, & \\ C_7^+ = M^8, & C_8^+ = M^8 \oplus M^8. & \end{array}$$

We now take up the case of the general signature (p, q) with $p \ge 1$. Once again it is a matter of ungraded algebras and tensor products. We write $C_{p,q}$ for the ungraded Clifford algebra of $\mathbf{R}^{p,q}$, namely, the algebra with generators e_i $(1 \le i \le D = p + q)$ and relations $e_i^2 = \varepsilon_i$, $e_ie_j + e_je_i = 0$ $(i \ne j)$; here the ε_i are all ± 1 and exactly q of them are equal to -1. We also write, for typographical reasons, $M(r)$ for $M^r(\mathbf{R})$, and $2A$ for $A \oplus A$. Since $\mathbf{R}^{1,1}$ is hyperbolic, $C_{1,1} = M^2(\mathbf{R})$. By convention $C_{0,0} = C_{0,0}^+ = \mathbf{R}$.

We first note that $C_{p,q}^+$ is generated by the $g_r = e_1e_r$ $(2 \le r \le D)$, with the relations

$$g_r^2 = -\varepsilon_1\varepsilon_r, \quad g_rg_s + g_sg_r = 0, \quad r \ne s.$$

If both p and q are ≥ 1 we can renumber the basis so that ε_1 takes both values ± 1, and in case one of them is 0, we have no choice about the sign of ε_1. Hence we have

$$C_{p,q}^+ = C_{q,p}^+ = C_{p,q-1} = C_{q,p-1}, \quad p, q \ge 0, \ p + q \ge 1,$$

with the convention that when p or q is 0 we omit the relation involving $C_{a,b}$ where one of a, b is less than 0.

First assume that D is even and ≥ 2. Then $C_{p,q}$ is a central simple algebra. As in the definite case we write

$$f_1 = e_1 \cdots e_{D-2}e_{D-1}, \quad f_2 = e_1 \cdots e_{D-2}e_D.$$

Then it is immediate that the f_j commute with all the e_i $(1 \le i \le D-2)$, while $f_1f_2 + f_2f_1 = 0$ and

$$f_1^2 = (-1)^{D/2-1}\varepsilon_1 \cdots \varepsilon_{D-2}\varepsilon_{D-1}, \quad f_2^2 = (-1)^{D/2-1}\varepsilon_1 \cdots \varepsilon_{D-2}\varepsilon_D.$$

If ε_j $(j = D-1, D)$ are of opposite signs, the algebra generated by f_1, f_2 is $C_{1,1}$ while the algebra generated by the e_i $(1 \le i \le D-2)$ is $C_{p-1,q-1}$. Hence we get

$$C_{p,q} = C_{p-1,q-1} \otimes M(2).$$

Repeating this process we get

$$C_{p,q} = \begin{cases} C_{0,q-p} \otimes M(2^p), & 1 \le p \le q, \ D = p + q \text{ is even}, \\ C_{p-q,0} \otimes M(2^q), & 1 \le q \le p, \ D = p + q \text{ is even}. \end{cases}$$

Let us now take up the case when D is odd. Let $\gamma = e_1 \cdots e_D$. By Propositions 5.3.1 and 5.3.2, $\gamma^2 = (-1)^{(p-q-1)/2}$ and $C_{p,q} = C_{p,q}^+ \otimes \operatorname{ctr}(C_{p,q})$ while

$\mathrm{ctr}(C_{p,q}) = \mathbf{R}[\gamma]$. We have already seen that $C^+_{p,q} = C_{p,q-1}$ while $\mathbf{R}[\gamma] = \mathbf{R} \oplus \mathbf{R}$ or $\mathbf{C}$ according as $q - p$ is of the form $4\ell + 3$ or $4\ell + 1$. Hence

$$C_{p,q} = \begin{cases} 2C_{p,q-1} & \text{if } q - p = 4\ell + 3 \\ C_{p,q-1} \otimes \mathbf{C} & \text{if } q - p = 4\ell + 1. \end{cases}$$

We are now in a position to determine the types of the simple modules of $C^+_{p,q}$ using our results for the algebras $C_{0,n}$ and $C^+_{0,n}$, especially the periodicity mod 8 established for them. It is enough to consider the case $p \le q$ because $C^+_{p,q} = C^+_{q,p}$. The fact that $\mathrm{Spin}(p, q) = \mathrm{Spin}(q, p)$ also points to the same restriction $p \le q$ being enough.

D odd. If $p < q$, then $C^+_{p,q} = C_{p,q-1}$ is a central simple algebra and so has a unique simple module. Since $C_{p,q-1} = C_{0,q-p-1} \otimes M(2^p)$, it is immediate that the type of the simple modules of $C^+_{p,q}$ is determined by $q - p \pmod 8$; it is $\mathbf{R}$ or $\mathbf{H}$ according as $q - p \equiv 1, 7 \pmod 8$ or $q - p \equiv 3, 5 \pmod 8$.

D even. We first assume that $0 < p < q$ so that $q \ge p + 2$. Then

$$C^+_{p,q} = C_{p,q-1} = \begin{cases} 2C_{p,q-2} & \text{if } q - p = 4\ell \\ C_{p,q-2} \otimes \mathbf{C} & \text{if } q - p = 4\ell + 2. \end{cases}$$

Since $C_{p,q-2} = C_{0,q-p-2} \otimes M(2^p)$, it is now clear that $C^+_{p,q}$ has two simple modules, both with the same commutant, when $q - p \equiv 0, 4 \pmod 8$, the commutant being $\mathbf{R}$ when $q - p \equiv 0 \pmod 8$, and $\mathbf{H}$ when $q - p \equiv 4 \pmod 8$. If $q - p \equiv 2, 6 \pmod 8$, there is a unique simple module with commutant $\mathbf{C}$.

There remains the case $p = q$. In this case $C^+_{p,p}$ is a direct sum of two copies of $M(2^{p-1})$ and so there are two simple modules of type $\mathbf{R}$.

We have thus reproven Theorem 6.4.2. At the same time this method also yields the structure of $C_{p,q}$ and $C^+_{p,q}$ when $p \le q$. For any integer $n \ge 0$ we write $[n]$ for the unique element of $\{0, 1, \dots, 7\}$ such that n is congruent to $[n] \bmod 8$, so that

$$n = [n] + 8r$$

for some integer $r \ge 0$. If $D = p + q$ is even, we have $q - p = [q - p] + 8r$ and $C_{p,q} = C_{0,q-p} \otimes M(2^p) = C_{0,[q-p]} \otimes M(2^{4r+p})$, which yields, for D even,

$$C_{p,q} = C_{0,[q-p]} \otimes M\left(2^{\frac{D-[q-p]}{2}}\right).$$

If D is odd, we get

$$C_{p,q} = \begin{cases} 2C_{0,[q-p]-1} \otimes M(2^{\frac{D-[q-p]}{2}}) & \text{if } q - p = 4\ell + 3 \\ C_{0,[q-p]-1} \otimes M(2^{\frac{D-[q-p]}{2}}) \otimes \mathbf{C} & \text{if } q - p = 4\ell + 3. \end{cases}$$

We have thus obtained the theorem below.

THEOREM 6.4.3 *The $C_{p,q}$, $C^+_{p,q}$ $(p \le q)$ are given in the following table*:

$[q-p]$	$C_{p,q}$ $(p \le q)$	$C^+_{p,q}$ $(p \le q)$
0	$M(2^{D/2})$	$2M(2^{(D-2)/2})$
1	$M(2^{(D-1)/2}) \otimes \mathbf{C}$	$M(2^{(D-1)/2})$
2	$M(2^{(D-2)/2}) \otimes \mathbf{H}$	$M(2^{(D-2)/2}) \otimes \mathbf{C}$
3	$2M(2^{(D-3)/2}) \otimes \mathbf{H}$	$M(2^{(D-3)/2}) \otimes \mathbf{H}$
4	$M(2^{(D-2)/2}) \otimes \mathbf{H}$	$2M(2^{(D-4)/2}) \otimes \mathbf{H}$
5	$M(2^{(D-1)/2}) \otimes \mathbf{C}$	$M(2^{(D-3)/2}) \otimes \mathbf{H}$
6	$M(2^{D/2})$	$M(2^{(D-2)/2}) \otimes \mathbf{C}$
7	$2M(2^{(D-1)/2})$	$M(2^{(D-1)/2)})$

Although Theorem 6.4.3 is enough for our purposes, we would like to complete the discussion with a brief sketch of the determination of $C_{p,q}$ and $C^+_{p,q}$ when $p \ge q$. Here the reduction is to the algebras $D_n = C_{n,0}$ for integers $n \ge 1$. Thus D_n is the algebra with generators e_i $(1 \le i \le n)$ and relations $e_i^2 = 1$, $e_ie_j = -e_je_i$ $(i \ne j)$. The D_n are determined in the same way as the C_n. We saw earlier that $D_1 = \mathbf{R} \oplus \mathbf{R}$ and $D_2 = M^2$. Then proceeding as in the case of C_n we find, in earlier notation, that $D_{2n} = D_{2n-2} \otimes A_{n-1}$. Thus

$$D_{2n} = \begin{cases} D_{2n-2} \otimes \mathbf{H} & \text{if } n \text{ is even} \\ D_{2n-2} \otimes M^2 & \text{if } n \text{ is odd.} \end{cases}$$

This gives as before the periodicity

$$D_{2n+8} = D_{2n} \otimes M^{16}$$

as well as

$$D_2 = M^2\,, \quad D_4 = M^2 \otimes \mathbf{H}\,, \quad D_6 = M^4 \otimes \mathbf{H}\,, \quad D_8 = M^{16}\,.$$

For odd dimensions $D = 2n+1$ we take $\gamma = e_0e_1 \cdots e_{2n}$ so that $\gamma^2 = (-1)^n$. Then $D_{2n+1} = D^+_{2n+1} \otimes \mathbf{R}[\gamma]$. But D^+_{2n+1} is generated by the e_0e_i $(1 \le i \le 2n)$ and is hence C_{2n}. Hence

$$D_{2n+1} = \begin{cases} C_{2n} \otimes (\mathbf{R} \oplus \mathbf{R}) & \text{if } n \text{ is even} \\ C_{2n} \otimes \mathbf{C} & \text{if } n \text{ is odd.} \end{cases}$$

The periodicity

$$D_{2n+9} = D_{2n+1} \otimes M^{16}$$

follows as before as well as

$$D_1 = \mathbf{R} \oplus \mathbf{R}\,, \quad D_3 = \mathbf{H} \otimes \mathbf{C} = M^2 \otimes \mathbf{C}\,, \quad D_5 = 2(M^2 \otimes \mathbf{H})\,, \quad D_7 = M^8 \otimes \mathbf{C}\,.$$

Hence we have

$$D_1 = \mathbf{R} \oplus \mathbf{R}, \qquad D_2 = M^2,$$
$$D_3 = M^2 \otimes \mathbf{C}, \qquad D_4 = M^2 \otimes \mathbf{H}, \quad D_{m+8} = D_m \otimes M^{16},$$
$$D_5 = 2(M^2 \otimes \mathbf{H}), \quad D_6 = M^4 \otimes \mathbf{H},$$
$$D_7 = M^8 \otimes \mathbf{C}, \qquad D_8 = M^{16}.$$

Moreover,

$$D^+_{n+1} = C_n$$

since the $e_0 e_j$ $(1 \leq j \leq n)$ generate D^+_{n+1}. Hence we have

$$D^+_1 = \mathbf{R}, \qquad D^+_2 = \mathbf{C},$$
$$D^+_3 = \mathbf{H}, \qquad D^+_4 = \mathbf{H} \oplus \mathbf{H}, \qquad D^+_{n+8} = D^+_n \otimes M^{16},$$
$$D^+_5 = M^2 \otimes \mathbf{H} \quad D^+_6 = M^4 \otimes \mathbf{C},$$
$$D^+_7 = M^8, \qquad D^+_8 = M^8 \oplus M^8.$$

Further, when $p \geq q$, we have $C_{p,q} = D_{p-q} \otimes M(2^q)$ so that

$$C_{p,q} = D_{[p-q]} \otimes M\big(2^{\frac{D-[p-q]}{2}}\big).$$

The determination of the $C^+_{p,q}$ is done as before using $C^+_{p,q} = C_{p,q-1}$.

THEOREM 6.4.4 *The $C_{p,q}$, $C^+_{p,q}$ are given by the following table*:

$[p-q]$	$C_{p,q}$ $(p \geq q)$	$C^+_{p,q}$ $(p \geq q)$
0	$M(2^{D/2})$	$2M(2^{(D-2)/2})$
1	$2M(2^{(D-1)/2})$	$M(2^{(D-1)/2})$
2	$M(2^{D/2})$	$M(2^{(D-2)/2}) \otimes \mathbf{C}$
3	$M(2^{(D-1)/2}) \otimes \mathbf{C}$	$M(2^{(D-3)/2}) \otimes \mathbf{H}$
4	$M(2^{(D-2)/2}) \otimes \mathbf{H}$	$2M(2^{(D-4)/2}) \otimes \mathbf{H}$
5	$2M(2^{(D-3)/2}) \otimes \mathbf{H}$	$M(2^{(D-3)/2}) \otimes \mathbf{H}$
6	$M(2^{(D-2)/2}) \otimes \mathbf{H}$	$M(2^{(D-2)/2}) \otimes \mathbf{C}$
7	$M(2^{(D-1)/2}) \otimes \mathbf{C}$	$M(2^{(D-1)/2)})$

6.5. Invariant Forms

For various purposes in physics one needs to know the existence and properties of morphisms

$$S_1 \otimes S_2 \longrightarrow \Lambda^r(V), \quad r \geq 0,$$

where S_1, S_2 are irreducible spin modules for a quadratic vector space V and $\Lambda^r(V)$ is the r^{th} exterior power with $\Lambda^0(V) = k$, k being the ground field. For applications to physics the results are needed over $k = \mathbf{R}$, but to do that we shall again find it convenient to work over $\mathbf{C}$ and then use descent arguments to come down to $\mathbf{R}$. Examples of questions we study are the existence of $\mathrm{Spin}(V)$-invariant forms on $S_1 \times S_2$ and whether they are symmetric or skew-symmetric; such facts are needed for writing the mass terms in the Lagrangian; the existence of symmetric morphisms $S \otimes S \longrightarrow V$ as well as $S \otimes S \longrightarrow \Lambda^r(V)$ needed for the construction of super Poincaré and superconformal algebras; and the existence of morphisms $V \otimes S_1 \longrightarrow S_2$ needed for defining the Dirac operators and writing down kinetic terms in the Lagrangians we need. Our treatment follows closely that of Deligne.[3]

We begin by studying the case $r = 0$, i.e., forms invariant under the spin groups (over $\mathbf{C}$). Right at the outset we remark that if S is an irreducible spin module, the nondegenerate forms on S, by which we mean nondegenerate bilinear forms on $S \times S$, define isomorphisms of S with its dual and so, by irreducibility, are unique up to scalar factors (whenever they exist). The basic lemma is the following:

LEMMA 6.5.1 *Let V be a real or complex quadratic vector space and S a spinorial module, i.e., a $C(V)^+$-module. Then a form $(\cdot\,,\cdot)$ is invariant under* $\mathrm{Spin}(V)$ *if and only if*

$$(*) \qquad (as, t) = (s, \beta(a)t)\,, \quad s, t \in S,\ a \in C(V)^+\,,$$

where β is the principal antiautomorphism of $C(V)$.

PROOF: We recall that β is the unique antiautomorphism of $C(V)$ that is the identity on V. If the above relation is true, then taking $a = g \in \mathrm{Spin}(V) \subset C(V)^+$ shows that $(gs, t) = (s, g^{-1}t)$ since $\beta(g) = g^{-1}$. In the other direction, if $(\cdot\,,\cdot)$ is invariant under $\mathrm{Spin}(V)$, we must have $(as, t) + (s, at) = 0$ for $a \in C^2 \simeq \mathrm{Lie}(\mathfrak{so}(V))$. But, for $a = uv - vu$ where $u, v \in V$, we have $\beta(a) = -a$, so that $(as, t) = (s, \beta(a)t)$ for $a \in C^2$. Since C^2 generates $C(V)^+$ as an associative algebra, we have $(*)$.

It is not surprising that information about invariant forms is controlled by antiautomorphisms. For instance, suppose that U is a purely even vector space and $A = \mathrm{End}(U)$; then there is a bijection between antiautomorphisms β of A and nondegenerate forms $(\cdot\,,\cdot)$ on U defined up to a scalar multiple such that

$$(as, t) = (s, \beta(a)t)\,, \quad s, t \in U,\ a \in A\,.$$

In fact, if $(\cdot\,,\cdot)$ is given, then for each $a \in A$ we can define $\beta(a)$ by the above equation and then verify that β is an antiautomorphism of A. The form can be changed to a multiple of it without changing β. In the reverse direction, suppose that β is an antiautomorphism of A. Then we can make the dual space U^* a module for A by writing

$$(as^*)[t] = s^*[\beta(a)t]\,,$$

and so there is an isomorphism $B_\beta : U \simeq U^*$ of A-modules. The form

$$(s, t) := B_\beta(s)[t]$$

then has the required relationship with β. Since B_β is determined up to a scalar, the form determined by β is unique up to a scalar multiple. If $(s, t)' := (t, s)$, it is immediate that $(as, t)' = (s, \beta^{-1}(a)t)'$ and so $(\cdot\,,\cdot)'$ is the form corresponding to β^{-1}. In particular, β is involutive if and only if $(\cdot\,,\cdot)'$ and $(\cdot\,,\cdot)$ are proportional, i.e., $(\cdot\,,\cdot)$ is either symmetric or skew-symmetric. Now suppose that U is a *super* vector space and $A = \mathbf{End}(U)$; then for *even* β, U^* is a supermodule for A and so is either isomorphic to U or its parity-reversed module ΠU, so that B_β above is even or odd. Hence the corresponding form is even or odd accordingly. Recall that for an even (odd) form we have $(s, t) = 0$ for unlike (like) pairs s, t. Thus we see that if $A \simeq M^{r|s}$ and β is an involutive even antiautomorphism of A, we can associate to (A, β) two invariants coming from the form associated to β, namely, the parity $\pi(A, \beta)$ of the form that is a number 0 or 1, and the symmetry $\sigma(A, \beta)$ that is a sign $\pm$, $+$ for symmetric and $-$ for skew-symmetric forms. □

In view of these remarks and the basic lemma above, we shall base our study of invariant forms for spin modules on the study of pairs $(C(V), \beta)$ where $C(V)$ is the Clifford algebra of a complex quadratic vector space and β is its principal antiautomorphism, namely, the one which is the identity on V. Inspired by the work in Section 6.4 we shall take a more general point of view and study pairs (A, β) where A is a CS superalgebra over $\mathbf{C}$ and β is an even involutive antiautomorphism of A. If $A = C(V)$, then the symbol β will be exclusively used for its principal antiautomorphism. The idea is to define the notion of a tensor product and a similarity relation for such pairs and obtain a group, in analogy with the super Brauer group, a group that we shall denote by $B(\mathbf{C})$. It will be proven that $B(\mathbf{C}) \simeq \mathbf{Z}_8$, showing that the theory of forms for spin modules is governed again by a periodicity mod 8; however, this time it is the *dimension* of the quadratic vector space mod 8 that will tell the story.

If (A_i, β_i) $(i = 1, 2)$ are two pairs, then

$$(A, \beta) = (A_1, \beta_1) \otimes (A_2, \beta_2)$$

is defined by

$$A = A_1 \otimes A_2\,, \qquad \beta = \beta_1 \otimes \beta_2\,,$$
$$\beta(a_1 \otimes a_2) = (-1)^{p(a_1)p(a_2)} \beta_1(a_1) \otimes \beta_2(a_2)\,.$$

The definition of β is made so that it correctly reproduces what happens for Clifford algebras. In fact, we have the following:

LEMMA 6.5.2 *If V_i $(i = 1, 2)$ are quadratic vector spaces and $V = V_1 \oplus V_2$, then*

$$(C(V), \beta) = (C(V_1), \beta) \otimes (C(V_2), \beta)\,.$$

PROOF: If u_i $(1 \le i \le p) \in V_1$, v_j $(1 \le j \le q) \in V_2$, then

$$\beta(u_1 \cdots u_p \cdots v_1 \cdots v_q) = v_q \cdots v_1 u_p \cdots u_1 = (-1)^{pq} \beta(u_1 \cdots u_p) \beta(v_1 \cdots v_q)\,,$$

which proves the lemma. □

We need to make a remark here. The definition of the tensor product of two β's violates the sign rule. One can avoid this by redefining it without altering the theory in any essential manner (see Deligne[3]), but this definition is more convenient for us. As a result, in a few places we shall see that the sign rule gets appropriately modified. The reader will notice these aberrations without any prompting.

For the pairs (A, β) the tensor product is associative and commutative as is easy to check. We now define the pair (A, β) to be *neutral* if $A \simeq M^{r|s}$ and the form corresponding to β that is defined over $\mathbf{C}^{r|s}$ is *even* and *symmetric*. We shall say that (A, β), (A', β') are *similar* if we can find neutral (B_1, β_1), (B_2, β_2) such that

$$(A, \beta) \otimes (B_1, \beta_1) \simeq (A', \beta') \otimes (B_2, \beta_2)\,.$$

If (A, β) is a pair where $A \simeq M^{r|s}$, we write $\pi(A, \beta), \sigma(A, \beta)$ for the parity and symmetry of the associated form on $\mathbf{C}^{r|s}$. When we speak of the parity and sign of a pair (A, β) it is implicit that A is a full matrix superalgebra. Notice that on a full matrix superalgebra we can have forms of arbitrary parity and symmetry. Indeed, nondegenerate forms are defined by invertible matrices x, symmetric or skew-symmetric, in the usual manner, namely $\varphi_x(s, t) = s^{\mathsf{T}}xt$. The involution β_x corresponding to x is

$$\beta_x(a) = x^{-1}a^{\mathsf{T}}x\,, \quad a \in M^{r|s}\,.$$

Note that β_x is even for x homogeneous and involutive if x is symmetric or skew-symmetric. We have the following where in all cases β_x is even and involutive:

$$A = M^{r|r}\,, \qquad x = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \varphi_x = \text{even and symmetric},$$

$$A = M^{r|r}\,, \qquad x = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \varphi_x = \text{odd and symmetric},$$

$$A = M^{2r|2r}\,, \qquad x = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \varphi_x = \text{even and skew-symmetric},$$

$$A = M^{2r|2r}\,, \qquad x = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \varphi_x = \text{odd and skew-symmetric}.$$

LEMMA 6.5.3 *Let π, σ_i be the parity and symmetry of (A_i, β_i) $(i = 1, 2)$. Then for the parity π and symmetry σ of the tensor product $(A_1 \otimes A_2, \beta_1 \otimes \beta_2)$ we have*

$$\pi = \pi_1 + \pi_2, \quad \sigma = (-1)^{\pi_1\pi_2}\sigma_1\sigma_2\,.$$

PROOF: It is natural to expect that the form corresponding to $\beta_1 \otimes \beta_2$ is the tensor product of the corresponding forms for the β_i. But because the definition of the tensor product has violated the sign rule, one should define the tensor product of forms with suitable sign factors so that this is true. Let $A_i = \mathbf{End}(S_i)$. Let us define

$$(s_1 \otimes s_2, t_1 \otimes t_2) = C(s_1, s_2, t_1, t_2)(s_1, t_1)(s_2, t_2)$$

where C is a sign factor depending on the parities of the s_i, t_j. The requirement that this corresponds to $\beta_1 \otimes \beta_2$ now leads to the equations

$$C(s_1, s_2, \beta_1(a_1)t_1, \beta_2(a_2)t_2) = (-1)^{p(a_2)[p(s_1)+p(t_1)+p(a_1)]}C(a_1s_1, a_2s_2, t_1, t_2)\,,$$

which is satisfied if we take

$$C(s_1, s_2, t_1, t_2) = (-1)^{p(s_2)[p(s_1)+p(t_1)]}\,.$$

Thus the correct definition of the tensor product of two forms is

$$(s_1 \otimes s_2, t_1 \otimes t_2) = (-1)^{p(s_2)[p(s_1)+p(t_1)]}(s_1, t_1)(s_2, t_2)\,.$$

If $(s, t) \neq 0$, then $\pi = p(s) + p(t)$, and so choosing $(s_i, t_i) \neq 0$ we have $\pi = p(s_1) + p(s_2) + p(t_1) + p(t_2) = \pi_1 + \pi_2$. For σ we get

$$\sigma = (-1)^{[p(s_1)+p(t_1)][p(s_2)+p(t_2)]}\sigma_1\sigma_2 = (-1)^{\pi_1\pi_2}\sigma_1\sigma_2\,.$$

It follows from this that if the (A_i, β_i) are neutral so is their tensor product. From this we see that similarity is an equivalence relation, obviously coarser than isomorphism, and that similarity is preserved under tensoring. In particular, we can speak of similarity classes and their tensor products. The similarity classes form a commutative semigroup, and the neutral elements form the identity element of this semigroup. We denote it by $B(\mathbf{C})$. Let $B_0(\mathbf{C})$ be the set of classes of pairs (A, β) where $A \simeq M^{r|s}$. Then $B_0(\mathbf{C})$ is a subsemigroup of $B(\mathbf{C})$ containing the neutral class. The parity and symmetry invariants do not change when tensored by a neutral pair so that they are really invariants of similarity classes in $B_0(\mathbf{C})$.

We wish to prove that $B(\mathbf{C})$ is indeed the cyclic group $\mathbf{Z}_8$ of order 8 and that $B_0(\mathbf{C})$ is a subgroup. Before doing this we define, for the parity group $P = \{0, 1\}$ and sign group $\Sigma = \{\pm 1\}$, their product $P \times \Sigma$ with the product operation defined by the lemma above:

$$(\pi_1, \sigma_1)(\pi_2, \sigma_2) = \left(\pi_1 + \pi_2, (-1)^{\pi_1\pi_2}\sigma_1\sigma_2\right).$$

It is a trivial calculation that $P \times \Sigma$ is a group isomorphic to $\mathbf{Z}_4$ and is generated by $(1, +)$. The map

$$\varphi : (A, \beta) \longmapsto (\pi, \sigma)$$

is then a homomorphism of $B_0(\mathbf{C})$ into $P \times \Sigma$. We assert that φ is surjective. It is enough to check that $(1, +)$ occurs in its image. Let V_2 be a two-dimensional quadratic vector space with basis $\{u, v\}$ where $\Phi(u, u) = \Phi(v, v) = 0$ and $\Phi(u, v) = 1/2$, so that $u^2 = v^2 = 0$ and $uv + vu = 1$. Then $C(V_2) \simeq M^{1|1}$ via the standard representation that acts on $\mathbf{C} \oplus \mathbf{C}v$ as follows:

$$v \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : 1 \mapsto v\,, \quad v \mapsto 0\,, \qquad u \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : 1 \mapsto 0\,, \quad v \mapsto 1\,.$$

The principal involution β is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The form corresponding to β is then defined by the invertible symmetric matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and so is odd and symmetric. Thus $(C(V_2), \beta)$ gets mapped to $(1, +)$ by φ. Thus φ is surjective. Moreover, the kernel of φ is just the neutral class. Hence $B_0(\mathbf{C})$ is already a group isomorphic to $\mathbf{Z}_4$ and is generated by the class of the Clifford algebra in dimension 2. In particular, the parity and symmetry of the forms determine the class of any element of $B_0(\mathbf{C})$. □

THEOREM 6.5.4 *$B(\mathbf{C})$ is a group isomorphic to the cyclic group $\mathbf{Z}_8$ of order* 8 *and is generated by the class of the Clifford algebra in dimension* 1, *namely* $(\mathbf{C}[\varepsilon], \beta)$, *where ε is odd and $\varepsilon^2 = 1$, and $\beta(\varepsilon) = \varepsilon$. The subgroup $B_0(\mathbf{C}) \simeq \mathbf{Z}_4$ is generated by the class of the Clifford algebra in dimension* 2.

PROOF: If A is a complex CS superalgebra that is not a full matrix algebra, then it is of the form $M^n \otimes \mathbf{C}[\varepsilon]$ and so $A \otimes A \simeq M^{n^2|n^2}$. Thus the square of any element x in $B(\mathbf{C})$ is in $B_0(\mathbf{C})$ and hence $x^8 = 1$. This proves that $B(\mathbf{C})$ is a group and $B(\mathbf{C})/B_0(\mathbf{C}) \simeq \mathbf{Z}_2$. The square of the class of the Clifford algebra in dimension 1 is the Clifford algebra in dimension 2, which has been shown to be a generator of $B_0(\mathbf{C})$. Thus $(\mathbf{C}[\varepsilon], \beta)$ generates $B(\mathbf{C})$ and has order 8. □

COROLLARY 6.5.5 *The inverse of the class of* $(\mathbf{C}[\varepsilon], \beta)$ *is the class of* $(\mathbf{C}[\varepsilon], \beta^0)$ *where* $\beta^0(\varepsilon) = -\varepsilon$.

PROOF: Since $\mathbf{C}[\varepsilon]$ is its own inverse in the super Brauer group $\mathrm{sBr}(\mathbf{C})$, the inverse in question has to be $(\mathbf{C}[\varepsilon], \beta')$ where $\beta' = \beta$ or β^0. The first alternative is impossible since $(\mathbf{C}[\varepsilon], \beta)$ has order 8, not 2. □

There is clearly a unique isomorphism of $B(\mathbf{C})$ with $\mathbf{Z}_8$ such that the class of $(\mathbf{C}[\varepsilon], \beta)$ corresponds to the residue class of 1. We shall identify $B(\mathbf{C})$ with $\mathbf{Z}_8$ through this isomorphism. We shall refer to the elements of $B_0(\mathbf{C})$ as the even classes and the elements of $B(\mathbf{C}) \setminus B_0(\mathbf{C})$ as the odd classes. For D-dimensional V_D the class of $(C(V_D), \beta)$ is in $B_0(\mathbf{C})$ if and only if D is even. Since the class of $(C(V_D), \beta)$ is the D^{th} power of the class of $(C(V_1), \beta) = (\mathbf{C}[\varepsilon], \beta)$, it follows that the class of $(C(V_8), \beta)$ is 1 and hence that $(C(V_D), \beta)$ and $(C(V_{D+8}), \beta)$ are in the same class, giving us the periodicity mod 8. The structure of invariant forms for the Clifford algebras is thus governed by the dimension mod 8. Table 6.1 gives for the even-dimensional cases the classes of the Clifford algebras in terms of the parity and symmetry invariants. Let $D = \dim(V)$ and let $\overline{D}$ be its residue class mod 8.

$\overline{D}$	π	σ
0	0	+
2	1	+
4	0	−
6	1	−

TABLE 6.1

However, for determining the nature of forms invariant under $\mathrm{Spin}(V)$ we must go from the Clifford algebra to its even part. We have the isomorphism $C(V_D) \simeq \mathbf{End}(S)$ where S is an irreducible supermodule for $C(V_D)$. Table 6.1 tells us that for $\overline{D} = 0, 4$ there is an even invariant form for S that is symmetric and skew-symmetric, respectively. Now under the action of $C(V_{2m})^+$ we have $S = S^+ \oplus S^-$ where $S^\pm$ are the semispin representations. So both of these have invariant forms that are symmetric for $\overline{D} = 0$ and skew-symmetric for $\overline{D} = 4$. For $\overline{D} = 2, 6$ the invariant form for S is odd, and so what we get is that $S^{\pm}$ are dual to each other. In this case there will be no invariant forms for $S^{\pm}$ individually; if, for example, S^{+} has an invariant form, then S^{+} would be isomorphic to its dual and so would be isomorphic to S^{-}, which is impossible. Since $S^{\pm}$ are irreducible, all invariant forms are multiples of the ones defined above. When the form is symmetric, the spin group is embedded inside the orthogonal group of the spin module, while in the skew case it is embedded inside the symplectic group. Later on we shall determine the imbeddings much more precisely when the ground field is **R**. Thus we have Table 6.2.

$\overline{D}$	Forms on $S^{\pm}$
0	Symmetric on $S^{\pm}$
2	$S^{\pm}$ dual to each other
4	Skew-symmetric on $S^{\pm}$
6	$S^{\pm}$ dual to each other

TABLE 6.2

We now examine the odd classes in $B(\mathbf{C})$. Here the underlying algebras A are of the form $M \otimes Z$ where M is a purely even, full matrix algebra and Z is the center (not supercenter) of the algebra, with $Z \simeq \mathbf{C}[\varepsilon]$:

$$A \simeq A^{+} \otimes Z\,, \quad A^{+} \simeq M^{r|s}\,, \quad Z = \mathbf{C}[\varepsilon],\ \varepsilon \text{ odd},\ \varepsilon^2 = 1\,.$$

Note that Z is a superalgebra. If β is an even involutive antiautomorphism of A, then β leaves Z invariant and hence also $Z^{\pm}$. It acts trivially on $Z^{+} = \mathbf{C}$ and as a sign $s(\beta)$ on Z^{-}. We now have the following key lemma:

LEMMA 6.5.6 *We have the following*:

(i) *Let* (A, β), (A', β') *be pairs representing an odd and even class, respectively, in* $B(\mathbf{C})$. *Then*

$$s(\beta \otimes \beta') = (-1)^{\pi'} s(\beta)$$

where π' *is the parity of the form corresponding to* β'. *In particular, the sign* $s(\beta)$ *depends only on the similarity class of* (A, β).

(ii) *With the identification* $B(\mathbf{C}) \simeq \mathbf{Z}_8$ *(written additively), the elements* x^{+}, x *of* $B(\mathbf{C})$ *corresponding to* (A^{+}, β), (A, β), *respectively, are related by*

$$x^{+} = x - s(\beta)1\,.$$

In particular, the similarity class of (A^{+}, β) *depends only on that of* (A, β).

PROOF: Let $(A'', \beta'') = (A, \beta) \otimes (A', \beta')$. The center of A'' is again of dimension $1|1$. If A' is purely even, then Z is contained in the center of A'' and so has to be its center and the actions of β, β'' are then the same. Suppose that $A' = M^{r|s}$ where $r, s > 0$. Let

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in A'\,.$$

It is trivial to check that η commutes with A'^{+} and anticommutes with A'^{-}, and that it is characterized by this property up to a scalar multiple. We claim that $\varepsilon \otimes \eta$ lies in the odd part of the center of A''. This follows from the fact that ε and η commute with $A \otimes 1$ and $1 \otimes A'^{+}$, while they anticommute with $1 \otimes A'^{-}$. Hence $\varepsilon \otimes \eta$ *spans* the odd part of the center of A''. Now

$$\beta''(\varepsilon \otimes \eta) = \beta(\varepsilon) \otimes \beta'(\eta).$$

The first factor on the right side is $s(\beta)\varepsilon$. On the other hand, by the characterization of η mentioned above, we must have $\beta'(\eta) = c\eta$ for some constant c, and so to prove (i) we must show that $c = (-1)^{\pi'}$. If the form corresponding to β' is even, there are even s, t such that $(s, t) \neq 0$; then $(s, t) = (\eta s, t) = (s, c\eta t) = c(s, t)$, so that $c = 1$. If the form is odd, then we can find even s and odd t such that $(s, t) \neq 0$; then $(s, t) = (\eta s, t) = (s, c\eta t) = -c(s, t)$ so that $c = -1$. This finishes the proof of (i).

For proving (ii) let x, x^{+}, z be the elements of $B(\mathbf{C})$ corresponding to (A, β), (A^{+}, β), (Z, β), respectively. Clearly, $x = x^{+} + z$. If $s(\beta) = 1$, (Z, β) is the class of the Clifford algebra in dimension 1 and so is given by the residue class of 1. Thus $x^{+} = x - 1$. If $s(\beta) = -1$, then (Z, β) is the inverse of the class of the Clifford algebra in dimension 1 by Corollary 6.5.5 and hence $x^{+} = x + 1$. □

For the odd classes of pairs (A, β) in $B(\mathbf{C})$ we thus have two invariants: the sign $s(\beta)$ and the symmetry $s(A^{+})$ of the form associated to the similarity class of (A^{+}, β). We then have Table 6.3.

Residue class	$s(A^{+})$	$s(\beta)$
1	$+$	$+$
3	$-$	$-$
5	$-$	$+$
7	$+$	$-$

TABLE 6.3

To get this table we start with $(\mathbf{C}[\varepsilon], \beta)$ with $\beta(\varepsilon) = \varepsilon$ for which the entries are $+, +$. For 7 the algebra remains the same but the involution is β^{0}, which takes ε to $-\varepsilon$, so that the entries are $+, -$. From Table 6.1 we see that the residue class 4 in $B_0(\mathbf{C})$ is represented by any full matrix superalgebra with an even invariant skew-symmetric form; we can take it to be the purely even matrix algebra $M = M^{2}$ in dimension 2 with the invariant form defined by the skew-symmetric matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let β_M be the corresponding involution. Then 5 is represented by $(M, \beta_M) \otimes (\mathbf{C}[\varepsilon], \beta)$. Using Lemma 6.5.3 we see that the signs of the form and the involution are $-, +$. To get the invariants of the residue class 3 we remark that since $3 = 4 - 1$, it is represented by $(M, \beta_M) \otimes (\mathbf{C}[\varepsilon], \beta^{0})$ and so its invariants are $-, -$.

If $D = 2m + 1$ is odd, $(C(V_D), \beta)$ is similar to the D^{th} power of the class of $(\mathbf{C}[\varepsilon], \beta)$. Hence there is periodicity in dimension mod 8 and the invariants for the residue classes of $D \bmod 8$ are the same as in Table 6.3. For the symmetry of the $\mathrm{Spin}(V)$-invariant forms, we simply read the first column of the table above. We thus have the following theorem:

THEOREM 6.5.7 *The existence and symmetry properties of forms on the spin modules associated to complex quadratic vector spaces V depend only on the residue class $\overline{D}$ of $D = \dim(V) \pmod 8$. The forms, when they exist, are unique up to scalar factors and their symmetry properties are given by Table* 6.4. *When $S^{\pm}$ are dual to each other, there are no forms on $S^{\pm}$ individually.*

$\overline{D}$	Forms on S, $S^{\pm}$
0	Symmetric on $S^{\pm}$
1, 7	Symmetric on S
2, 6	$S^{\pm}$ dual to each other
3, 5	Skew-symmetric on S
4	Skew-symmetric on $S^{\pm}$

TABLE 6.4

Forms in the Real Case. We shall now extend the above results to the case of *real* spin modules. The results are now governed by *both* the dimension and the signature mod 8.

We are dealing with the following situation: $S_{\mathbf{R}}$ is a real irreducible module for $C(V)^+$ where $C(V)$ is the Clifford algebra of a real quadratic vector space V; equivalently, $S_{\mathbf{R}}$ is an irreducible module for $\mathrm{Spin}(V)$. The integers p, q are such that $V \simeq \mathbf{R}^{p,q}$, $D = p + q$, $\Sigma = p - q$, D and Σ having the same parity. $\overline{D}, \overline{\Sigma}$ are the residue classes of $D, \Sigma \bmod 8$. We write σ for the conjugation of $S_{\mathbf{C}}$ that defines $S_{\mathbf{R}}$, $S_{\mathbf{C}}$ being the complexification of $S_{\mathbf{R}}$. If $S_{\mathbf{R}}$ is of type $\mathbf{R}$, then $S_{\mathbf{C}}$ is the irreducible spin module S or $S^{\pm}$; if $S_{\mathbf{R}}$ is of type $\mathbf{H}$, then $S_{\mathbf{C}} = S_0 \otimes W$ where S_0 is an irreducible complex spin module and $\dim(W) = 2$, $C(V)^+$ acting on $S_{\mathbf{C}}$ only through the first factor. If $S_{\mathbf{R}}$ is of type $\mathbf{C}$, this case occurring only when D is even, then $S_{\mathbf{C}} = S^+ \oplus S^-$.

Let $\mathbf{A}$ be the commutant of the image of $C(V)^+$ in $\mathrm{End}(S_{\mathbf{R}})$. Then $\mathbf{A} \simeq \mathbf{R}, \mathbf{H}$, or $\mathbf{C}$. We write $\mathbf{A}_1$ for the group of elements of norm 1 in $\mathbf{A}$. Notice that this is defined independently of the choice of the isomorphism of $\mathbf{A}$ with these algebras, and

$$\mathbf{A}_1 \simeq \{\pm 1\}, \quad \mathrm{SU}(2), \quad T,$$

in the three cases, T being the multiplicative group of complex numbers of absolute value 1. If β is any invariant form for $S_{\mathbf{R}}$, and $a \in \mathbf{A}_1$,

$$a \cdot \beta : (u, v) \longmapsto \beta(a^{-1}u, a^{-1}v)$$

is also an invariant form for $S_{\mathbf{R}}$. Thus we have an action of $\mathbf{A}_1$ on the space of invariant forms for $S_{\mathbf{R}}$. We shall also determine this action below. Actually, when

$\mathbf{A} = \mathbf{R}$, we have $\mathbf{A}_1 = \{\pm 1\}$ and the action is trivial, so that only the cases $\mathbf{A} = \mathbf{H}, \mathbf{C}$ need to be considered insofar as this action is concerned.

The simplest case is when $S_{\mathbf{R}}$ is of type $\mathbf{R}$. This occurs when $\overline{\Sigma} = 0, 1, 7$. If $\overline{\Sigma} = 1, 7$ then D is odd and the space of invariant bilinear forms for $S_{\mathbf{C}}$ is of dimension 1. It has a conjugation $B \mapsto B^\sigma$ defined by $B^\sigma(s,t) = B(s^\sigma, t^\sigma)^{\text{conj}}$, and if B is a real nonzero element, then B spans this space and is an invariant form for $S_{\mathbf{R}}$. The symmetry of the form does not change, and the conclusions are given by the first column of the first table of Theorem 6.5.10 below. If $\overline{\Sigma} = 0$ the conclusions are again the same for the spin modules $S_{\mathbf{R}}^{\pm}$ for $\overline{D} = 0, 4$. When $\overline{D} = 2, 6$, $S^{\pm}$ are in duality, which implies that $S_{\mathbf{R}}^{\pm}$ are also in duality. We have thus verified the first column of the second table of Theorem 6.5.10.

To analyze the remaining cases we need some preparation. For any complex vector space U, we define a *pseudoconjugation* to be an antilinear map τ of U such that $\tau^2 = -1$. For example, if $U = \mathbf{C}^2$ with standard basis $\{e_1, e_2\}$, then

$$\tau : e_1 \longmapsto e_2\,, \qquad e_2 \longmapsto -e_1\,,$$

defines a pseudoconjugation. For an arbitrary U, if τ is a pseudoconjugation or an ordinary conjugation, we have an induced conjugation on $\operatorname{End}(U)$ defined by $a \longmapsto \tau a \tau^{-1}$ (conjugations of $\operatorname{End}(U)$ have to preserve the product by definition). If we take τ to be the conjugation of $\mathbf{C}^2$ that fixes the e_i, then the induced conjugation on $\operatorname{End}(\mathbf{C}^2) = M^2(\mathbf{C})$ is just $a \longmapsto a^{\text{conj}}$ with the fixed-point algebra $M^2(\mathbf{R})$, while for the pseudoconjugation τ defined above, the induced conjugation is given by

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \overline{\delta} & -\overline{\gamma} \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

so that its fixed points form the algebra of matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \qquad \alpha, \beta \in \mathbf{C}\,.$$

If $\alpha = a_0 + i a_1$, $\beta = a_2 + i a_3$ ($a_j \in \mathbf{R}$), then

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \longmapsto a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

is an isomorphism of the above algebra with $\mathbf{H}$, and the elements of $\mathbf{H}_1$ correspond to SU(2). If $U = \mathbf{C}^{2m}$, then, with (e_i) as the standard basis,

$$\tau : e_j \mapsto e_{m+j}\,, \quad e_{m+j} \mapsto -e_j\,, \quad 1 \le j \le m\,,$$

is a pseudoconjugation. Pseudoconjugations cannot exist if the vector space is odd dimensional. Indeed, on $\mathbf{C}$, any antilinear transformation is of the form $z \mapsto cz^{\text{conj}}$, and its square is $z \mapsto |c|^2 z$, showing that it can never be -1. The same argument shows that no pseudoconjugation of an arbitrary vector space U can fix a line. If T is a pseudoconjugation on U, then for any nonzero u, the span U' of u, Tu is of dimension 2 and stable under T, and so T induces a pseudoconjugation on U/U'; an induction on dimension shows that pseudoconjugations do not exist when U is odd dimensional. Any two conjugations of U are equivalent; if U_i are the real forms defined by two conjugations σ_i ($i = 1, 2$), then for any element $g \in \operatorname{GL}(U)$

that takes U_1 to U_2, we have $g\sigma_1 g^{-1} = \sigma_2$. In particular, any conjugation is equivalent to the standard conjugation on $\mathbf{C}^m$. The same is true for pseudoconjugations also. Indeed, if $\dim(U) = 2m$ and τ is a pseudoconjugation, let W be a maximal subspace of U such that $W \cap \tau(W) = 0$; we claim that $U = W \oplus \tau(W)$. Otherwise, if $u \notin W' := W \oplus \tau(W)$, and L is the span of u and τu, then $L \cap W'$ is τ-stable and so has to have dimension 0 or 2, and hence has dimension 0 because otherwise we would have $L \subset W'$. The span W_1 of W and u then would have the property that $W_1 \cap \tau(W_1) = 0$, a contradiction. So $U = W \oplus \tau(W)$. It is then clear that τ is isomorphic to the pseudoconjugation of $\mathbf{C}^{2m}$ defined earlier.

LEMMA 6.5.8 *Any conjugation of* $\mathrm{End}(U)$ *is induced by a conjugation or a pseudoconjugation of U that is unique up to a scalar factor of absolute value* 1.

PROOF: Choose some conjugation θ of U and let $a \mapsto a^\theta$ be the induced conjugation of $\mathrm{End}(U)$:

$$a^\theta = \theta a \theta\,, \quad a^\theta u = (au^\theta)^\theta\,, \quad u \in U,\ a \in \mathrm{End}(U)\,.$$

Let $a \mapsto a^*$ be the given conjugation of $\mathrm{End}(U)$. Then $a \mapsto (a^\theta)^*$ is an automorphism, and so we can find an $x \in \mathrm{GL}(U)$ such that $(a^\theta)^* = xax^{-1}$. Replacing a by a^θ gives $a^* = xa^\theta x^{-1}$. So $a = (a^*)^* = xx^\theta a(xx^\theta)^{-1}$, showing that $xx^\theta = c1$ for a constant c, and hence that $xx^\theta = x^\theta x = c1$. Thus c is real, and replacing x by $|c|^{-1/2}x$, we may assume that $xx^\theta = \pm 1$. Let τ be defined by $u^\tau = xu^\theta (u \in U)$. Then τ is antilinear and $\tau^2 = xx^\theta = \pm 1$. Clearly, $*$ is induced by τ. If τ' is another such, then $\tau'^{-1}\tau$ induces the identity automorphism on $\mathrm{End}(U)$, and so $\tau = c\tau'$ where c is a scalar. Since $\tau^2 = |c|^2\tau'^2$, we must have $|c| = 1$. □

For any conjugation or pseudoconjugation α of U, we write $\widehat{\alpha}$ for the induced conjugation $a \mapsto \alpha a \alpha^{-1}$ of $\mathrm{End}(U)$.

LEMMA 6.5.9 *Let $S_{\mathbf{R}}$ be of type* **H** *and let $S_{\mathbf{C}}$, S_0, W, σ be as above. Then $\sigma = \tau \otimes \tau_1$ where τ (resp., τ_1) is a pseudoconjugation of S_0 (resp., W). τ and τ_1 are unique up to scalar factors of absolute value* 1, *and τ commutes with the action of $C(V)^+$. Conversely, if S_0 is an irreducible spin module for* $\mathrm{Spin}(V_{\mathbf{C}})$ *and* $\mathrm{Spin}(V)$ *commutes with a pseudoconjugation, any real irreducible spin module of* $\mathrm{Spin}(V)$ *is of type* **H**.

PROOF: The complexifications of the image of $C(V)^+$ in $\mathrm{End}(S_{\mathbf{R}})$ and its commutant are $\mathrm{End}(S_0) \simeq \mathrm{End}(S_0) \otimes 1$ and $\mathrm{End}(W) \simeq 1 \otimes \mathrm{End}(W)$, respectively. Hence the conjugation $\widehat{\sigma}$ of $\mathrm{End}(S_{\mathbf{C}})$ induced by σ leaves both $\mathrm{End}(S_0)$ and $\mathrm{End}(W)$ invariant. So, by the above lemma there are conjugations or pseudoconjugations τ, τ_1 on S_0, W inducing the restrictions of $\widehat{\sigma}$ on $\mathrm{End}(S_0)$ and $\mathrm{End}(W)$, respectively. Since $\mathrm{End}(S_0)$ and $\mathrm{End}(W)$ generate $\mathrm{End}(S_{\mathbf{C}})$, we have $\widehat{\sigma} = \widehat{\tau} \otimes \widehat{\tau_1} = (\tau \otimes \tau_1)\widehat{}$. It follows that for some $c \in \mathbf{C}$ with $|c| = 1$, we must have $\sigma = c(\tau \otimes \tau_1)$. Replacing τ_1 by $c\tau_1$, we may therefore assume that $\sigma = \tau \otimes \tau_1$. Since σ commutes with the action of $C(V)^+$, and $C(V)^+$ acts on $S_0 \otimes W$ only through the first factor, it follows easily that τ commutes with the action of $C(V)^+$. Now the subalgebra of $\mathrm{End}(W)$ fixed by $\widehat{\tau_1}$ is **H** and so τ_1 must be a pseudoconjugation. Therefore, since σ is a conjugation, τ must also be a pseudoconjugation.

For the converse, choose a W of dimension 2 with a pseudoconjugation τ_1. Let τ be the pseudoconjugation on S commuting with $\mathrm{Spin}(V)$. Then $\sigma = \tau \otimes \tau_1$ is a conjugation on $S_0 \otimes W$ commuting with $\mathrm{Spin}(V)$ and so $2S$ has a real form $S'_{\mathbf{R}}$. This real form must be irreducible; for otherwise, if $S'_{\mathbf{R}}$ is a proper irreducible constituent, then $S'_{\mathbf{C}} \simeq S$, which would imply that S_0 has a real form. So $\mathrm{Spin}(V)$ would have to commute with a conjugation also, an impossibility. This proves the entire lemma. □

Suppose now that $S_{\mathbf{R}}$ is of type **H** and S_0 has an invariant form. The space of these invariant forms is of dimension 1, and τ, since it commutes with $C(V)^+$, induces a conjugation $B \mapsto B^\tau$ on this space where $B^\tau(s,t) = B(s^\tau, t^\tau)^{\mathrm{conj}}$. Hence we may assume that S_0 has an invariant form $B = B^\tau$. The space of invariant forms for $S_0 \otimes W$ is now $B \otimes J$, where J is the space of bilinear forms for W, which is a natural module for $\mathbf{A}_1$ and which carries a conjugation, namely, the one induced by τ_1. We select a basis e_1, e_2 for W so that $\tau_1(e_1) = e_2$, $\tau_1(e_2) = -e_1$. Then $\mathbf{A}_1 = \mathrm{SU}(2)$, and its action on W commutes with τ_1. Clearly $J = J_1 \oplus J_3$ where J_k carries the representation $\mathbf{k}$ of dimension k, where J_1 is spanned by skew-symmetric forms while J_3 is spanned by symmetric forms, and both are stable under τ_1. Hence

$$\mathrm{Hom}(S_{\mathbf{R}} \otimes S_{\mathbf{R}}, \mathbf{C}) = B \otimes J^{\tau_1},$$

where $B = B^\tau$ is an invariant form for S and J^{τ_1} is the subspace of J fixed by the conjugation induced by τ_1. For a basis of J_1 we can take the symplectic form $b_0 = b_0^{\tau_1}$ given by $b_0(e_1, e_2) = 1$. Then $B_{\mathbf{R},0} = B \otimes b_0$ is invariant under σ and defines an invariant form for $S_{\mathbf{R}}$, fixed by the action of $\mathbf{A}_1$, and is unique up to a scalar factor. If b_j, $j = 1, 2, 3$, are a basis for $J_3^{\tau_1}$, then $B_{\mathbf{R},j} = B \otimes b_j$ are symmetric invariant forms for $S_{\mathbf{R}}$, defined up to a transformation of SO(3). The symmetry of $B_{\mathbf{R},0}$ is the *reverse* of that of B while those of the $B_{\mathbf{R},j}$ are the *same* as that of B. This takes care of the cases $\overline{\Sigma} = 3, 5$, $\overline{D}$ arbitrary, and $\overline{\Sigma} = 4$, $\overline{D} = 0, 4$. In the latter case the above argument applies to $S^\pm_{\mathbf{R}}$.

Suppose that $\overline{\Sigma} = 4$, $\overline{D} = 2, 6$. Then S^+ and S^- are dual to each other. We have the irreducible spin modules $S^\pm_{\mathbf{R}}$ with complexifications $S^\pm_{\mathbf{C}} = S^\pm_0 \otimes W^\pm$ and conjugations $\sigma^\pm = \tau^\pm \otimes \tau_1^\pm$ (with the obvious notation). The invariant form

$$B : S^+_{\mathbf{C}} \times S^-_{\mathbf{C}} \longrightarrow \mathbf{C}$$

is unique up to a scalar factor, and so, as before, we may assume that $B = B^{\mathrm{conj}}$ where

$$B^{\mathrm{conj}}(s^+, s^-) = B((s^+)^{\tau^+}, \quad (s^-)^{\tau^-})^{\mathrm{conj}}, \quad s^\pm \in S^\pm_{\mathbf{C}}.$$

For any form $b(W^+ \times W^- \longrightarrow \mathbf{C})$ such that $b^{\mathrm{conj}} = b$ where the conjugation is with respect to $\tau_1^\pm$,

$$B \otimes b : S^+_{\mathbf{C}} \otimes W^+ \times S^-_{\mathbf{C}} \otimes W^- \longrightarrow \mathbf{C}$$

is an invariant form fixed by σ and so restricts to an invariant form

$$S^+_{\mathbf{R}} \otimes S^-_{\mathbf{R}} \longrightarrow \mathbf{R}.$$

Thus $S_{\mathbf{R}}^{+}$ and $S_{\mathbf{R}}^{-}$ are in duality. As before there are no invariant forms on $S_{\mathbf{R}}^{\pm} \times S_{\mathbf{R}}^{\pm}$ separately. We have thus verified the second column in the two tables of Theorem 6.5.10.

The case $\overline{\Sigma} = 2, 6$ when the real spin modules $S_{\mathbf{R}}$ are of type $\mathbf{C}$ remains. In this case $S_{\mathbf{C}} = S^{+} \oplus S^{-}$ and is self-dual. We have a conjugate linear isomorphism $u \longmapsto u^{*}$ of S^{+} with S^{-} and

$$\sigma : (u, v^{*}) \longmapsto (v, u^{*})$$

is the conjugation of $S_{\mathbf{C}}$ that defines $S_{\mathbf{R}}$. The space of maps from $S_{\mathbf{C}}$ to its dual is of dimension 2, and so the space spanned by the invariant forms for $S_{\mathbf{C}}$ is of dimension 2. This space as well as its subspaces of symmetric and skew-symmetric elements are stable under the conjugation induced by σ. Hence the space of invariant forms for $S_{\mathbf{R}}$ is also of dimension 2 and spanned by its subspaces of symmetric and skew-symmetric forms. If $\overline{D} = 0$ (resp., 4), $S^{\pm}$ admit symmetric (resp., skew-symmetric) forms, and so all invariant forms for $S_{\mathbf{R}}$ are symmetric (resp., skew-symmetric). If $\overline{D} = 2, 6$, $S^{\pm}$ are dual to each other. The pairing between $S^{\pm}$ then defines two invariant forms on $S^{+} \oplus S^{-}$, one symmetric and the other skew-symmetric. Hence both the symmetric and skew-symmetric subspaces of invariant forms for $S_{\mathbf{C}}$ have dimension 1. So $S_{\mathbf{R}}$ has both symmetric and skew-symmetric forms.

It remains to determine the action of $\mathbf{A}_1 = T$ on the space of invariant forms for $S_{\mathbf{R}}$. For $b \in T$ its action on $S_{\mathbf{R}}$ is given by

$$(u, u^{*}) \longmapsto (bu, b^{\mathrm{conj}} u^{*}) .$$

$\overline{D} = 0, 4$. In this case the space of invariant forms for S^{+} is nonzero and has a basis β. The form

$$\beta^{*} : (u^{*}, v^{*}) \longmapsto \beta(u, v)^{\mathrm{conj}}$$

is then a basis for the space of invariant forms for S^{-}. The space of invariant forms for $S_{\mathbf{C}}$ is spanned by β, β^{*}, and the invariant forms for $S_{\mathbf{R}}$ are those of the form

$$\beta_c = c\beta + c^{\mathrm{conj}}\beta^{*} (c \in \mathbf{C}) .$$

The induced action of T is then given by

$$\beta_c \longmapsto \beta_{b^{-2}c} .$$

Thus the space of invariant forms for $S_{\mathbf{R}}$ is the module $\mathbf{2}$ for T given by

$$\mathbf{2} : e^{i\theta} \longmapsto \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$$

with respect to the basis β_1, β_i. In particular, there are no forms fixed by T.

$\overline{D} = 2, 6$. In this case we have a bilinear duality $\langle \cdot , \cdot \rangle$ of $S^{\pm}$ given by

$$u^{+}, u^{-} \longmapsto \langle u^{+}, u^{-} \rangle .$$

The action of $b \in T$ on $S_{\mathbf{C}}$ is

$$u^{+}, u^{-} \longmapsto bu^{+}, \overline{b}u^{-} .$$

The space of invariant forms for $S_{\mathbf{C}}$ is spanned by

$$((u^{+}, u^{-}), (v^{+}, v^{-})) \longmapsto \langle u^{+}, v^{-} \rangle \pm \langle v^{+}, u^{-} \rangle ,$$

which are, respectively, symmetric and skew-symmetric; they are clearly invariant under T. Hence both the real symmetric and skew-symmetric forms are invariant under T.

We now have the following theorem: Note that when $S^{\pm}_{\mathbf{R}}$ are dual, there are no forms on $S^{\pm}_{\mathbf{R}}$ individually. For the second columns in the two tables $\mathbf{k}$ denotes the representation of dimension k for SU(2), while for the third column in the second table the number $\mathbf{k}$ denotes the representation of T in which $e^{i\theta}$ goes over to the rotation by $2k\theta$. The notation $\pm[\mathbf{k}]$ means that the space of forms with symmetry $\pm$ carries the representation $[\mathbf{k}]$ of $\mathbf{A}_1$. When there is no number attached to a symmetry, it means that the form is unique up to a real scalar factor.

THEOREM 6.5.10 *The forms for the real irreducible spin modules are given by the following tables. Here* $\overline{D}, \overline{\Sigma}$ *denote the residue class of* D, Σ (mod 8), d.p. *means that* $S^{\pm}_{\mathbf{R}}$ *are dual to each other, and* $+$ *and* $-$ *denote symmetric and skew-symmetric forms.*

$\overline{D}\backslash\overline{\Sigma}$	1, 7(**R**)	3, 5(**H**)
1, 7	$+$	$-[\mathbf{1}], +[\mathbf{3}]$
3, 5	$-$	$+[\mathbf{1}], -[\mathbf{3}]$

$\overline{D}\backslash\overline{\Sigma}$	0(**R**, **R**)	4(**H**, **H**)	2, 6(**C**)
0	$+$	$-[\mathbf{1}], +[\mathbf{3}]$	$+[\mathbf{2}]$
4	$-$	$+[\mathbf{1}], -[\mathbf{3}]$	$-[\mathbf{2}]$
2, 6	d.p.	d.p.$[\mathbf{1}] \oplus [\mathbf{3}]$	$+[\mathbf{0}], -[\mathbf{0}]$

REMARK. The vector space of bilinear forms

$$\Gamma : S_{\mathbf{R},1} \times S_{\mathbf{R},2} \longrightarrow \mathbf{R},$$

where $S_{\mathbf{R},j}$ are real irreducible spin modules, is isomorphic to the space of maps

$$\gamma : S_{\mathbf{R},1} \longrightarrow S'_{\mathbf{R},2}$$

where $'$ denotes dual. These are left vector spaces over $\mathbf{A}$ if we define

$$(z\Gamma)(s_1, s_2) = \Gamma(zs_1, s_2), \quad (z\gamma)(s_1)(s_2) = \gamma(zs_1)(s_2).$$

Any $\gamma \neq 0$ is an isomorphism, and if γ' is any map, $\gamma^{-1}\gamma' \in \mathbf{A}$ and so $\gamma' = z\gamma$ for some $z \in \mathbf{A}$. So these vector spaces, if they are nonzero, are of dimension 1 over $\mathbf{A}$, hence of dimension equal to $\dim_{\mathbf{R}}(\mathbf{A})$ over $\mathbf{R}$. The computation of the separate dimensions of the symmetric and skew-symmetric parts, however, needs special arguments such as what we have used.

6.6. Morphisms from Spin Modules to Vectors and Exterior Tensors

As mentioned earlier, we need to know, for real irreducible spin modules S_1, S_2, the existence of symmetric morphisms $S_1 \otimes S_2 \longrightarrow V$ in the construction of super spacetimes, and more generally morphisms $S_1 \otimes S_2 \longrightarrow \Lambda^r(V)$ in the construction of super Poincaré and superconformal algebras. We shall now study this question, which is also essentially the same as the study of morphisms $\Lambda^r(V) \otimes S_1 \longrightarrow S_2$ (for $r = 1$ these morphisms allow one to define the Dirac or Weyl operators). Here again we first work over **C** and then come down to the real case. For $r = 0$ we are in the context of Section 6.5.

Let $D = \dim(V)$. We shall first assume that D is even. In this case we have the two semispin modules $S^\pm$ and their direct sum S_0, which is a simple supermodule for the full Clifford algebra. Write ρ for the isomorphism

$$\rho : C(V) \simeq \mathbf{End}(S_0)\,, \quad \dim(S_0) = 2^{(D/2)-1}|2^{(D/2)-1}\,.$$

Since $(S^\pm)^* = S^\pm$ or $S^\mp$ where $*$ denotes duals, it is clear that S_0 is self-dual. Since $-1 \in \mathrm{Spin}(V)$ goes to -1 in S_0, it follows that -1 goes to 1 in $S_0 \otimes S_0$ and so $S_0 \otimes S_0$ is an $\mathrm{SO}(V)$-module. We have, as $\mathrm{Spin}(V)$-modules,

$$S_0 \otimes S_0 \simeq S_0 \otimes S_0^* \simeq \mathrm{End}(S_0)$$

where $\mathrm{End}(S_0)$ is viewed as an ungraded algebra on which $g \in \mathrm{Spin}(V)$ acts by $t \longmapsto \rho(g)t\rho(g)^{-1} = \rho(gtg^{-1})$. Since ρ is an isomorphism of $C(V)$ with $\mathrm{End}(S_0)$ (ungraded), it follows that the action of $\mathrm{Spin}(V)$ on $C(V)$ is by inner automorphisms and so is the one coming from the action of the image of $\mathrm{Spin}(V)$ in $\mathrm{SO}(V)$. Thus

$$S_0 \otimes S_0 \simeq C(V)\,.$$

LEMMA 6.6.1 *If D is even, then S_0 is self-dual and*

$$S_0 \otimes S_0 \simeq 2\Bigg(\bigoplus_{0\le r\le D/2-1} \Lambda^r(V)\Bigg) \oplus \Lambda^{D/2}(V)\,.$$

In particular, because the $\Lambda^r(V)$ are irreducible for $r \le D/2 - 1$, we have

$$\dim\big(\mathrm{Hom}(S_0 \otimes S_0, \Lambda^r(V))\big) = 2\,, \quad 0 \le r \le \frac{D}{2} - 1\,.$$

PROOF: In view of the last relation above, it is a question of determining the $\mathrm{SO}(V)$-module structure of $C(V)$. This follows from the results of Section 5.2. The Clifford algebra $C = C(V)$ is filtered, and the associated graded algebra is isomorphic to $\Lambda = \Lambda(V)$. The skew-symmetrizer map (see Section 5.2)

$$\lambda : \Lambda \longrightarrow C$$

is manifestly equivariant with respect to $\mathrm{SO}(V)$, and so we have that, with $\Lambda^r = \Lambda^r(V)$,

$$\lambda : \Lambda = \bigoplus_{0\le r\le D} \Lambda^r \simeq C \tag{6.1}$$

is an isomorphism of $\mathrm{SO}(V)$-modules. If we now observe that $\Lambda^r \simeq \Lambda^{D-r}$ and that the Λ^r are irreducible for $0 \le r \le D/2 - 1$, the lemma follows immediately. □

Suppose now A, B, L are three modules for a group G. Then $\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(A \otimes B^*, \mathbf{C})$, where $\alpha(A \longrightarrow B)$ corresponds to the map (also denoted by α) of $A \otimes B^* \longrightarrow \mathbf{C}$ given by

$$\alpha(a \otimes b^*) = b^*(\alpha(a)).$$

So

$$\mathrm{Hom}(A \otimes B, L) \simeq \mathrm{Hom}(A \otimes B \otimes L^*, \mathbf{C}) \simeq \mathrm{Hom}(B \otimes L^*, A^*).$$

If A and L have invariant forms, we can use these to identify them with their duals and obtain a correspondence

$$\mathrm{Hom}(A \otimes B, L) \simeq \mathrm{Hom}(L \otimes B, A), \quad \gamma' \leftrightarrow \gamma,$$

where the corresponding elements γ', γ of the two Hom spaces are related by

$$(\gamma(\ell \otimes b), a) = (\gamma'(a \otimes b), \ell), \quad a \in A,\ b \in B,\ \ell \in L.$$

We remark that the correspondence $\gamma' \leftrightarrow \gamma$ depends on the choices of invariant forms on A and L. We now apply these considerations to the case when $G = \mathrm{Spin}(V)$ and $A = B = S_0$, $L = \Lambda^r$. The invariant form on V lifts to one on Λ^r. Now the Clifford algebra $C = C(V)$ is isomorphic to $\mathbf{End}(S_0)$, and so the theory of the B-group discussed earlier associates to (C, β) the invariant form $(\cdot, \cdot)$ on $S_0 \times S_0$, for which we have $(as, t) = (s, \beta(a)t)(a \in C)$. We then have a correspondence

$$\gamma' \leftrightarrow \gamma, \quad \gamma' \in \mathrm{Hom}(S_0 \otimes S_0, \Lambda^r),\ \gamma \in \mathrm{Hom}(\Lambda^r \otimes S_0, S_0),$$

such that

$$(\gamma'(s \otimes t), v) = (\gamma(v \otimes s), t), \quad s, t \in S,\ v \in \Lambda^r.$$

Let (see Section 5.2) λ be the skew-symmetrizer map (which is $\mathrm{Spin}(V)$-equivariant) of Λ onto C. The action of C on S_0 then gives a $\mathrm{Spin}(V)$-morphism

$$\gamma_0 : v \otimes s \longmapsto \lambda(v)s.$$

Let Γ_0 be the element of $\mathrm{Hom}(S_0 \otimes S_0, \Lambda^r)$ that corresponds to γ_0. We then have, with respect to the above choices of invariant forms,

$$(*) \qquad (\Gamma_0(s \otimes t), v) = (\lambda(v)s, t) = (s, \beta(\lambda(v))t), \quad s, t \in S,\ v \in \Lambda^r.$$

Note that Γ_0, γ_0 are both nonzero since $\lambda(v) \neq 0$ for $v \neq 0$. To the form on S_0 we can associate its parity π and the symmetry σ. Since $\lambda(v)$ has parity $p(r)$, it follows that $(\lambda(v)s, t) = 0$ when $p(r) + p(s) + p(t) + \pi = 1$. Thus $\Gamma_0(s \otimes t) = 0$ under the same condition. In other words, Γ_0 is even or odd, and

$$\text{parity } (\Gamma_0) = p(r) + \pi.$$

Since

$$\beta(\lambda(v)) = (-1)^{r(r-1)/2}\lambda(v), \quad v \in \Lambda^r,$$

it follows that Γ_0 is symmetric or skew-symmetric and

$$\text{symmetry } (\Gamma) = (-1)^{r(r-1)/2}\sigma.$$

The parity and symmetry of Γ_0 are thus dependent only on $\overline{D}$.

In case Γ_0 is even, i.e., when $\pi = p(r)$, Γ_0 restricts to nonzero maps

$$\Gamma^{\pm} : S^{\pm} \times S^{\pm} \longrightarrow \Lambda^r.$$

To see why these are nonzero, suppose for definiteness that $\Gamma^+ = 0$. Then $\Gamma_0(s \otimes t) = 0$ for $s \in S^+$, $t \in S^\pm$, and so $(\lambda(v)s, t) = 0$ for $s \in S^+$, $t \in S_0$, $v \in \Lambda^r$. Then $\lambda(v) = 0$ on S^+ for all $v \in \Lambda^r$, which is manifestly impossible because if (e_i) is an ON basis for V and $v = e_{i_1} \wedge \cdots \wedge e_{i_r}$, then $\lambda(v) = e_{i_1} \cdots e_{i_r}$ is invertible and so cannot vanish on S^+. The maps $\Gamma^\pm$ may be viewed as linearly independent elements of $\mathrm{Hom}(S_0 \otimes S_0, \Lambda^r)$. Since this Hom space has dimension 2, it follows that Γ^+, Γ^- form a basis of this Hom space. It follows that

$$\mathrm{Hom}(S^\pm \otimes S^\pm, \Lambda^r) = \mathbf{C}\Gamma^\pm\,, \quad \mathrm{Hom}(S^\pm \otimes S^\mp, \Lambda^r) = 0\,.$$

If $\pi = p(r) = 0$, then $S^\pm$ are self-dual. Let $\gamma^\pm$ be the restrictions of γ_0 to $S^\pm$ and

$$\mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\pm) = \mathbf{C}\gamma^\pm\,, \quad \mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\mp) = 0\,.$$

From Table 6.1 in Section 6.5 we see that $\pi = 0$ when $\overline{D} = 0, 4$, and then $\sigma = +, -$, respectively. Thus $\Gamma^\pm$ have the symmetry $(-1)^{r(r-1)/2}$ and $-(-1)^{r(r-1)/2}$, respectively, in the two cases.

If $\pi = p(r) = 1$, then $S^\pm$ are dual to each other, $\gamma^\pm$ map $\Lambda^r \otimes S^\pm$ to $S^\mp$, and we argue similarly that

$$\mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\mp) = \mathbf{C}\gamma^\pm\,, \quad \mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\pm) = 0\,.$$

We see from Table 6.1 in Section 6.5 that $\pi = 1$ when $\overline{D} = 2, 6$ with $\sigma = +, -$ respectively. Thus $\Gamma^\pm$ have the symmetry $(-1)^{r(r-1)/2}$ and $-(-1)^{r(r-1)/2}$, respectively, in the two cases.

If Γ is odd, i.e., when $\pi = p(r) + 1$, the discussion is entirely similar. Then Γ_0 is 0 on $S^\pm \otimes S^\pm$, and it is natural to define $\Gamma^\pm$ as the restrictions of Γ_0 to $S^\pm \otimes S^\mp$. Thus

$$\Gamma^\pm : S^\pm \times S^\mp \longrightarrow \Lambda^r\,,$$

and these are again seen to be nonzero. We thus obtain as before

$$\mathrm{Hom}(S^\pm \otimes S^\mp, \Lambda^r) = \mathbf{C}\Gamma^\pm\,, \quad \mathrm{Hom}(S^\pm \otimes S^\pm, \Lambda^r) = 0\,.$$

If $\pi = 1$, $p(r) = 0$, then $S^\pm$ are dual to each other, and

$$\mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\pm) = \mathbf{C}\gamma^\pm\,, \quad \mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\mp) = 0\,.$$

This happens when $\overline{D} = 2, 6$ and there is no symmetry.

If $\pi = 0$, $p(r) = 1$, then $S^\pm$ are self-dual, γ maps $\Lambda^r \otimes S^\pm$ to $S^\mp$, and

$$\mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\mp) = \mathbf{C}\gamma^\pm\,, \quad \mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\pm) = 0\,.$$

This happens when $\overline{D} = 0, 4$ and there is no symmetry.

This completes the treatment of the case when D, the dimension of V, is even.

We turn to the case when D is odd. As usual, the center Z of $C(V)$ now enters the picture. We have $Z = \mathbf{C}[\varepsilon]$ where ε is odd, $\varepsilon^2 = 1$, and $C(V) = C = C^+ \otimes Z$. The even algebra C^+ is isomorphic to $\mathrm{End}(S)$ where S is the spin module and $S \oplus S$ is the simple supermodule for C in which C^+ acts diagonally and ε acts as the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The basic lemma here is the following:

LEMMA 6.6.2 *If D is odd, then S is self-dual and*

$$S \otimes S \simeq \bigoplus_{0 \le r \le (D-1)/2} \Lambda^r$$

is the decomposition of $S \otimes S$ into irreducible components under SO(V). *In particular, the maps*

$$S \otimes S \longrightarrow \Lambda^r , \quad \Lambda^r \otimes S \longrightarrow S ,$$

are unique up to a scalar factor.

PROOF: The skew-symmetrizer isomorphism λ of $\Lambda(V)$ with C takes $\Lambda^{\text{even}} := \bigoplus_{0 \le k \le (D-1)/2} \Lambda^{2k}$ onto C^+. We have

$$S \otimes S \simeq \text{End}(S) \simeq C^+ \simeq \Lambda^{\text{even}} .$$

But now r and $D - r$ have opposite parity, and so exactly one of them is even. Hence

$$\Lambda^{\text{even}} \simeq \bigoplus_{0 \le r \le (D-1)/2} \Lambda^r .$$

This proves the decomposition formula for $S \otimes S$ and gives

$$\dim(\text{Hom}(S \otimes S, \Lambda^r)) = \dim(\text{Hom}(\Lambda^r \otimes S, S)) = 1 .$$

□

The rest of the discussion is essentially the same as in the case of even D. The form $(\cdot, \cdot)$ on S is such that $(as, t) = (s, \beta(a)t)$ for all $a \in C^+$, $s, t \in S$.

If r is even, we have $\lambda(v) \in C^+$ for all $v \in \Lambda^r$, and so the map $\gamma : v \otimes s \longmapsto \lambda(v)s$ is a nonzero element of $\text{Hom}(\Lambda^r \otimes S, S)$. We then obtain $\Gamma \in \text{Hom}(S \otimes S, \Lambda^r)$ defined by

$$(\Gamma(s \otimes t), v) = (\lambda(v)s, t) , \quad s, t \in S, \ v \in \Lambda^r, \ r \text{ even.}$$

There is no question of parity as S is purely even and

$$\text{symmetry } (\Gamma) = (-1)^{r(r-1)/2}\sigma$$

where σ is the symmetry of $(\cdot, \cdot)$. We use Table 6.3 of Section 6.5 for the values of σ that depend only on $\overline{D}$. Since $\text{Hom}(S \otimes S, \Lambda^r)$ has dimension 1 by Lemma 6.6.2, we must have

$$\text{Hom}(S \otimes S, \Lambda^r) = \mathbf{C}\Gamma , \quad \text{Hom}(\Lambda^r \otimes S, S) = \mathbf{C}\gamma .$$

The symmetry of Γ is $(-1)^{r(r-1)/2}$ or $-(-1)^{r(r-1)/2}$ according as $\overline{D} = 1, 7$ or $3, 5$.

If r is odd, $\varepsilon\lambda(v) \in C^+$ for all $v \in \Lambda^r$, and so if we define

$$\gamma_\varepsilon : v \otimes s \longmapsto \varepsilon\lambda(v)s ,$$

then

$$0 \neq \gamma_\varepsilon \in \text{Hom}(\Lambda^r \otimes S, S) .$$

We now define Γ_ε by

$$(\Gamma_\varepsilon(s \otimes t), v) = (\varepsilon\lambda(v)s, t) , \quad s, t \in S, \ v \in \Lambda^r, \ r \text{ odd,}$$

and obtain as before

$$\mathrm{Hom}(S \otimes S, \Lambda^r) = \mathbf{C}\Gamma_\varepsilon\,, \quad \mathrm{Hom}(\Lambda^r \otimes S, S) = \mathbf{C}\gamma_\varepsilon\,.$$

To calculate the symmetry of Γ_ε, we must note that β acts on ε by $\beta(\varepsilon) = s(\beta)\varepsilon$, and so

$$(\varepsilon\lambda(v)s, t) = s(\beta)(-1)^{r(r-1)/2}(s, \varepsilon\lambda(v)t)\mathbf{M}\,.$$

Hence

$$\text{symmetry }(\Gamma) = (-1)^{r(r-1)/2}s(\beta)\sigma\,.$$

We now use Table 6.3 for the values of σ and $s(\beta)$. The symmetry of γ_ε is either $(-1)^{r(r-1)/2}$ or $-(-1)^{r(r-1)/2}$ according as $\overline{D} = 1, 3$ or $5, 7$.

We can summarize our results in the following theorem. Here S_1, S_2 denote the irreducible spin modules $S^\pm$ when D is even and S when D is odd. Also, $r \le D/2 - 1$ or $r \le (D-1)/2$ according as D is even or odd. Let

$$\sigma_r = (-1)^{r(r-1)/2}\,.$$

THEOREM 6.6.3 *For complex quadratic vector spaces V the existence and symmetry properties of maps*

$$\Gamma : S_1 \otimes S_2 \longrightarrow \Lambda^r(V)\,, \quad \gamma : \Lambda^r \otimes S_1 \longrightarrow S_2\,,$$

and the symmetry properties of the maps Γ depend only on the residue class $\overline{D}$ of $D = \dim(V) \pmod 8$. The maps, when they exist, are unique up to scalar factors and are related by

$$(\Gamma(s_1 \otimes s_2), v) = (\gamma(v \otimes s_1), s_2)\,.$$

The maps γ exist only when $S_1 = S_2 = S$ (D odd), $S_1 = S_2 = S^\pm$ (D, r both even), $S_1 = S^\pm$, $S_2 = S^\mp$ (D even and r odd). In all cases the γ are given up to a scalar factor by the following table:

$r\backslash D$	Even	Odd
Even	$\gamma(v \otimes s^\pm) = \lambda(v)s^\pm$	$\gamma(v \otimes s) = \lambda(v)s$
Odd	$\gamma(v \otimes s^\pm) = \lambda(v)s^\pm$	$\gamma_\varepsilon(v \otimes s) = \varepsilon\lambda(v)s$

Here ε is a nonzero odd element in the center of $C(V)$ with $\varepsilon^2 = 1$. The maps Γ do not exist except in the cases described in the table below, which also gives their symmetry properties.

Morphisms over the Reals. The story goes along the same lines as it did for the forms. V is now a real quadratic vector space, and the modules Λ^r are real and define conjugations on their complexifications. For a real irreducible spin module $S_{\mathbf{R}}$, the space of morphisms $S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow \Lambda^r$ carries, as in the case of forms, an action by $\mathbf{A}_1$. In this case the space of morphisms $\Lambda^r \otimes S_{\mathbf{R}} \longrightarrow S_{\mathbf{R}}$ also carries an action of $\mathbf{A}_1$, and the identification of these two Hom spaces respects this action.

Let $S_{\mathbf{R}}$ be of type $\mathbf{R}$, i.e., $\overline{\Sigma} = 1, 7, 0$. The morphisms from $S \otimes S$, $S^\pm \otimes S^\pm$, $S^\pm \otimes S^\mp$ to Λ^r over $\mathbf{C}$ span one-dimensional spaces stable under conjugation. Hence we can choose basis elements for them that are real. The morphisms $\Lambda^r \otimes S_{\mathbf{R}} \longrightarrow S_{\mathbf{R}}$ defined in Theorem 6.6.3 make sense over $\mathbf{R}$ (we must take ε to be

	$\overline{D}$	Maps	Symmetry
r even	0	$S^{\pm} \otimes S^{\pm} \to \Lambda^r$	σ_r
	1, 7	$S \otimes S \to \Lambda^r$	σ_r
	2, 6	$S^{\pm} \otimes S^{\mp} \to \Lambda^r$	
	3, 5	$S \otimes S \to \Lambda^r$	$-\sigma_r$
	4	$S^{\pm} \otimes S^{\pm} \to \Lambda^r$	$-\sigma_r$
r odd	0, 4	$S^{\pm} \otimes S^{\mp} \to \Lambda^r$	
	1, 3	$S \otimes S \to \Lambda^r$	σ_r
	2	$S^{\pm} \otimes S^{\pm} \to \Lambda^r$	σ_r
	5, 7	$S \otimes S \to \Lambda^r$	$-\sigma_r$
	6	$S^{\pm} \otimes S^{\pm} \to \Lambda^r$	$-\sigma_r$

real) and span the corresponding Hom space over **R**. The results are then the same as in the complex case. The symmetries remain unchanged.

Let $S_{\mathbf{R}}$ be of type **H**, i.e., $\overline{\Sigma} = 3, 5, 4$. Let $B(A_1, A_2 : R)$ be the space of morphisms $A_1 \otimes A_2 \longrightarrow R$. The relevant observation is that if S_1, S_2 are complex irreducible spin modules and U is a Spin($V_{\mathbf{C}}$)-module such that $\dim(B(S_1, S_2 : U)) = 0$ or 1, then the space of morphisms $(S_1 \otimes W_1) \otimes (S_2 \otimes W_2) \longrightarrow U$ is just $B(S_1, S_2 : U) \otimes B(W_1, W_2 : \mathbf{C})$. The arguments are now the same as in the case of scalar forms since the second term $B(W_1, W_2 : \mathbf{C})$ is the same as in the scalar case. The symmetries follow the same pattern as in the case of $r = 0$.

The last case is when $S_{\mathbf{R}}$ is of type **C**, i.e., $\overline{\Sigma} = 2, 6$. Then $S_{\mathbf{C}} = S^+ \oplus S^-$ and so the morphisms $S_{\mathbf{C}} \otimes S_{\mathbf{C}} \longrightarrow \Lambda^r(V_{\mathbf{C}})$ form a space of dimension 2, and this space, as well as its subspaces of symmetric and skew-symmetric elements, are stable under the conjugation on the Hom space. From this point on the argument is the same as in the case $r = 0$.

THEOREM 6.6.4 (Odd Dimension) *For a real quadratic vector space V of odd dimension D, the symmetry properties of morphisms $S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow \Lambda^r$ are governed by the residue classes $\overline{D}, \overline{\Sigma}$ as in the following table. If no number is attached to a symmetry sign, then the morphism is determined uniquely up to a real scalar factor.*

	$\overline{D} \backslash \overline{\Sigma}$	1, 7(**R**)	3, 5(**H**)
r even	1, 7	σ_r	$-\sigma_r[1], \sigma_r[3]$
	3, 5	$-\sigma_r$	$\sigma_r[1], -\sigma_r[3]$
r odd	1, 3	σ_r	$-\sigma_r[1], \sigma_r[3]$
	5, 7	$-\sigma_r$	$\sigma_r[1], -\sigma_r[3]$

THEOREM 6.6.5 (Even Dimension) *For real quadratic vector spaces V of even dimension D, the symmetry properties of the maps $S^{\pm}_{\mathbf{R}} \otimes S^{\pm}_{\mathbf{R}}$, $S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow \Lambda^r$*

are described in Table 6.5. The notation d.p. *means that the morphism goes from* $S_{\mathbf{R}}^{\pm} \otimes S_{\mathbf{R}}^{\mp}$ *to* Λ^r. *If no number is attached to a symmetry sign, then the morphism is determined uniquely up to a real scalar factor.*

	$\overline{D}\backslash\Sigma$	$0(\mathbf{R},\mathbf{R})$	$4(\mathbf{H},\mathbf{H})$	$2,6(\mathbf{C})$
r even	0	σ_r	$-\sigma_r[\mathbf{1}]$, $\sigma_r[\mathbf{3}]$	$\sigma_r[\mathbf{2}]$
	2, 6	d.p.	d.p.	$\sigma_r[\mathbf{0}]$, $-\sigma_r[\mathbf{0}]$
	4	$-\sigma_r$	$\sigma_r[\mathbf{1}]$, $-\sigma_r[\mathbf{3}]$	$-\sigma_r[\mathbf{2}]$
r odd	0, 4	d.p.	d.p.	$\sigma_r[\mathbf{0}]$, $-\sigma_r[\mathbf{0}]$
	2	σ_r	$-\sigma_r[\mathbf{1}]$, $\sigma_r[\mathbf{3}]$	$\sigma_r[\mathbf{2}]$
	6	$-\sigma_r$	$\sigma_r[\mathbf{1}]$, $-\sigma_r[\mathbf{3}]$	$-\sigma_r[\mathbf{2}]$

TABLE 6.5

From the two theorems above we obtain the following theorem by inspection.

THEOREM 6.6.6 *The space of bilinear morphisms*

$$\Gamma : S_{\mathbf{R},1} \times S_{\mathbf{R},2} \longrightarrow \Lambda^r(V),$$

is either 0 *or has real dimension* 1, 2, *or* 4 *according as* $\mathbf{A} = \mathbf{R}$, $\mathbf{C}$, *or* $\mathbf{H}$. *It becomes a left vector space over* $\mathbf{A}$ *with the definition*

$$(z\Gamma)(s_1, s_2) = \Gamma(zs_1, s_2), \quad s_i \in S_{\mathbf{R},i}, z \in \mathbf{A}.$$

and this vector space has dimension 1 *over* $\mathbf{A}$. *In particular, if* Γ *is any nonzero element, all the elements of the space are of the form*

$$s_1, s_2 \longmapsto \Gamma(zs_1, s_2).$$

6.7. The Minkowski Signature and Extended Supersymmetry

In physics the special signatures $(1, D-1)$ and $(2, D-2)$ are important, especially the Minkowski signature $(1, D-1)$; the signature $(2, D-2)$ plays a role in super conformal theories since the corresponding orthogonal group is the conformal extension of the super Poincaré group. We shall now examine the Minkowski case more closely. In this case the signature is given by $\Sigma = D-2$ and so signature and dimension are coupled. This means that the entire theory is governed by a single periodicity, namely that of the dimension mod 8.

Let us consider this specialization for the morphisms

$$S_{\mathbf{R},1} \times S_{\mathbf{R},2} \longrightarrow V, \quad S_{\mathbf{R},1} \times S_{\mathbf{R},2} \longrightarrow \Lambda^r(V).$$

An examination of the tables in Theorems 6.6.4 and 6.6.5 reveals that when V has signature $(1, D-1)$, $S_{\mathbf{R}}$ is a real spin module irreducible over $\mathbf{R}$, and r is odd, there is *always* a unique (up to a real scalar factor) nontrivial morphism $\Gamma : S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow \Lambda^r(V)$ of symmetry type σ_r, and furthermore this morphism is invariant with respect to the action of $\mathbf{A}_1$. Indeed, when r is odd, the cases where there is a projectively unique morphism $S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow \Lambda^r(V)$ of symmetry type σ_r are given by

$$\begin{aligned} \overline{\Sigma} &= 1, 7\,, & \overline{D} &= 1, 3\,; \\ \overline{\Sigma} &= 3, 5\,, & \overline{D} &= 5, 7\,; \\ \overline{\Sigma} &= 2, 6\,, & \overline{D} &= 0, 4\,; \\ \overline{\Sigma} &= 0\,, & \overline{D} &= 2\,; \\ \overline{\Sigma} &= 4\,, & \overline{D} &= 6\,, \end{aligned}$$

which include all the cases when the signature is Minkowski since this case corresponds to the relations $\overline{D} \pm \overline{\Sigma} = 2$. In particular, in the Minkowski case, there is a projectively unique symmetric morphism

$$S_{\mathbf{R}} \times S_{\mathbf{R}} \longrightarrow V$$

and it is $\mathbf{A}_1$-invariant. It turns out (see Deligne[3]) that this morphism is positive definite in a natural sense. Let $V^{\pm}$ be the sets in V where the quadratic form Q of V is greater than 0 or less than 0, and let $(\cdot\,,\cdot)$ be the bilinear form associated to Q.

THEOREM 6.7.1 *Let V be a real quadratic vector space of dimension D and signature $(1, D-1)$, and let $S_{\mathbf{R}}$ be a real spin module irreducible over $\mathbf{R}$. Then, for r odd, there is a projectively unique morphism*

$$\Gamma : S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow \Lambda^r(V)$$

of symmetry type σ_r, and it is $\mathbf{A}_1$-invariant. In particular, there is a projectively unique nontrivial symmetric morphism

$$\Gamma : S_{\mathbf{R}} \times S_{\mathbf{R}} \longrightarrow V$$

and it is $\mathbf{A}_1$-invariant; i.e., $\Gamma(zs_1, zs_2) = \Gamma(s_1, s_2)$ for $s_i \in S_{\mathbf{R}}$, $z \in \mathbf{A}_1$. Moreover, we can normalize the sign of the scalar factor so that Γ is positive in the following sense: for $0 \neq s \in S_{\mathbf{R}}$ we have

$$(v, \Gamma(s,s)) > 0\,, \qquad v \in V^+\,.$$

Finally, whether $S_{\mathbf{R}}$ is irreducible or not, there is always a nontrivial symmetric morphism $S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow V$.

PROOF: We have already remarked that the existence and (projective) uniqueness of Γ follows from the tables of Theorems 6.6.4 and 6.6.5. It is thus a question of proving the positivity when $r = 1$. Write $S = S_{\mathbf{R}}$ for brevity.

For this we give the argument of Deligne.[3] First of all, we claim that the form $b_v(s,t) = (\Gamma(s,t), v)$ cannot be identically 0 for any $v \neq 0$; for if this were true for some v, it would be true for all $g \cdot v (g \in \mathrm{Spin}(V))$ and so, by irreducibility of S, for all elements of V. This is a contradiction. Fix now a $v \in V$ such that $Q(v) > 0$. Then b_v is invariant with respect to the stabilizer K of v in $\mathrm{Spin}(V)$. Because V is of Minkowski signature, it follows that $K \simeq \mathrm{Spin}(D-1)$ and is a maximal compact of $\mathrm{Spin}(V)$. If $\overline{D} = 2$ so that $V \simeq \mathbf{R}^{1,8k+1}$, $\overline{\Sigma} = 0$ so that we have two simple spin modules for $\mathrm{Spin}(V)$, $S^{\pm}_{\mathbf{R}}$, of type $\mathbf{R}$. The dimensions of $S^{\pm}_{\mathbf{R}}$ are equal to 2^{4k}, which is also the dimension of the spin module of $\mathrm{Spin}(8k+1)$. Since spin modules restrict on quadratic subspaces to spinorial modules, the restrictions to K of $S^{\pm}_{\mathbf{R}}$ are *irreducible*. But K is compact and so leaves a unique (up to a scalar) definite form invariant, and hence b_v is definite. We are thus done when $\overline{D} = 2$. In the general case we consider $V_0 = V \oplus V_1$ where V_1 is a negative definite quadratic space so that $\dim(V_0) \equiv 2 \pmod 8$. By the above result there are positive symmetric morphisms $\Gamma^{\pm}_0 : S^{\pm}_{0,\mathbf{R}} \longrightarrow V_0$. Let P be the projection $V_0 \longrightarrow V$. Now the representation $S^{+}_{0,\mathbf{R}} \oplus S^{-}_{0,\mathbf{R}}$ is faithful on $C(V_0)^+$, hence on $C(V)^+$. We claim that $S_{\mathbf{R}}$ is contained in $2S^{+}_{0,\mathbf{R}} \oplus 2S^{-}_{0,\mathbf{R}}$. Indeed, let U be $S^{+}_{0,\mathbf{R}} \oplus S^{-}_{0,\mathbf{R}}$ viewed as a $C(V)^+$-module. Then $U_{\mathbf{C}}$, being faithful on $C(V_{\mathbf{C}})^+$, contains all the complex irreducibles of $C(V_{\mathbf{C}})^+$. If $S_{\mathbf{R}}$ is of type $\mathbf{R}$ or $\mathbf{C}$, we have $S_{\mathbf{C}} = S, S^{\pm}, S^{+} \oplus 5$ and so $\mathrm{Hom}(S_{\mathbf{C}}, U_{\mathbf{C}}) \neq 0$, showing that $\mathrm{Hom}(S_{\mathbf{R}}, U) \neq 0$. If $S_{\mathbf{R}}$ is of type $\mathbf{H}$, then $S_{\mathbf{C}} = 2S, 2S^{\pm}$, and so $\mathrm{Hom}(S_c, 2U_{\mathbf{C}}) \neq 0$. Thus we have $S \hookrightarrow S^{+}_{0,\mathbf{R}}$ or $S \hookrightarrow S^{-}_{0,\mathbf{R}}$. Then we can define

$$\Gamma(s,t) = P\Gamma^{\pm}_0(s,t), \quad s,t \in S \hookrightarrow S^{\pm}_{0,\mathbf{R}}.$$

It is obvious that Γ is positive and hence nontrivial. An appeal to the projective uniqueness of Γ finishes the proof in the general case. □

If $S_{\mathbf{R}}$ is not irreducible, write $S_{\mathbf{R}} = \oplus_j S_{\mathbf{R},j}$ where the $S_{\mathbf{R},j}$ are irreducible. Choose $\Gamma_j : S_{\mathbf{R},j} \times S_{\mathbf{R},j} \longrightarrow V$ positive in the sense described above, and define $\Gamma = \oplus_j \Gamma_j$:

$$\Gamma : \sum_j s_j, \sum_k t_k \longmapsto \sum_r \Gamma_r(s_r, t_r).$$

Then Γ is positive and hence nontrivial.

Extended Supersymmetry. The above discussion of symmetric morphisms

$$S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow V$$

when $S_{\mathbf{R}}$ is not irreducible is very inadequate because it does not address the question of whether there are other ways of defining positive morphisms. Moreover, the requirement of positivity is introduced ad hoc, without motivation. We shall now discuss these points. The positivity comes from the fact that morphisms $S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow V$ determine super Lie algebras in which S is the odd part. The energy operator then gets expressed in terms of the odd elements (which are called *spinorial charges* in the physics literature), and the positivity described above is essentially the same as requiring that *the energy be positive*. We shall now discuss the second point mentioned above, namely, to determine all the symmetric morphisms in question when $S_{\mathbf{R}}$ is not reducible. The theories corresponding to the super Poincaré algebras whose odd parts are not irreducible are said to have *extended supersymmetry*. If N is the number of irreducible components of S, one speaks of *N-extended supersymmetry*. Occasionally N may denote the number of irreducible components in $S_{\mathbf{C}}$ rather than in S.

Our goal is to obtain a view of all symmetric morphisms

$$S \otimes S \longrightarrow V$$

where S is a module for $\mathrm{Spin}(V)$, V being of signature $(1, D-1)$. There is a natural group that acts on the set of these morphism. For a given choice of Γ we then have the stabilizer of Γ with respect to this action. This is the so-called *R-group*. We shall see that it is *compact* for Γ positive in the sense described above. In the physical literature these questions are either not treated properly or at best in a perfunctory manner. In particular the R-group is introduced without any naturalness, and its compactness is treated as a mystery. I follow the beautiful and transparent treatment of Deligne.[3]

Let $D \geq 3$. The real irreducible modules for $\mathrm{Spin}(V)$ are of type $\mathbf{R}$ in the cases $D \equiv 1, 2, 3 \pmod 8$, of type $\mathbf{H}$, i.e., quaternionic if $D \equiv 5, 6, 7 \pmod 8$, and of type $\mathbf{C}$ if $D \equiv 0, 4 \pmod 8$. In the even case and for the real and quaternionic types one can have a finer classification of extended supersymmetry: if $S_{\mathbf{R}}$ decomposes into N^+ copies of $S_{\mathbf{R}}^+$ and N^- copies of $S_{\mathbf{R}}^-$, we speak of *(N^+, N^-)-extended supersymmetry*. When N^+ or N^- is 0, the supersymmetry is called *chiral*; we should obviously extend this terminology to include all cases when $N^+ \neq N^-$.

Let S_0 be an irreducible spin module of $\mathrm{Spin}(V)$ where V is a quadratic vector space of signature $(1, D-1)$. We denote by Γ_0 a symmetric morphism

$$S_0 \otimes S_0 \longrightarrow V\,,$$

which is invariant under the action of the group $\mathbf{A}_1$ of elements of norm 1 in the commutant $\mathbf{A}$ in S, and which is positive. $\mathbf{A}$ is a division algebra $\simeq \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, and the invariance means that

$$\Gamma_0(zs, zt) = \Gamma_0(s, t)\,, \quad s, t \in S,\ z \in \mathbf{A}_1\,.$$

The positivity means that

$$(v, \Gamma_0(s, s)) > 0 \quad 0 \neq s \in S,\ v \in C^+,$$

where C^+ is the forward light cone. One can choose an identification of V with $\mathbf{R}^{1,D-1}$ and define C^+ in terms of the coordinates $v_i (0 \leq i \leq D-1)$ by

$$C^+ = \{v \in V \mid v_0 > 0, (v, v) > 0\}\,.$$

Since

$$(v, u) > 0 \quad \text{for all } v \in C^+ \Leftrightarrow u \in (C^+)^0$$

where $(C^+)^0$ is the interior of C^+, it follows that the positivity of Γ is equivalent to saying that for $0 \neq s \in V$, $\Gamma_0(s, s)$ lies in the interior of the forward light cone.

Case 1. $S = N \cdot S_0$ $(N \geq 1)$. Note that S_0 can be of any type. Recall that the morphisms

$$S_0 \otimes S_0 \longrightarrow V$$

are all of the form

$$\Gamma_{0,z} : s, t \longmapsto \Gamma_0(zs, t)$$

for suitable choices of $z \in \mathbf{A}$. Choose a decomposition

$$S = S_0 \oplus S_0 \oplus \cdots \oplus S_0$$

so that any morphism $\Gamma : S \otimes S \longrightarrow V$ can be represented by a matrix (Γ_{ij}) where the Γ_{ij} are morphisms $S_0 \otimes S_0 \longrightarrow V$. Hence we have a matrix (z_{ij}) of elements $z_{ij} \in \mathbf{A}$ such that

$$\Gamma \sim (\Gamma_{0,z_{ij}}).$$

We write

$$\Gamma \sim (z_{ij}).$$

This is a bijection between $\mathrm{Hom}(S \otimes S, V)$ and $N \times N$ matrices of elements of $\mathbf{A}$. It is now a question of identifying the matrices that correspond to symmetric and positive symmetric morphisms.

To find out how the symmetry condition translates in terms of matrices, we need the following simple formula:

$$(*) \qquad \Gamma_0(zs, t) = \Gamma_0(s, z^* t)$$

where $z \longmapsto z^*$ is the natural involution on $\mathbf{A}$. In fact, write $z = rg$ where r is real and > 0 and g has norm 1. Then

$$\Gamma_0(zs, t) = r\Gamma(gs, t) = r\Gamma_0(s, g^{-1}t) = r\Gamma_0(s, g^* t) = \Gamma_0(s, z^* t).$$

This said, let $s = (s_i)$, $t = (t_j) \in S$, where $s_i, t_j \in S_0$. Then

$$\begin{aligned}\Gamma(s, t) = \sum_{ij} \Gamma_0(z_{ij}s_i, t_j) &= \sum_{ij} \Gamma_0(s_i, z_{ij}^* t_j) \\ &= \sum_{ij} \Gamma_0(z_{ij}^* t_j, s_i) = \sum_{ij} \Gamma_0(z_{ji}t_i, s_j).\end{aligned}$$

But

$$\Gamma(s, t) = \Gamma(t, s) = \sum_{ij} \Gamma_0(z_{ij}t_i, s_j).$$

Choosing $t_i = \delta_{ip}t$, $s_j = \delta_{jq}s$, we get $\Gamma_0(z_{pq}t, s) = \Gamma_0(z_{qp}^* t, s)$, i.e., $z_{qp}^* = z_{pq}$, which gives the Hermitian nature of the matrix (z_{ij}). The converse that a Hermitian matrix (z_{ij}) defines a symmetric morphism is a retracing of these steps. Thus we get the result that in the bijection between $\mathrm{Hom}(S \otimes S, V)$ and $N \times N$ matrices over $\mathbf{A}$, the symmetric morphisms correspond to Hermitian matrices.

It is possible to do this in an invariant manner without starting from a (non-canonical) decomposition of S as a multiple of S_0. Write $W = \mathrm{Hom}_{\mathbf{R}}(S_0, S)$. It is obvious that W is a right vector space over $\mathbf{A}$ in a natural manner. We have a map

$$W \otimes_{\mathbf{A}} S_0 \longrightarrow S, \qquad w \otimes s \longmapsto w(s).$$

It is not difficult to show that this is an isomorphism. To prove this, write $S = \oplus_{1 \le i \le N} S_i$ with $w_i : S_0 \simeq S_i$ an isomorphism. Let P_i be the projections $S \longrightarrow S_i$. If $w \in W$, then $P_i w$ is a morphism from S_0 to S_i and so $P_i w = w_i z_i$ for a unique $z_i \in \mathbf{A}$. Thus $w = \sum_i w_i z_i$, showing that $\dim_{\mathbf{A}} W = N$. Since $W \otimes_{\mathbf{A}} S_0 = \oplus_i w_i \otimes S_0$ it follows that $\dim_{\mathbf{R}}(W \otimes_{\mathbf{A}} S_0) = N \dim_{\mathbf{R}}(S_0) = \dim_{\mathbf{R}}(S)$. On the other hand, as $w_i \otimes s$ goes to $w_i(s)$, the map in question is surjective. Hence it is an isomorphism.

We may therefore identify S with $W \otimes_{\mathbf{A}} S_0$ and define, for any $\Gamma(S \times S, V)$, the element $\psi = \psi_\Gamma(W \times W \longrightarrow \mathbf{A})$ by

$$\Gamma(w_1' \otimes s, w_2' \otimes t) = \Gamma_0(\psi(w_1', w_2')s, t) .$$

If $w_1' = w_i, w_2' = w_j$, we get (with appropriate identification) $\psi(w_i, w_j) = z_{ij}$. The correspondence $\Gamma \leftrightarrow \psi_\Gamma$ is then just the correspondence $\Gamma \leftrightarrow (z_{ij})$. As before we conclude that this gives a bijection between symmetric morphisms Γ and *Hermitian* forms ψ_Γ on $W \times W$ with values in $\mathbf{A}$.

The theory of Hermitian $\mathbf{A}$-valued forms on right vector spaces over the division algebra $\mathbf{H}$ does not differ in any essential detail from the theory of real symmetric or complex Hermitian forms. From this theory it follows that if ψ is a Hermitian form on $W \times W$, there is a basis (w_r) for W over $\mathbf{A}$ such that $\psi(w_p, w_q) = \varepsilon_p \delta_{pq}$ with $\varepsilon_p = 0, \pm 1$. All the ε's are ± 1 if and only if ψ is nondegenerate in the usual sense; in this case, the numbers a, b of ε's that are $+1$ and -1, respectively, are invariants of ψ and (a, b) is the *signature* of ψ. The *positive definite* ψ are those for which all the ε_p are $+1$; they are precisely those ψ for which $\psi(w, w)$ is real and > 0 for $0 \neq w \in W$.

The group $\mathrm{GL}(W)$ acts on the space of Hermitian forms:

$$g\psi(w, w') = \psi(g^{-1}w, g^{-1}w') , \quad w, w' \in W,\ g \in \mathrm{GL}(W) .$$

The orbits for this action are classified by the signature. In particular, the positive definite ones are all in one orbit, namely, the orbit of the form ψ_0 defined by

$$\psi_0(w_p, w_q) = \delta_{pq} .$$

Let us return to the space of morphisms $S \times S \longrightarrow V$. The isomorphism $W \otimes_{\mathbf{A}} S_0 \simeq S$ is compatible with the action of $\mathrm{Spin}(V)$: $h \in \mathrm{Spin}(V)$ acts on $W \otimes_{\mathbf{A}} S_0$ as $\mathrm{id} \otimes h$. The group $\mathrm{GL}(W)$ acts on $W \otimes_{\mathbf{A}} S_0$ as $g \otimes \mathrm{id}$. The action of $\mathrm{GL}(W)$ gives rise to an action on the space of Γ's by

$$g\Gamma(s, t) = \Gamma(g^{-1}s, g^{-1}t) , \quad s, t \in S,\ g \in \mathrm{GL}(W) .$$

It is to be expected that

$$\psi_{g\Gamma} = g\psi_\Gamma ,$$

and this is indeed true and easy to check.

We shall now show that Γ is positive in the sense described earlier if and only if ψ_Γ is positive definite. Let Γ be positive and choose a basis (w_i) of W such that $\psi_\Gamma(w_i, w_j) = \varepsilon_i \delta_{ij}$ with $\varepsilon_i \in \{0, \pm 1\}$. We have

$$0 < (v, \Gamma(w_i \otimes s, w_i \otimes s)) = \varepsilon_i (v, \Gamma_0(s, s))$$

showing that $\varepsilon_i = 1$ for all i. So ψ_Γ is positive definite. Suppose conversely that ψ_γ is positive definite. Choose a basis (w_i) for W such that $\psi_\Gamma(w_i, w_j) = \delta_{ij}$. Then for $s = \sum_i w_i \otimes s_i$ we have

$$\Gamma(s, s) = \sum_i \Gamma_0(s_i, s_i) ,$$

showing that Γ is positive. With this basis $W \simeq \mathbf{A}^N$, ψ_Γ becomes the form

$$\sum_i z_i'^* z_i$$

and its stabilizer is $\mathrm{U}(\mathbf{A}^N)$. Notice that in this case there is an isomorphism $S \simeq \oplus_{1\le i\le N} S_0$ such that

$$\Gamma\Big(\sum_i s_i, \sum_j t_j\Big) = \sum_{1\le i\le N} \Gamma_0(s_i, t_i)\,.$$

Thus every positive Γ arises in the manner suggested by the example above, i.e., is a direct sum of N copies of Γ_0.

Case 2. $S \simeq N^+S_0^+ \oplus N^-S_0^-$. Of course, this case arises only when D is even and the real irreducible spin modules are of real or quaternionic type, i.e., $D \equiv 2, 6 \pmod 8$. In this case the morphisms $S_0^\pm \times S_0^\mp \longrightarrow V$ are 0. We thus have bijections

$$\Gamma \leftrightarrow (\Gamma^+, \Gamma^-) \leftrightarrow (\psi^+, \psi^-)$$

between symmetric morphisms Γ, symmetric morphisms $\Gamma^\pm(S^\pm \times S^\pm \longrightarrow N^\pm S_0)$, and Hermitian forms $\psi^\pm(W^\pm \times W^\pm \longrightarrow \mathbf{A})$, where $W^\pm = \mathrm{Hom}(S_0^\pm, S)$. The stability group of a positive element is $\simeq \mathrm{U}(\mathbf{A}^{N^+}) \times \mathrm{U}(\mathbf{A}^{N^-})$.

We summarize this discussion in the following theorem:

THEOREM 6.7.2

(i) *If $S = N \cdot S_0$ where S_0 is a real irreducible spin module, the symmetric morphisms $\Gamma(S \times S \longrightarrow V)$ are in natural bijection with Hermitian forms $\psi_\Gamma(W \times W \longrightarrow \mathbf{A})$ where $W = \mathrm{Hom}(S_0, S)$; the bijection is compatible with the action of $\mathrm{GL}(W)$. Γ is positive if and only if ψ_Γ is positive definite. In this case there is an isomorphism $S \simeq \oplus_{1\le i\le N} S_0$ such that*

$$\Gamma\Big(\sum_i s_i, \sum_j t_j\Big) = \sum_{1\le i\le N} \Gamma_0(s_i, t_i)\,.$$

The stabilizer of Γ inside $\mathrm{GL}(W)$ is $\simeq$ the unitary group $\mathrm{U}(\mathbf{A}^N)$.

(ii) *Suppose D is even and $S = N^+ \cdot S_0^+ \oplus N^- \cdot S_0^-$ where $S_0^\pm$ are the real irreducible spin modules of type $\mathbf{R}$ or $\mathbf{H}$. Then the Γ are in natural bijection with the set of pairs (ψ^+, ψ^-) where $\psi^\pm$ are Hermitian forms $W^\pm \times W^\pm \longrightarrow \mathbf{A}$. The bijection is compatible with the action of $\mathrm{GL}(W)$. Γ is positive if and only if $\psi^\pm$ are positive definite. The stability group of Γ is $\simeq \mathrm{U}(\mathbf{A}^{N^+}) \times \mathrm{U}(\mathbf{A}^{N^-})$.*

6.8. Image of the Real Spin Group in the Complex Spin Module

From Theorem 6.5.10 we find that when $\overline{D} = 1$, $\overline{\Sigma} = 1$, the spin module $S_{\mathbf{R}}$ is of type $\mathbf{R}$ and has a symmetric invariant form, and so the spin group $\mathrm{Spin}(V)$ is embedded in a real orthogonal group. The question naturally arises as to what the *signature* of this orthogonal group is. More generally, it is a natural question to ask what can be said about the image of the real spin group in the spinor space. This question makes sense even when the complex spin representation does not have a real form. In this section we shall try to answer this question. The results discussed here can be found in D'Auria et al.[2] They are based on E. Cartan's classification of the real forms of complex simple Lie algebras[6] and a couple of simple lemmas.

Let V be a real quadratic vector space. A complex irreducible representation of $\mathrm{Spin}(V)$ is said to be *strict* if it does not factor through to a representation of $\mathrm{SO}(V)^0$. The spin and semispin representations are strict but so are many others. Indeed, the strict representations are precisely those that send the nontrivial central element of the kernel of $\mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V)^0$ to -1 in the representation space. If $D = \dim(V) = 1$, $\mathrm{Spin}(V)$ is $\{\pm 1\}$ and the only strict representation is the spin representation that is the nontrivial character. In dimension 2, if V is definite, we have $\mathrm{Spin}(V) = \mathrm{U}(1)$ with $\mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V)^0 \simeq \mathrm{U}(1)$ as the map $z \longmapsto z^2$, and the strict representations are the characters $z \longrightarrow z^n$ where n is an odd integer; the spin representations correspond to $n = \pm 1$. If V is indefinite, $\mathrm{Spin}(V) = \mathbf{R}^\times$, $\mathrm{SO}(V)^0 = \mathbf{R}_+^\times$, and the covering map is $t \longmapsto t^2$; the strict representations are the characters $t \longrightarrow \mathrm{sgn}(t)|t|^z$ where $z \in \mathbf{C}$, and the spin representations correspond to $z = \pm 1$. In dimension 3 when $\mathrm{Spin}(V) = \mathrm{SL}(2, \mathbf{C})$, the strict representations are the nontrivial representations of even dimension; the spin representation is the one with dimension 2.

LEMMA 6.8.1 *If $D > 2$, the spin representations are precisely the strict representations of minimal dimension; i.e., if a representation is strict and different from the spin representation, its dimension is strictly greater than the dimension of the spin representation.*

PROOF: We go back to the discussion of the basic structure of the orthogonal Lie algebras in Section 5.3. Let $\mathfrak{g} = \mathfrak{so}(V)$.

$\mathfrak{g} = D_\ell$. The positive roots are

$$a_i - a_j, \quad 1 \le i < j \le \ell, \quad a_p + a_q, \quad 1 \le p < q \le \ell .$$

If $b_1, \ldots, b_\ell$ are the fundamental weights, then we have

$$b_i = a_1 + \cdots + a_i, \quad 1 \le i \le \ell - 2,$$

while

$$b_{\ell-1} = \frac{1}{2}(a_1 + \cdots + a_{\ell-1} - a_\ell), \quad b_\ell = \frac{1}{2}(a_1 + \cdots + a_{\ell-1} + a_\ell).$$

For any dominant integral linear form λ, we write π_λ for the irreducible representation with highest weight λ. The weights of V are $(\pm a_i)$, and it is not difficult to verify (see Varadarajan,[7] chap. 4) that

$$\Lambda^r \simeq \pi_{b_r}, \quad 1 \le r \le \ell - 2, \quad \Lambda^{\ell-1} \simeq \pi_{b_{\ell-1}+b_\ell}, \quad \Lambda^\ell \simeq \pi_{2b_{\ell-1}} \oplus \pi_{2b_\ell}.$$

The most general highest weight is $\lambda = m_1 b_1 + \cdots + m_\ell b_\ell$ where the m_i are integers ≥ 0. Expressing it in terms of the a_i, we see that it is an integral linear combination of the a_i if and only if $m_{\ell-1}$ and m_ℓ have the same parity, and this is the condition that the representation π_λ occurs among the tensor spaces over V. So the strictness condition is that $m_{\ell-1}$ and m_ℓ have opposite parities. The semispin representations correspond to the choices where $m_i = 0$ for $1 \le i \le \ell - 2$ and $(m_{\ell-1}, m_\ell) = (1, 0)$

or (0, 1). If $m_{\ell-1}$ and m_ℓ have opposite parities, then one of $m_{\ell-1}, m_\ell$ is odd and so ≥ 1. Hence

$$(m_1, \dots, m_\ell) = \begin{cases} (m_1, \dots, m_{\ell-2}, m_{\ell-1} - 1, m_\ell) + (0, \dots, 0, 1, 0), & m_{\ell-1} \geq 1, \\ (m_1, \dots, m_{\ell-1}, m_\ell - 1) + (0, \dots, 0, 1), & m_\ell \geq 1. \end{cases}$$

The result follows if we remark that the Weyl dimension formula for π_μ implies that

$$\dim(\pi_{\mu+\nu}) > \dim(\pi_\mu), \quad \nu \neq 0,$$

where μ, ν are dominant integral.

$\mathfrak{g} = B_\ell$. The positive roots are

$$a_i - a_j, \ 1 \leq i < j \leq \ell, \quad a_p + a_q, \ 1 \leq p < q \leq \ell, \quad a_i, \ 1 \leq i \leq \ell.$$

If $b_1, \dots, b_\ell$ are the fundamental weights, then we have

$$b_i = a_1 + \cdots + a_i, \ 1 \leq i \leq \ell - 1, \quad b_\ell = \frac{1}{2}(a_1 + \cdots + a_\ell).$$

For a dominant integral $\lambda = m_1 b_1 + \cdots + m_\ell b_\ell$ we find that it is an integral linear combination of the a_i's if and only if m_ℓ is even. So the strictness condition is that m_ℓ should be odd. If m_ℓ is odd, we can write

$$(m_1, \dots, m_\ell) = (m_1, \dots, m_{\ell-1}, m_\ell - 1) + (0, \dots, 0, 1)$$

from which the lemma follows again by using Weyl's dimension formula. □

Let $d_0 = 1$ and let d_p ($p \geq 1$) be the dimension of the spin module(s) of Spin(p). Recall from Section 5.3 that

$$d_p = 2^{[\frac{p+1}{2}]-1}, \quad p \geq 1.$$

LEMMA 6.8.2 *Let π be a representation of* Spin(p, q) *in a vector space U with the property that $\pi(\varepsilon) = -1$ where ε is the nontrivial element in the kernel of* Spin$(p, q) \longrightarrow$ SO$(p, q)^0$, *and let $K_{p,q}$ be the maximal compact subgroup of* Spin(p, q) *lying above $K_0 =$* SO$(p) \times$ SO(q). *If W is any nonzero subspace of U invariant under $\pi(K_{p,q})$, then*

$$\dim(W) \geq d_p d_q.$$

In particular, if H is a real, connected, semisimple Lie subgroup of GL(U) *such that $\pi(K_{p,q}) \subset H$, and L a maximal compact subgroup of H, then for any nonzero subspace W of U invariant under L, we have*

$$\dim(W) \geq d_p \, d_q.$$

PROOF: The cases $p = 0, 1$, $q = 0, 1, 2$, are trivial since the right side of the inequality to be established is 1.

$p = 0, 1$, $q \geq 3$. Then $K_{p,q} =$ Spin(q). We may obviously assume that W is irreducible. Then we have a strict irreducible representation of Spin(q) in W and hence, by Lemma 6.8.1, we have the desired inequality.

$2 \leq p \leq q$. In this case we use the description of $K_{p,q}$ given in the remark following Theorem 5.3.7 so that ε_r maps on ε for $r = p, q$. We can view the restriction of π to $K_{p,q}$ as a representation ρ of $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$ acting irreducibly on W. Then $\rho \simeq \rho_p \times \rho_q$ where ρ_r is an irreducible representation of $\mathrm{Spin}(r)$ ($r = p, q$). Since $\rho(\varepsilon_p) = \rho(\varepsilon_q) = -1$, by our hypothesis it follows that $\rho_p(\varepsilon_p) = -1$, $\rho_q(\varepsilon_q) = -1$. Hence ρ_r is a strict irreducible representation of $\mathrm{Spin}(r)$ ($r = p, q$), so that $\dim(\rho_r) \geq d_r$ ($r = p, q$). But then

$$\dim(W) = \dim(\rho_p)\dim(\rho_q) \geq d_p d_q \,.$$

This proves the first statement.

For the second statement we proceed as follows. Choose a maximal compact M of H containing $\pi(K_{p,q})$; this is always possible because $\pi(K_{p,q})$ is a compact, connected subgroup of H. There is an element $h \in H$ such that $hLh^{-1} = M$. Since W is invariant under L if and only if $h[W]$ is invariant under hLh^{-1}, and $\dim(W) = \dim(h[W])$, it is clear that we may replace L by M. But then W is invariant under $\pi(K_{p,q})$ and the result follows from the first assertion. This finishes the proof of the lemma. □

COROLLARY 6.8.3 *Suppose π is the irreducible complex spin representation. Let $N = \dim(\pi)$ and let H, L be as in the lemma. Then, for any nonzero subspace W of U invariant under L we have*

$$\dim(W) \geq \begin{cases} \frac{N}{2} & \text{if one of } p, q \text{ is even} \\ N & \text{if both } p, q \text{ are odd.} \end{cases}$$

In particular, when both p and q are odd, the spin module of $\mathrm{Spin}(p, q)$ *is already irreducible when restricted to its maximal compact subgroup.*

PROOF: We can assume p is even for the first case since everything is symmetric between p and q. Let $p = 2k$, $q = 2\ell$ or $2\ell + 1$; we have $d_p = 2^{k-1}$, $d_q = 2^{\ell-1}$ or 2^{ℓ} while $N = 2^{k+\ell-1}$ or $2^{k+\ell}$ and we are done. If $p = 2k + 1$, $q = 2\ell + 1$, then $d_p = 2^k$, $d_q = 2^\ell$, $N = 2^{k+\ell}$, and hence $d_p d_q = N$. This implies at once that U is already irreducible under $K_{p,q}$. □

Review of Real Forms of Complex Semisimple Lie Algebras. If $\mathfrak{g}$ is a complex Lie algebra, by a *real form* of $\mathfrak{g}$ we mean a real Lie algebra $\mathfrak{g}_0 \subset \mathfrak{g}$ such that $\mathfrak{g} \simeq \mathbf{C} \otimes_{\mathbf{R}} \mathfrak{g}_0$. This amounts to requiring that there is a basis of $\mathfrak{g}_0$ over $\mathbf{R}$ that is a basis of $\mathfrak{g}$ over $\mathbf{C}$. Then the map $X + iY \longmapsto X - iY$ ($X, Y \in \mathfrak{g}$) is a *conjugation* of $\mathfrak{g}$, i.e., a conjugate linear map of $\mathfrak{g}$ onto itself preserving brackets such that $\mathfrak{g}_0$ is the set of fixed points of this conjugation. If G is a connected complex Lie group, a connected real Lie subgroup $G_0 \subset G$ is called a *real form* of G if $\mathrm{Lie}(G_0)$ is a real form of $\mathrm{Lie}(G)$. E. Cartan determined all real forms of complex simple Lie algebras $\mathfrak{g}$ up to conjugacy by the adjoint group of $\mathfrak{g}$, leading to a classification of real forms of the complex classical Lie groups. We begin with a summary of Cartan's results.[6] Note that if ρ is any conjugate linear transformation of $\mathbf{C}^n$, we can write $\rho(z) = Rz^\sigma$ where R is a linear transformation and $\sigma : z \mapsto z^{\mathrm{conj}}$ is the standard conjugation of $\mathbf{C}^n$; if $R = (r_{ij})$, then the r_{ij} are defined by $\rho e_j = \sum_i r_{ij} e_i$. We have $R\overline{R} = \pm 1$ according as ρ is a conjugation or a pseudoconjugation. We say ρ corresponds to R; the standard conjugation corresponds to $R = I_n$. If we take $R = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, we get the standard pseudo conjugation τ of $\mathbf{C}^{2n}$ given by $\tau : (z, w) \longmapsto (-\overline{w}, \overline{z})$. If L is an endomorphism of $\mathbf{C}^n$, then L commutes with the antilinear transformation defined by R if and only if $LR = R\overline{L}$.

$G = \mathrm{SL}(n, \mathbf{C})$. The real forms are

$$(\sigma)\,\mathrm{SL}(n, \mathbf{R}), \quad \mathrm{SU}(a, b), \quad a \le b,\ a + b = n,$$
$$(\tau)\,\mathrm{SU}^*(2m) \simeq \mathrm{SL}(m, \mathbf{H}), \quad n = 2m,$$

where the notation is the usual one and the symbol placed before the real form means that it is the subgroup commuting with the conjugation or pseudoconjugation described. We write $\mathrm{SU}(n)$ for $\mathrm{SU}(0, n)$. It is the unique (up to conjugacy) *compact* real form.

The isomorphism

$$\mathrm{SU}^*(2m) \simeq \mathrm{SL}(m, \mathbf{H})$$

needs some explanation. If we identify $\mathbf{C}^2$ with the quaternions $\mathbf{H}$ by $(z, w) \longmapsto z + \mathbf{j}w$, then the action of $\mathbf{j}$ from the right on $\mathbf{H}$ corresponds to the pseudoconjugation $(z, w) \longmapsto (-\overline{w}, \overline{z})$. If we make the identification of $\mathbf{C}^{2m}$ with $\mathbf{H}^m$ by

$$(z_1, \dots, z_m, w_1, \dots, w_m) \longmapsto (z_1 + \mathbf{j}w_1, \dots, z_m + \mathbf{j}w_m),$$

we have an isomorphism between $\mathrm{GL}(m, \mathbf{H})$ and the subgroup G of $\mathrm{GL}(2m, \mathbf{C})$ commuting with the pseudoconjugation τ. It is natural to call the subgroup of $\mathrm{GL}(m, \mathbf{H})$ that corresponds to $G \cap \mathrm{SL}(2m, \mathbf{C})$ under this isomorphism as $\mathrm{SL}(m, \mathbf{H})$. The group G is a direct product of $H = G \cap \mathrm{U}(2m)$ and a vector group. If J is as above, then H is easily seen to be the subgroup of $\mathrm{U}(2m)$ preserving the symplectic form with matrix J, and so H coincides with $\mathrm{Sp}(2m)$; hence H is connected. So G is connected. On the other hand, the condition $gJ = J\overline{g}$ implies that $\det(g)$ is real for all elements of G. Hence the determinant is greater than 0 for all elements of G. It is clear then that G is the direct product of $G \cap \mathrm{SL}(2m, \mathbf{C})$ and the positive homotheties, i.e., $G \simeq G \cap \mathrm{SL}(2m, \mathbf{C}) \times \mathbf{R}_+^\times$. Thus $\mathrm{GL}(m, \mathbf{H}) \simeq \mathrm{SL}(m, \mathbf{H}) \times \mathbf{R}_+^\times$.

$G = \mathrm{SO}(n, \mathbf{C})$. The real forms are

$$(\sigma_a)\,\mathrm{SO}(a, b), \quad a \le b,\ a + b = n,$$
$$(\tau)\,\mathrm{SO}^*(2m), \quad n = 2m.$$

σ_a is the conjugation corresponding to $R_a = \begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}$; if $x = \begin{pmatrix} I_a & 0 \\ 0 & iI_b \end{pmatrix}$, then it is easily verified that $x\,\mathrm{SO}(a, b)x^{-1}$ is the subgroup of $\mathrm{SO}(n, \mathbf{C})$ fixed by σ_a. It is also immediate that

$$g^{\mathsf{T}}g = I_{2m}, \quad gJ_{2m} = J_{2m}\overline{g} \iff g^{\mathsf{T}}g = I_{2m}, \quad \overline{g}^{\mathsf{T}}J_{2m}g = J_{2m},$$

so that $\mathrm{SO}^*(2m)$ is also the group of all elements of $\mathrm{SO}(2m, \mathbf{C})$ that leave invariant the skew Hermitian form

$$-z_1\overline{z}_{m+1} + z_{m+1}\overline{z}_1 - z_2\overline{z}_{m+2} + z_{m+2}\overline{z}_2 - \cdots - z_m\overline{z}_{2m} + z_{2m}\overline{z}_m.$$

We write $\mathrm{SO}(n)$ for $\mathrm{SO}(0, n)$; it is the compact form.

$G = \mathrm{Sp}(2n, \mathbf{C})$. We remind the reader that this is the group of all elements g in $\mathrm{GL}(2n, \mathbf{C})$ such that $g^{\mathsf{T}} J_{2n} g = J_{2n}$ where J_{2n} is as above. It is known that $\mathrm{Sp}(2n, \mathbf{C}) \subset \mathrm{SL}(2n, \mathbf{C})$. Its real forms are

$$(\sigma)\ \mathrm{Sp}(2n, \mathbf{R}),$$
$$(\tau_a)\ \mathrm{Sp}(2a, 2b), \quad a \le b,\ a + b = n,$$

where τ_a is the pseudoconjugation

$$\tau_a : z \longmapsto J_a \bar{z}, \quad J_a = \begin{pmatrix} 0 & 0 & I_a & 0 \\ 0 & 0 & 0 & -I_b \\ -I_a & 0 & 0 & 0 \\ 0 & I_b & 0 & 0 \end{pmatrix},$$

and it can be shown as in the previous case that the subgroup in question is also the subgroup of $\mathrm{Sp}(2n, \mathbf{C})$ preserving the invariant Hermitian form $\bar{z}^{\mathsf{T}} B_{a,b} z$ where

$$B_{a,b} = \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & -I_b & 0 & 0 \\ 0 & 0 & I_a & 0 \\ 0 & 0 & 0 & -I_b \end{pmatrix}.$$

We write $\mathrm{Sp}(2n)$ for $\mathrm{Sp}(0, 2n)$. It is the compact form.

The groups listed above are all connected, and the fact that they are real forms is verified at the Lie algebra level. Cartan's theory shows that there are no others.

LEMMA 6.8.4 *Let G be a connected real Lie group and let $G \subset M$ where M is a complex connected Lie group. If $M = \mathrm{SO}(n, \mathbf{C})$ (resp., $\mathrm{Sp}(2n, \mathbf{C})$), then for G to be contained in a real form of M, it is necessary that G commute with either a conjugation or a pseudoconjugation of $\mathbf{C}^n$ (resp., $\mathbf{C}^{2n}$); if G acts irreducibly on $\mathbf{C}^n$ (resp., $\mathbf{C}^{2n}$), this condition is also sufficient, and then the real form containing G is unique and is isomorphic to $\mathrm{SO}(a, b)$ (resp., $\mathrm{Sp}(a, b)$). If $M = \mathrm{SL}(n, \mathbf{C})$, then for G to be contained in a real form of M it is necessary that G commute with either a conjugation or a pseudoconjugation of $\mathbf{C}^n$ or leave invariant a nondegenerate Hermitian form on $\mathbf{C}^n$. If G acts irreducibly on $\mathbf{C}^n$ and does not leave a nondegenerate Hermitian form invariant, then the above condition is also sufficient and the real form, which is isomorphic to either $\mathrm{SL}(n, \mathbf{R})$ or $\mathrm{SU}^*(n) (n = 2m)$, is then unique.*

PROOF: The first assertion is clear since the real forms of $\mathrm{SO}(n, \mathbf{C})$ and $\mathrm{Sp}(2n, \mathbf{C})$ are those that commute with either a conjugation or a pseudoconjugation of the underlying vector space. Let $M = \mathrm{SO}(n, \mathbf{C})$ or $\mathrm{Sp}(2n, \mathbf{C})$ and suppose that G acts irreducibly. If G commutes with a conjugation σ, then the space of invariant forms for G is one dimensional, and so this space is spanned by the given form on $\mathbf{C}^n$ or $\mathbf{C}^{2n}$ in the two cases. This means that the given form transforms into a multiple of itself under σ and hence M is fixed by σ. But then $G \subset M^{\sigma}$, showing that G is contained in a real form of M. If there is another real form containing G, let λ be the conjugation or pseudoconjugation commuting with G. Then $\sigma^{-1}\lambda$ is an automorphism of $\mathbf{C}^n$ or $\mathbf{C}^{2n}$ commuting with G and so must be a scalar c since G

acts irreducibly. Thus $\lambda = c\sigma$, showing that $M^\sigma = M^\lambda$. Let $M = \mathrm{SL}(n, \mathbf{C})$. The necessity and sufficiency are proven as before, and the uniqueness also follows as before since we exclude the real forms $\mathrm{SU}(a, b)$. □

THEOREM 6.8.5 *Let V be a real quadratic space of dimension D. When $D = 1$ the spin group is $\{\pm 1\}$ and its image is* $\mathrm{O}(1)$. *If $D = 2$ we have* $\mathrm{Spin}(2) \simeq \mathrm{U}(1)$ *and the spin representations are the characters $z \longmapsto z, z^{-1}$, while* $\mathrm{Spin}(1,1) \simeq \mathrm{GL}(1, \mathbf{R}) \simeq \mathbf{R}^\times$ *and the spin representations are the characters $a \longmapsto a, a^{-1}$. In all other cases the image of the restriction of the complex spin representation(s) to* $\mathrm{Spin}(V)$ *is contained in a unique real form of the appropriate classical group of the spinor space according to the following tables (N = dimension of the complex spin module(s)):*

Spin(V) noncompact			
	Real	Quaternionic	Complex
Orthogonal	$\mathrm{SO}(\frac{N}{2}, \frac{N}{2})$	$\mathrm{SO}^*(N)$	$\mathrm{SO}(N, \mathbf{C})_{\mathbf{R}}$
Symplectic	$\mathrm{Sp}(N, \mathbf{R})$	$\mathrm{Sp}(\frac{N}{2}, \frac{N}{2})$	$\mathrm{Sp}(N, \mathbf{C})_{\mathbf{R}}$
Dual pair	$\mathrm{SL}(N, \mathbf{R})$	$\mathrm{SU}^*(N)$	$\mathrm{SU}(\frac{N}{2}, \frac{N}{2})$

Spin(V) compact		
Real	Quaternionic	Complex
$\mathrm{SO}(N)$	$\mathrm{Sp}(N)$	$\mathrm{SU}(N)$

PROOF: The arguments are based on the lemmas and corollary above. Let us consider first the case when the Spin group is noncompact so that $V \simeq \mathbf{R}^{p,q}$ with $1 \le p \le q$. Let Γ be the image of $\mathrm{Spin}(V)$ in the spinor space.

Spin representation(s) orthogonal (orthogonal spinors). This means $\overline{D} = 0, 1, 7$. Then Γ is inside the complex orthogonal group and commutes with either a conjugation or a pseudoconjugation according as $\overline{\Sigma} = 0, 1, 7$ or $\overline{\Sigma} = 3, 4, 5$. In the second case $\Gamma \subset \mathrm{SO}^*(N)$ where N is the dimension of the spin representation(s). In the first case $\Gamma \subset \mathrm{SO}(a, b)^0$, and we claim that $a = b = N/2$. Indeed, we first note that p and q cannot both be odd; for if $\overline{D} = 1, 7$, $p - q$ is odd, while for $\overline{D} = 0$, both $p + q$ and $p - q$ have to be divisible by 8, which means that p and q are both divisible by 4. For $\mathrm{SO}(a, b)^0$ a maximal compact is $\mathrm{SO}(a) \times \mathrm{SO}(b)$, which has invariant subspaces of dimension a and b, and so, by Corollary 6.8.3 above we must have $a, b \ge N/2$. Since $a + b = N$, we see that $a = b = N/2$. There still remains the case $\overline{\Sigma} = 2, 6$, i.e., when the real spin module is of the complex type. But the real forms of the complex orthogonal group commute either with a conjugation or a pseudoconjugation and this cannot happen by Lemma 6.5.9. So there is no real form of the complex orthogonal group containing Γ. The best we can apparently do is to say that the image is contained in $\mathrm{SO}(N, \mathbf{C})_{\mathbf{R}}$ where the suffix $\mathbf{R}$ means that it is the real Lie group underlying the complex Lie group.

Spin representation(s) symplectic (symplectic spinors). This means that $\overline{D} = 3, 4, 5$. Here Γ is inside the complex symplectic group of spinor space. Then Γ commutes with either a conjugation or a pseudoconjugation according as $\overline{\Sigma} = 0, 1, 7$ or $\overline{\Sigma} = 3, 4, 5$. In the first case $\Gamma \subset \mathrm{Sp}(N, \mathbf{R})$. In the second case we have $\Gamma \subset \mathrm{Sp}(2a, 2b)$ with $2a + 2b = N$. The group $\mathrm{S}(\mathrm{U(a)} \times \mathrm{U(b)})$ is a maximal compact of $\mathrm{Sp}(2a, 2b)$ and leaves invariant subspaces of dimension $2a$ and $2b$. Moreover, in this case both p, q cannot be odd; for if $\overline{D} = 3, 5$, $p - q$ is odd, while for $\overline{D} = 4$ both $p - q$ and $p + q$ are divisible by 4 so that p and q have to be even. By Corollary 6.8.3 above we have $2a, 2b \geq N/2$ so that $2a = 2b = N/2$. Once again in the complex case $\Gamma \subset \mathrm{Sp}(N, \mathbf{C})_{\mathbf{R}}$. We shall see below that there is equality for $\mathrm{Spin}(1, 3)$.

Dimension even and spin representations dual to each other (linear spinors). Here $\overline{D} = 2, 6$. If the spin representations are real, then they admit no invariant bilinear forms and the only inclusion we have is that they are inside the special linear group of the spinor space. Hence, as they commute with a conjugation, we have, by the lemma above, $\Gamma \subset \mathrm{SL}(N, \mathbf{R})$. If the spin representations are quaternionic, Γ commutes with a pseudoconjugation τ while admitting no invariant bilinear form. We claim that Γ does not admit an invariant Hermitian form either. In fact, if h is an invariant Hermitian form, then $s, t \longmapsto h(s, \tau(t))$ is an invariant bilinear form, which is impossible. So we must have $\Gamma \subset \mathrm{SU}^*(N)$. If the real spin representation is of the complex type, the argument is more interesting. Let S be the real irreducible spin module so that $S_{\mathbf{C}} = S^+ \oplus S^-$. Let J be the conjugation in $S_{\mathbf{C}}$ that defines S. Then $JS^{\pm} = S^{\mp}$. There exists a pairing $(\cdot, \cdot)$ between $S^{\pm}$. Define $b(s^+, t^+) = (s^+, Jt^+)$, $(s^+, t^+ \in S^+)$. Then b is a $\mathrm{Spin}^+(V)$-invariant sesquilinear form; because S^+ is irreducible, the space of invariant sesquilinear forms is of dimension 1 and so b is a basis for this space. Since this space is stable under adjoints, b is either Hermitian or skew-Hermitian, and replacing b by ib if necessary we may assume that S^+ admits a Hermitian-invariant form. Hence $\Gamma \subset \mathrm{SU}(a, b)$. The maximal compact argument using Corollary 6.8.3 above implies as before that $a, b \geq N/2$. Hence $\Gamma \subset \mathrm{SU}(\frac{N}{2}, \frac{N}{2})$. This finishes the proof of the theorem when $\mathrm{Spin}(V)$ is noncompact.

Spin group compact. This means that $p = 0$ so that $\overline{D} = -\overline{\Sigma}$. So we consider the three cases when the real spin module is of the real, quaternionic, or complex type. If the type is real, the spin representation is orthogonal and so $\Gamma \subset \mathrm{SO}(N)$. If the type is quaternionic, Γ is contained in a compact form of the complex symplectic group and so $\Gamma \subset \mathrm{Sp}(N)$. Finally, if the real spin module is of the complex type, the previous discussion tells us that Γ admits a Hermitian-invariant form. Since the action of γ is irreducible, this form has to be definite (since the compactness of γ implies that it admits an invariant definite Hermitian form anyway). Hence $\gamma \subset \mathrm{SU}(N)$. This finishes the proof of the theorem. □

Low-Dimensional Isomorphisms. In dimensions $D = 3, 4, 5, 6, 8$ the dimension of the spin group is the same as the dimension of the real group containing its image in spinor space and so the spin representation(s) defines a covering map. We need the following lemma.

LEMMA 6.8.6 *Let V be a real quadratic space of dimension $D \neq 4$. Then the spin representation(s) is (are) faithful except when $D = 4k$ and $V \simeq \mathbf{R}^{a,b}$ where both a and b are even. In this case the center of* $\mathrm{Spin}(V) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$ *in such a way that the diagonal subgroup is the kernel of the covering map* $\mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V)^0$, *and the two semispin representations have kernels as the two subgroups of order 2 in the center that are different from the diagonal subgroup.*

PROOF: If D is odd, $\mathrm{Spin}(V_\mathbf{C})$ has center $C \simeq \mathbf{Z}_2$. Since $\mathfrak{so}(V_\mathbf{C})$ is simple, the kernel of the spin representation is contained in C. It cannot be C since then the spin represention would descend to the orthogonal group. So the spin representation is faithful.

For D even, the situation is more delicate. Let C be the center of $\mathrm{Spin}(V_\mathbf{C})$ (see the end of Section 5.3). If $D = 4k + 2$, we have $C \simeq \mathbf{Z}_4$ and the nontrivial element of the kernel of $\mathrm{Spin}(V_\mathbf{C}) \longrightarrow \mathrm{SO}(V_\mathbf{C})$ is the unique element of order 2 in C; this goes to -1 under the (semi)spin representations. It is then clear that they are faithful on C, and the simplicity argument above (which implies that their kernels are contained in C) shows that they are faithful on the whole group.

If $D = 4k$, then $C \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$. From our description of the center of $\mathrm{Spin}(V_\mathbf{C})$ in Section 5.3, we see that after identifying $\mathbf{Z}_2$ with $\{0, 1\}$, the nontrivial element z of the kernel of the covering map $\mathrm{Spin}(V_\mathbf{C}) \longrightarrow \mathrm{SO}(V_\mathbf{C})$ is $(1, 1)$. Let $z_1 = (1, 0)$, $z_2 = (0, 1)$. Since $z = z_1 z_2$ goes to -1 under the semispin representations $S^\pm$, each of $S^\pm$ must map exactly one of z_1, z_2 to 1. They cannot both map the same z_i to 1 because the representation $S^+ \oplus S^-$ of $C(V_\mathbf{C})^+$ is faithful. Hence the kernels of $S^\pm$ are the two subgroups of order 2 inside C other than the diagonal subgroup.

We now consider the restriction to $\mathrm{Spin}(V)$ of $S^\pm$. Let $V = \mathbf{R}^{a,b}$ with $a + b = D$. If a, b are both odd and I is the identity endomorphism of V, $-I \notin \mathrm{SO}(a) \times \mathrm{SO}(b)$, and so the center of $\mathrm{SO}(V)^0$ is trivial. This means that the center of $\mathrm{Spin}(V)$ is $\mathbf{Z}_2$ and is $\{1, z\}$. So the semispin representations are again faithful on $\mathrm{Spin}(V)$. Finally, suppose that both a and b are even. Then $-I \in \mathrm{SO}(a) \times \mathrm{SO}(b)$, and so the center of $\mathrm{Spin}(V)^0$ consists of $\pm I$. Hence the center of $\mathrm{Spin}(V)$ has four elements and so coincides with C, the center of $\mathrm{Spin}(V_\mathbf{C})$. Thus the earlier discussion for complex quadratic spaces applies without change, and the two spin representations have as kernels the two $\mathbf{Z}_2$ subgroups of C that do not contain z. This finishes the proof of the lemma. □

The case $D = 4$ is a little different because the orthogonal Lie algebra in dimension 4 is not simple but splits into two simple algebras. Nevertheless, the table remains valid and we have

$$\mathrm{Spin}(0,4) \longrightarrow \mathrm{SU}(2)\,, \qquad \mathrm{Spin}(2,2) \longrightarrow \mathrm{SL}(2,\mathbf{R})\,.$$

The groups on the left have dimension 6 while those on the right are of dimension 3, and so the maps are not covering maps. The case of $\mathrm{Spin}(1, 3)$ is more interesting. We can identify it with $\mathrm{SL}(2, \mathbf{C})_\mathbf{R}$ where the suffix $\mathbf{R}$ means that the group is the underlying real Lie group of the complex group. Let $\mathcal{H}$ be the space of 2×2 Hermitian matrices viewed as a quadratic vector space with the metric

$h \longmapsto \det(h)(h \in \mathcal{H})$. If we write

$$h = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}, \quad x_\mu \in \mathbf{R},$$

then

$$\det(h) = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

so that $\mathcal{H} \simeq \mathbf{R}^{1,3}$. The action of SL(2, **C**) on $\mathcal{H}$ is given by

$$g, h \longmapsto gh\overline{g}^{\mathrm{T}},$$

which defines the covering map

$$\mathrm{SL}(2, \mathbf{C})_{\mathbf{R}} \longrightarrow \mathrm{SO}(1,3)^0.$$

The spin representations are

$$\mathbf{2} : g \longmapsto g, \quad \overline{\mathbf{2}} : g \longmapsto \overline{g},$$

and their images are exactly $\mathrm{SL}(2, \mathbf{C})_{\mathbf{R}}$.

The following special isomorphisms follow from the lemma above. The symbol $A \xrightarrow{2} B$ means that A is a double cover of B.

$D = 3$	$\mathrm{Spin}(1,2) \simeq \mathrm{SL}(2, \mathbf{R})$
	$\mathrm{Spin}(3) \simeq \mathrm{SU}(2)$
$D = 4$	$\mathrm{Spin}(1,3) \simeq \mathrm{SL}(2, \mathbf{C})_{\mathbf{R}}$
$D = 5$	$\mathrm{Spin}(2,3) \simeq \mathrm{Sp}(4, \mathbf{R})$
	$\mathrm{Spin}(1,4) \simeq \mathrm{Sp}(2,2)$
	$\mathrm{Spin}(5) \simeq \mathrm{Sp}(4)$
$D = 6$	$\mathrm{Spin}(3,3) \simeq \mathrm{SL}(4, \mathbf{R})$
	$\mathrm{Spin}(2,4) \simeq \mathrm{SU}(2,2)$
	$\mathrm{Spin}(1,5) \simeq \mathrm{SU}^*(4) \simeq \mathrm{SL}(2, \mathbf{H})$
	$\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$
$D = 8$	$\mathrm{Spin}(4,4) \xrightarrow{2} \mathrm{SO}(4,4)$
	$\mathrm{Spin}(2,6) \xrightarrow{2} \mathrm{SO}^*(8)$
	$\mathrm{Spin}(8) \xrightarrow{2} \mathrm{SO}(8)$

Finally, the case $D = 8$ deserves special attention. In this case the Dynkin diagram has three extreme nodes, and so there are three fundamental representations of $\mathrm{Spin}(V)$ where V is a complex quadratic vector space of dimension 8. They are the vector representation and the two spin representations. They are *all* of dimension 8, and their kernels are the three subgroups of order 2 inside the center C of $\mathrm{Spin}(V)$. In this case the group of automorphisms of the Dynkin diagram is $\mathfrak{S}_3$, the group of permutations of $\{1, 2, 3\}$. This is the group of automorphisms of $\mathfrak{g}$ modulo the group of inner automorphisms and so is also the group of automorphisms of $\mathrm{Spin}(V)$ modulo the inner automorphisms. Thus $\mathfrak{S}_3$ itself operates on the set of equivalence classes of irreducible representations. Since it acts transitively on the extreme nodes, it permutes transitively the three fundamental representations. Thus the three fundamental representations are all on the same footing. This is the famous *principle of triality*, first discovered by E. Cartan.[8] Actually, $\mathfrak{S}_3$ *itself* acts on $\mathrm{Spin}(V)$.

6.9. References

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CHAPTER 7

Superspacetimes and Super Poincaré Groups

7.1. Super Lie Groups and Their Super Lie Algebras

This last chapter is a very brief introduction to the theory of super Lie groups and its applications, mainly to the construction of superspacetimes and super Poincaré groups. It is not my intention to develop the super Lie theory systematically. Instead I have chosen to illustrate some of the basic aspects of this theory by means of examples that are of great interest from the point of view of physics.

The definition of a super Lie group within the category of supermanifolds imitates the definition of Lie groups within the category of classical manifolds. A *real super Lie group* G is a real supermanifold with morphisms

$$m : G \times G \longrightarrow G\,, \quad i : G \longrightarrow G\,,$$

which are multiplication and inverse, and

$$1 : \mathbf{R}^{0|0} \longrightarrow G$$

defining the unit element, such that the usual group axioms are satisfied. However, in formulating the axioms we must take care to express them entirely in terms of the maps $m, i, 1$. To formulate the associativity law in a group, namely, $a(bc) = (ab)c$, we observe that $a, b, c \longmapsto (ab)c$ may be viewed as the map $I \times m : a, (b, c) \longmapsto a, bc$ of $G \times (G \times G) \longrightarrow G \times G$ (I is the identity map), followed by the map $m : x, y \longmapsto xy$. Similarly, one can view $a, b, c \longmapsto (ab)c$ as $m \times I$ followed by m. Thus the associativity law becomes the relation

$$m \circ (I \times m) = m \circ (m \times I)$$

between the two maps from $G \times G \times G$ to G. We leave it to the reader to formulate the properties of the inverse and the identity. The identity of G is a point of G_{red}. It follows almost immediately from this that if G is a super Lie group, then G_{red} is a Lie group in the classical sense. Also, we have defined real super Lie groups above without specifying the smoothness type. One can define smooth or analytic super Lie groups by simply taking the objects and maps to be those in the category of smooth or analytic super manifolds; the same holds for complex super Lie groups.

The functor of points associated to a super Lie group reveals the true character of a super Lie group. Let G be a super Lie group. For any supermanifold S let $G(S)$ be the set of morphisms from S to G. The maps $m, i, 1$ then give rise to maps

$$m_S : G(S) \times G(S) \longrightarrow G(S)\,, \quad i_S : G(S) \longrightarrow G(S)\,, \quad 1_S : 1 \longrightarrow G(S)\,,$$

such that the group axioms are satisfied. This means that the functor

$$S \longmapsto G(S)$$

takes values in groups. Moreover, if T is another supermanifold and we have a map $S \longrightarrow T$, the corresponding map $G(T) \longrightarrow G(S)$ is a homomorphism of groups. Thus $S \longmapsto G(S)$ is a *group-valued functor*. One can also therefore define a super Lie group as a representable functor

$$S \longmapsto G(S)$$

from the category of supermanifolds to the category of groups. If G is the supermanifold that represents this functor, the maps $m_S : G(S) \times G(S) \longrightarrow G(S)$, $i_S : G(S) \longrightarrow G(S)$, and 1_S then define, by Yoneda's lemma, maps $m, i, 1$ that convert G into a super Lie group, and $S \longmapsto G(S)$ is the functor of points corresponding to G. A morphism of super Lie groups $G \longrightarrow H$ is now one that commutes with $m, i, 1$. It corresponds to homomorphisms

$$G(S) \longrightarrow H(S)$$

that are functorial in S. If G and H are already given, Yoneda's lemma assures us that morphisms $G \longrightarrow H$ correspond one-to-one to homomorphisms $G(S) \longrightarrow H(S)$ that are functorial in S.

The actions of super Lie groups on supermanifolds are defined exactly in the same way. Thus if G is a super Lie group and M is a supermanifold, actions are defined either as morphisms $G \times M \longrightarrow M$ with appropriate axioms or as actions $G(S) \times M(S) \longrightarrow M(S)$ that are functorial in S; again Yoneda's lemma makes such actions functorial in S to be in canonical bijection with actions $G \times M \longrightarrow M$.

Subsuper Lie groups are defined exactly as in the classical theory. A super Lie group H is a subgroup of a super Lie group if H_{red} is a Lie subgroup of G_{red} and the inclusion map of H into G is a morphism that is an immersion everywhere. One of the most common ways of encountering subsuper Lie groups is as stabilizers of points in actions. Suppose that G acts on M and m is a point of M_{red}. Then, for any supermanifold S, we have the stabilizer $H(S)$ of the action of $G(S)$ on $M(S)$ at the point m_S of $M(S)$. The assignment $S \longmapsto H(S)$ is clearly functorial in S. It can then be shown that this functor is representable, and that the super Lie group it defines is a closed subsuper Lie group of G. We shall not prove this result here.

As in the classical case, products of super Lie groups are super Lie groups. The opposite of a super Lie group is also a super Lie group.

At this time we mention two examples.

EXAMPLE. $G = \mathbf{R}^{p|q}$. If $(x, \theta) = (x^1, \dots, x^p, \theta^1, \dots, \theta^q)$ are the global coordinates on G, then for any supermanifold S the set $G(S)$ of morphisms are in one-to-one correspondence with the set of vectors (f, g) where $f = (f^1, \dots, f^p)$, $g = (g^1, \dots, g^q)$, the f^i (resp., g^α) being even (resp., odd) global sections of the structure sheaf $\mathcal{O}_S$. On $G(S)$ we have the additive group structure

$$(f, g) + (f', g') = (f + f', g + g').$$

So $S \longmapsto G(S)$ is the group-valued functor that defines $\mathbf{R}^{p|q}$ as an abelian super Lie group. In symbolic notation (see Chapter 4) the "group law" is written as

$$(x,\theta)+(x'\theta')=(x'',\theta''), \quad x''=x+x', \;\; \theta''=\theta+\theta'.$$

EXAMPLE. $G=\mathrm{GL}(p|q)$. We start with $M^{p|q}\simeq \mathbf{R}^{p^2+q^2|2pq}$ whose coordinates are written as the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix};$$

here $A=(a_{ij})_{1\le i,j\le p}$, $D=(d_{\alpha\beta})_{1\le\alpha,\beta\le q}$ are matrices of even elements while $B=(b_{i\beta})_{1\le i\le p,1\le\beta\le q}$, $C=(c_{\alpha j})_{1\le\alpha\le q,1\le j\le p}$ are matrices of odd elements. The underlying reduced manifold for G is the open set $\mathrm{GL}(p)\times\mathrm{GL}(q)$ in $M^p\times M^q$, and $\mathrm{GL}(p|q)$ is the open subsupermanifold of $M^{p|q}$ defined by $\mathrm{GL}(p)\times\mathrm{GL}(q)$. A morphism of a supermanifold S into $M^{p|q}$ is then defined by a matrtix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a_{ij}, d_{\alpha\beta}\in\mathcal{O}(S)_0, \;\; b_{i\beta}, c_{\alpha j}\in\mathcal{O}(S)_1,$$

as before, and such a morphism defines a morphism into $\mathrm{GL}(p|q)$ if and only if $\det(a)\det(d)\in\mathcal{O}(S)^*$, i.e., a unit of $\mathcal{O}(S)$. Under matrix multiplication the set $G(S)$ of these morphisms is a group, and this group-valued functor is represented by the supermanifold $\mathrm{GL}(p|q)$. In symbolic notation the group law is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}=\begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix}.$$

Super Lie Algebra of a Super Lie Group. In the classical theory the Lie algebra of a Lie group is defined as the Lie algebra of left (or right) invariant vector fields on the group manifold with the bracket as the usual bracket of vector fields. The left invariance guarantees that the vector field is uniquely determined by the tangent vector at the identity; one starts with a given tangent vector at the identity and then translates it to each point to obtain the vector field. In the case of a super Lie group we follow the same procedure, but much more care is required because we have to consider not only the topological points but others also.

For a super Lie group it is now a question of making precise what is a left invariant vector field. If we are dealing with a classical Lie group G, the left invariance of a vector field X is the relation $\ell_x\circ X=X\circ\ell_x$ for all $x\in G$ where ℓ_x is left translation by x, i.e.,

$$X_y f(xy)=(Xf)(xy)$$

for all $x,y\in G$ where X_y means that X acts only on the second variable y. This can also be written as

$$(I\otimes X)\circ m^*=m^*\circ X$$

where m^* is the sheaf morphism from $\mathcal{O}_G$ to $\mathcal{O}_{G\times G}$ corresponding to the multiplication $m: G\times G\longrightarrow G$.

It is convenient to reformulate this slightly. The family of left translations $\ell_x(x\in G)$ defines a diffeomorphism

$$L: G\times G\longrightarrow G\times G, \quad (x,y)\longmapsto (x,xy).$$

If X is a vector field on G, its left invariance can also be seen as equivalent to the invariance of the vector field $I \otimes X$ on $G \times G$ with respect to the diffeomorphism L. Similarly, the right invariance of X is equivalent to the invariance of $I \otimes X$ with respect to the diffeomorphism

$$R : G \times G \longrightarrow G \times G, \quad (x, y) \longmapsto (x, yx).$$

It is in this form that we extend the concept of left or right invariance of vector fields on a super Lie group G. We treat only the case of left invariance. First of all, if X is a vector field on G, the vector field $I \otimes X$ on $G \times G$ is defined in the obvious manner: if $(a, b) \in G_{\mathrm{red}} \times G_{\mathrm{red}}$, and (x^i), (y^i) are coordinates at a, b, respectively (we do not distinguish between even and odd coordinates), and $X = \sum_j a_j \partial/\partial y^j$ in the coordinates (y^j), then $I \otimes X = \sum_j a_j \partial/\partial y^j$ in the coordinates (x^i, y^j).

We shall now define L. In view of Yoneda's lemma we need to define only the bijections of $G(S) \times G(S)$ functorial in S for S a supermanifold; these are simply the maps $(x, y) \longmapsto (x, xy)$. Thus X is left invariant if and only if $I \otimes X$ is invariant under L:

$$(I \otimes X)L^* = L^*(I \otimes X). \tag{7.1}$$

It is clear from this that the set of (homogeneous) left invariant vector fields spans a super Lie algebra.

Before formulating the main theorem, let us define, for each homogeneous tangent vector $\tau \in T_e(G)$, a vector field X_τ of the same parity as τ as follows: If f is the germ of a section of $\mathcal{O}_G$ at e and we write $m^* f$ for the germ of the section at (a, e), $m(G \times G \longrightarrow G)$ being multiplication, then

$$X_\tau f = (I \otimes \tau) m^* f$$

viewed as a germ at a. Let (x^i) be coordinates at a and (y^j) coordinates at e, and let m be symbolically written as $(x^i, y^j) \longrightarrow (m^1(x, y), \ldots, m^r(x, y))$; let $\tau = (\partial/\partial y^k)_e$. Then

$$X_\tau = \sum_j \left(\frac{\partial m^j}{\partial y^k}\right)_e \frac{\partial}{\partial x^j} \tag{7.1ℓ}$$

We can also define another vector field ${}_\tau X$ by

$${}_\tau X = \sum_j \left(\frac{\partial m'^j}{\partial y^k}\right)_e \frac{\partial}{\partial x^j} \tag{7.1r}$$

where $m'(x, y) = m(y, x)$.

THEOREM 7.1.1 *The Lie algebra* $\mathfrak{g} = \mathrm{Lie}(G)$ *of a super Lie group G is spanned by the set of all vector fields X on G satisfying* (7.1). *It is a super Lie algebra of the same dimension as G. The map $X \longmapsto X_1$ that sends $X \in \mathfrak{g}$ to the tangent vector at the identity point* 1 *is a linear isomorphism of super vector spaces. If τ is a tangent vector to G at* 1, *the vector field $X \in \mathfrak{g}$ such that $X_1 = \tau$ is given by*

$$X_\tau = (I \otimes \tau) \circ m^*. \tag{7.2}$$

Finally, the even part of $\mathfrak{g}$ *is the Lie algebra of the classical Lie group underlying* G, *i.e.,*

$$\mathfrak{g}_0 = \mathrm{Lie}(G_{\mathrm{red}})\,.$$

SKETCH OF PROOF: We must prove that each X_τ is left invariant and that the X_τ are precisely all the left invariant vector fields. Notice that equation (7.2) can be interpreted formally as

$$(Xf)(x) = (\tau_y)(f(xy))\,. \tag{7.3}$$

We shall use the symbolic method to prove that

$$(I\otimes)L^* = L^*(I \otimes X_\tau)\,.$$

It is enough to show that both sides have the same effect on polynomials $f(x)g(y)$ of local coordinates. The left side is

$$f(x)\tau_z(g(x(yz))$$

while the right side is

$$f(x)\tau_z(g((xy)z))\,.$$

These are the same by the associativity law $x(yz) = (xy)z$.

Suppose now that $\tau_1, \ldots, \tau_n$ is a homogeneous basis for $T_e(G)$. The vector fields $X_{\tau_1}, \ldots, X_{\tau_n}$ are linearly independent at e and so at all points of G_{red}. From our results in Section 4.7 it follows that $X_{\tau_1}, \ldots, X_{\tau_n}$ is an $\mathcal{O}_G$ basis for the $\mathcal{O}_G$-module of vector fields on G. If X is now any left invariant vector field on G, we must have

$$X = \sum_i f_i X_{\tau_i}\,, \qquad f_i \in \mathcal{O}(G)\,.$$

Hence

$$I \otimes X = \sum_i F_i(I \otimes X_{\tau_i})\,, \qquad F_i = f_i \circ \mathrm{pr}_2\,.$$

Here pr_2 is the projection $x, y \longmapsto y$ of $G \times G$ on G. Since the vector fields $I \otimes X_{\tau_i}$ are linearly independent over $\mathcal{O}_{G\times G}$, the left invariance of both sides implies that the F_i are invariant under L. So $f_i(y) = f_i(xy)$. Evaluation at $y = e$ implies that $f_i(x) = f_i(e)$. So X is a *constant* linear combination of the X_{τ_i}.

The last statement of the theorem is obvious. This proves the theorem. □

REMARK. The same argument proves that the ${}_\tau X$ are right invariant and are precisely all the right invariant vector fields on G.

The formulae (7.1ℓ) and (7.1r) for X_τ and ${}_\tau X$ in local coordinates are very handy for calculations. As a first example consider $G = \mathbf{R}^{1|1}$ with global coordinates x, θ. We introduce the group law

$$(x, \theta)(x', \theta') = (x + x' + \theta\theta', \theta + \theta')$$

with the inverse

$$(x, \theta)^{-1} = (-x, -\theta)\,.$$

The Lie algebra is of dimension $1|1$. If D_x, D_θ are the left invariant vector fields that define the tangent vectors $\partial_x = \partial/\partial x$, $\partial_\theta = \partial/\partial\theta$ at the identity element 0, and D_x^r, D_θ^r are the corresponding right invariant vector fields, then (7.1ℓ), (7.1r) yield

$$\begin{aligned} D_x &= \partial_x\,, & D_x^r &= \partial_x\,,\\ D_\theta &= -\theta\partial_x + \partial_\theta\,, & D_\theta^r &= \theta\partial_x + \partial_\theta\,. \end{aligned}$$

It is now an easy check that

$$[D_x, D_\theta] = 2D_x$$

(all other commutators are zero) giving the structure of the Lie algebra. A similar method yields the Lie algebras of $\mathrm{GL}(p|q)$; it is $\mathfrak{gl}(p|q)$. The calculation is very similar to the classical case of $\mathrm{GL}(n)$. We use the matrix of coordinates and formula (7.1ℓ) to get

$$\mathrm{Lie}(\mathrm{GL}(p|q) = \mathfrak{gl}(p|q)$$

with $[\,\cdot\,,\cdot\,]$ as the usual bracket in $\mathfrak{gl}(p|q)$.

THEOREM 7.1.2 *For a morphism $f : G \longrightarrow G'$ of super Lie groups G, G' we have its differential Df, which is a morphism of the corresponding super Lie algebras, i.e., $Df : \mathfrak{g} \longrightarrow \mathfrak{g}'$. It is uniquely determined by the relation $Df(X)_{1'} = df_1(X_1)$ where 1, $1'$ are the identity elements of G, G' and df_1 is the tangent map $T_1(G) \longrightarrow T_{1'}(G')$. Moreover, f_{red} is a morphism $G_{\mathrm{red}} \longrightarrow G'_{\mathrm{red}}$ of classical Lie groups.*

The proof is left to the reader.

The fundamental theorems of Lie go over to the super category without change. All topological aspects are confined to the classical Lie groups underlying the super Lie groups. Thus, a morphism $\alpha : \mathfrak{g} \longrightarrow \mathfrak{g}'$ comes from a morphism $G \longrightarrow G'$ if and only if $\alpha_0 : \mathfrak{g}_0 \longrightarrow \mathfrak{g}'_0$ comes from a morphism $G_{\mathrm{red}} \longrightarrow G'_{\mathrm{red}}$. The story is the same for the construction of a super Lie group corresponding to a given super Lie algebra: given a classical Lie group H with Lie algebra $\mathfrak{g}_0$, there is a unique super Lie group G with $\mathfrak{g}$ as its super Lie algebra such that $G_{\mathrm{red}} = H$. The classification of super Lie algebras over **R** and **C** and their representation theory thus acquire a geometric significance that plays a vital role in supersymmetric physics. For additional discussion, see Deligne and Morgan.[1]

Super Affine Algebraic Groups. There is another way to discuss Lie theory in the supersymmetric context, namely, as algebraic groups. In the classical theory algebraic groups are defined as groups of matrices satisfying polynomial equations. Examples are $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{SO}(n)$, $\mathrm{Sp}(2n)$, and so on. They are affine algebraic varieties that carry a group structure such that the group operations are morphisms. If R is a commutative k-algebra with unit element, $G(R)$ is the set of solutions to the defining equations; thus we have $\mathrm{GL}(n, R)$, $\mathrm{SL}(n, R)$, $\mathrm{SO}(n, R)$, $\mathrm{Sp}(2n, R)$. In general, an affine algebraic group scheme defined over k is a representable functor $R \longmapsto G(R)$ from the category of commutative k-algebras with units to the category of groups. Representability means that there is a commutative algebra with unit, $k[G]$ say, such that

$$G(R) = \mathrm{Hom}(k[G], R)$$

for all R. By Yoneda's lemma the algebra $k[G]$ acquires a coalgebra structure, an antipode, and a co-unit, converting it into a *Hopf algebra*. The generalization to the super context is almost immediate: a super affine algebraic group defined over k is a functor

$$R \longmapsto G(R)$$

from the category of supercommutative k-algebras to the category of groups that is representable; i.e., there is a supercommutative k-algebra with unit, $k[G]$ say, such that

$$G(R) = \mathrm{Hom}(k[G], R)$$

for all R. The algebra $k[G]$ then acquires a super Hopf structure. The theory can be developed in parallel with the transcendental theory. Of course, in order to go deeper into the theory we need to work with general super schemes, for instance, when we deal with homogeneous spaces that are very often not affine but projective. The Borel subgroups and the super flag varieties are examples of these.[2]

7.2. The Poincaré-Birkhoff-Witt Theorem

The analogue for super Lie algebras of the Poincaré-Birkhoff-Witt (PBW) theorem is straightforward to formulate. Let $\mathfrak{g}$ be a super Lie algebra and T the tensor algebra over $\mathfrak{g}$. We denote by I the two-sided ideal generated by

$$x \otimes y - (-1)^{p(x)p(y)} y \otimes x - [x, y]1\,, \quad x, y \in \mathfrak{g}\,,$$

and define

$$\mathcal{U} = \mathcal{U}(\mathfrak{g}) = T/I\,.$$

Then $\mathcal{U}$ is a superalgebra, since I is homogeneous in the $\mathbf{Z}_2$ grading of T inherited from that on $\mathfrak{g}$, and we have a natural map $p : \mathfrak{g} \longrightarrow \mathcal{U}$. The pair $(\mathcal{U}, p)$ has the following universal property: if A is a superalgebra with associated super Lie algebra $A_L([x, y] = xy-(-1)^{p(x)p(y)}yx)$ and $f : \mathfrak{g} \longrightarrow A_L$ is a morphism of super Lie algebras, there is a unique morphism $f^{\sim} : \mathcal{U} \longrightarrow A$ such that $f^{\sim}(p(X)) = f(X)$ for all $X \in \mathfrak{g}$. It is clear that $(\mathcal{U}, p)$ is uniquely determined up to a unique isomorphism by this universality requirement. $(\mathcal{U}, p)$ is called the *universal enveloping algebra of* $\mathfrak{g}$. The PBW theorem below implies that p is injective. So it is usual to identify $\mathfrak{g}$ with its image by p inside $\mathcal{U}$ and refer to $\mathcal{U}$ itself as the universal enveloping algebra of $\mathfrak{g}$. The generators of I are homogeneous in the $\mathbf{Z}_2$-grading induced on T, and so $\mathcal{U}$ is a superalgebra.

We select a homogeneous basis (X_a, X_α) for $\mathfrak{g}$ where the X_a are even and the X_α are odd. We also assume that the indices a, α are linearly ordered; this is not a restriction since any set can be linearly ordered, even well-ordered, by the axiom of choice. In most applications the dimension of $\mathfrak{g}$ is countable, and so we can use $\{1, 2, \dots, N\}$ or $\{1, 2, \dots\}$ as sets of indices.

THEOREM 7.2.1 (PWB) *Let k be a commutative ring with unit in which* 2 *and* 3 *are invertible. Let $\mathfrak{g}$ be a super Lie algebra over k that is a free k-module with a homogeneous basis. Let the notation be as above. Then the map p of $\mathfrak{g}$ into $\mathcal{U}$ is* *an imbedding. If* (X_a), (X_α) *are bases for* $\mathfrak{g}_0, \mathfrak{g}_1$, *respectively, then the standard monomials*

$$X_{a_1}^{b_1} \cdots X_{a_r}^{b_r} X_{\alpha_1} \cdots X_{\alpha_s}, \quad a_1 \leq \cdots \leq a_r,\ \alpha_1 < \cdots < \alpha_s,$$

form a basis for $\mathcal{U}$. *In particular,*

$$\mathcal{U} \simeq \mathcal{U}(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1)$$

as super vector spaces.

REMARK. In recent times, as the notion of the Lie algebra has been generalized to include Lie superalgebras and quantum groups, the PBW theorem has also been generalized to these contexts. It seems useful to point out that one can formulate and prove a single result from which the PBW theorems in the various contexts follow quite simply. The following treatment is nothing more than a synopsis of a paper by George M. Bergman;[3] see also Corwin, Ne'eman, and Sternberg.[4]

We work over a commutative ring k with unit. We wish to construct a basis for an associative k-algebra A given by a set of generators with relations of a special type. Let T be the tensor algebra over k determined by the generators and I the two-sided ideal generated by the relations. In the special contexts mentioned above there is a natural k-module of tensors spanned by the so-called *standard monomials* and denoted by S. The problem is to find conditions such that $T = S \oplus I$; then the images of a basis of S in $A = T/I$ will furnish a basis for A. Following Bergman we speak of *words* instead of monomial tensors.

Let X be a set whose elements are called *letters* and let W be the set of *words* formed from the letters, with 1 as the null word; W is a semigroup with 1 as unit, the product ww' of the words w, w' being the word in which w is followed by w'. T is the free k-module spanned by W whose elements will be called *tensors*. We are given a family $(w_\sigma)_{\sigma \in \Sigma}$ of words and for each $b \in B$ a tensor $f_\sigma \in T$; we assume that for $\sigma \neq \sigma'$, $w_\sigma \neq w_{\sigma'}$. Our interest is in the algebra generated by the elements of X with relations

$$w_\sigma = f_\sigma, \quad \sigma \in \Sigma.$$

A word is called *standard* if it does not contain any of the words w_σ $(\sigma \in \Sigma)$ as a subword. Let S be the free k-module spanned by the standard words. Elements of S will be called the *standard tensors*. We write I for the two-sided ideal in T generated by the elements $w_\sigma - f_\sigma$, namely, the k-span of all tensors of the form

$$u(w_\sigma - f_\sigma)v, \quad \sigma \in \Sigma,\ u, v \in W.$$

The theorem sought is the statement that

$$T = S \oplus I.$$

We shall refer to this as the *basic* or the *PBW theorem*. To see how this formulation includes the classical PBW theorem, let $X = (x_i)$ be a basis of a Lie algebra over k where the indices i are linearly ordered. Then Σ is the set of pairs i, j with $i > j$.

The words w_σ are $x_i x_j$ $(i > j)$ and f_σ is $x_j x_i + [x_i, x_j]$ so that the relations defining the universal enveloping algebra are

$$x_i x_j = x_j x_i + [x_i, x_j], \quad i > j .$$

A word is then standard if it is of the form $x_{i_1} x_{i_2} \cdots x_{i_r}$ where $i_1 \leq \cdots \leq i_r$ and S is the usual k-span of standard monomials in the basis elements (x_i).

The natural way to prove the basic theorem is to show that every word is congruent to a standard tensor mod I and that this standard tensor is uniquely determined. We shall say that the standard tensor is a *reduced expression* of the original word and the process of going from the given word to its reduced expression a *reduction procedure*. The procedure of reduction is quite simple. We check if the given word is already standard, and if it is not, then it must contain a subword w_σ $(\sigma \in \Sigma)$ which we replace by f_σ; we call this an *elementary reduction*. We repeat this process for the words in the tensor thus obtained. We hope that this process ends in a finite number of steps, necessarily in a standard tensor, and that the standard tensor thus obtained is independent of the reduction algorithm. The ambiguity of the reduction process stems from the fact that a given word may contain several words w_σ $(\sigma \in \Sigma)$ as subwords, and any one of them can be replaced by f_σ in the next step. If the reduction process exists and is unambiguous, we have an operator R from T to S that is a projection on S. We shall see below that the existence and uniqueness of the reduction to standard form is equivalent to the basic theorem.

Before going ahead, let us look at an example where X has three elements x_i $(i = 1, 2, 3)$ and we start with the relations

$$[x_i, x_j] := x_i x_j - x_j x_i = x_k , \quad (ijk) \text{ is an even permutation of } (123).$$

These are the commutation rules of the rotation Lie algebra, and we know that the PBW theorem is valid where the standard words are the ones $x_1^{r_1} x_2^{r_2} x_3^{r_3}$. But suppose we change these relations slightly so that the Jacobi identity is not valid; for instance, let

$$[x_1, x_2] = x_3 , \quad [x_2, x_3] = x_1 , \quad [x_3, x_1] = x_3 .$$

Let us consider two ways of reducing the nonstandard word $x_3 x_2 x_1$. We have

$$x_3 x_2 x_1 \equiv x_2 x_3 x_1 - x_1^2 \equiv x_2 x_1 x_3 + x_2 x_3 - x_1^2 \equiv x_1 x_2 x_3 - x_1^2 + x_2 x_3 - x_1 - x_3^2$$

where we start by an elementary reduction of $x_3 x_2$. If we start with $x_2 x_1$ we get

$$x_3 x_2 x_1 \equiv x_3 x_1 x_2 - x_3^2 \equiv x_1 x_3 x_2 + x_3 x_2 - x_3^2 \equiv x_1 x_2 x_3 - x_1^2 + x_2 x_3 - x_1 - x_3^2 .$$

Hence we have $x_1 \in I$. The PBW theorem has already failed. From the commutation rules we get that $x_3 \in I$ so that $I \supset I'$ where I' is the two-sided ideal generated by x_1, x_3. On the other hand, all the relations are in I' so that $I \subset I'$. Hence $I = I'$, showing that $T = k[x_2] \oplus I$. Thus $A \simeq k[x_2]$.

We shall now make a few definitions. Words containing a w_σ $(\sigma \in \Sigma)$ as a subword are of the form $u w_\sigma v$ where $u, v \in W$; for any such we define the *elementary reduction operator* $R_{u w_\sigma v}$ as the linear operator $T \longrightarrow T$ that fixes any word $\neq u w_\sigma v$ and sends $u w_\sigma v$ to $u f_\sigma v$. If $w_\sigma \neq f_\sigma$, then this operator fixes a tensor if and only if it is a linear combination of words different from $u w_\sigma v$.

We shall assume from now on that $w_\sigma \neq f_\sigma$ *for all* $\sigma \in \Sigma$. A finite product of elementary reduction operators is called simply a *reduction operator.*

A tensor t is *reduction finite* if for any sequence R_i of elementary reduction operators the sequence $R_1 t, R_2 R_1 t, \ldots, R_k R_{k-1} \cdots R_1 t$ stabilizes, i.e., for some n, $R_k \cdots R_1 t = R_n \cdots R_1 t$ for all $k > n$. Clearly, the set T_f of reduction finite tensors is a k-module that is stable under all elementary reduction operators. The set of tensors that is the k-span of words different from any word of the form $uw_\sigma v$ ($u, v \in W, \sigma \in \Sigma$) is denoted by S and its elements are called *standard.* These are the tensors that are fixed by all the reduction operators. If $t \in T_f$ it is easy to see that there is a reduction operator R such that $Rt = s \in S$; s is said to be a *reduced form of t.* If all standard reduced forms of t are the same, t is called *reduction unique*, and the set of all such tensors is denoted by T_u. T_u is also a k-module, $S \subset T_u \subset T_f$, T_u is stable under all reduction operators, and the map that sends $t \in T_u$ to its unique reduced standard form is a well-defined linear operator that is a *projection* from T_u to S. We shall denote it by $\mathcal{R}$. Clearly, if $t \in T_u$ and L is a reduction operator, $\mathcal{R}(Lt) = \mathcal{R}t$. To see that T_u is closed under addition, let $t, t' \in T_u$ and let t_0, t_0' be their reduced forms. Then $t + t' \in T_f$; if M is a reduction operator such that $M(t + t') = u_0 \in S$, we can find reduction operators L, L' such that $LMt = t_0$, $L'LMt' = t_0'$, so that $u_0 = L'LM(t + t') = t_0 + t_0'$, showing that $t + t' \in T_u$ and $\mathcal{R}(t + t') = \mathcal{R}t + \mathcal{R}t'$.

We shall now show that when $T = T_f$, the basic theorem, namely, the assertion that $T = S \oplus I$, is equivalent to the statement that every word is reduction unique, i.e., $T_u = T$. Suppose first that $T = S \oplus I$. If $t \in T$ and R is an elementary reduction operator, it is immediate that $t \equiv Rt \pmod I$. We claim that this is true for R any reduction operator, elementary or not. Indeed, assume that $s - R_q \cdots R_1 s \in I$ for all $s \in I$. If $R = R_{q+1} R_q \cdots R_1$, $t \in I$, and $s = R_1 t$, we have $t - R_{q+1} \cdots R_1 t = t - R_1 t + (s - R_{q+1} \cdots R_2 s) \in I$. Any reduced form s of t satisfies $t \equiv s \pmod I$. But then s must be the projection of t on $S \pmod I$. Hence s is uniquely determined by t, showing that $t \in T_u$. Conversely, suppose that $T_u = T$. Then $\mathcal{R}$ is a projection operator on S so that $T = S \oplus K$ where K is the kernel of $\mathcal{R}$. It is now a question of showing that $K = I$. Suppose that $t \in K$. Since $t \equiv Rt \pmod I$ for any reduction operator R and $0 = \mathcal{R}t = Rt$ for some reduction operator R, it follows that $t \in I$, showing that $K \subset I$. On the other hand, consider $t = uw_\sigma v$ where $\sigma \in \Sigma$. If R is the elementary reduction operator $R_{uw_\sigma v}$, we know that $\mathcal{R}t = \mathcal{R}(Rt) = \mathcal{R}(uf_\sigma v)$. Hence $\mathcal{R}(u(w_\sigma - f_\sigma)v) = 0$, showing that $\mathcal{R}$ vanishes on I. Thus $I \subset K$. So $K = I$ and we are done.

We now have the following simple but important lemma.

LEMMA 7.2.2 *Let* $u, v \in W$ *and* $t \in T$. *Suppose that* utv *is reduction unique and* R *is a reduction operator. Then* $u(Rt)v$ *is also reduction unique and* $\mathcal{R}(utv) = \mathcal{R}(u(Rt)v)$.

PROOF: It is clearly sufficient to prove this when R is an elementary reduction operator $R_{aw_\sigma c}$ where $a, c \in W$ and $\sigma \in \Sigma$. Let R' be the elementary reduction operator $R_{uaw_\sigma cv}$. Then $R'(utv) = u(Rt)v$. Since $utv \in T_u$, we have $u(Rt)v = R'(utv) \in T_u$ also and $\mathcal{R}(u(Rt)v) = \mathcal{R}(R'(utv)) = \mathcal{R}(utv)$. □

The basic question is now clear: when can we assert that every tensor is reduction unique? However, it is not obvious that the process of reduction of a tensor terminates in a finite number of steps in a standard tensor. To ensure this we consider a partial order on the words such that for any $\sigma \in \Sigma$, f_σ is a linear combination of words *strictly less than* w_σ; it is then almost obvious that any tensor can be reduced to a standard form in a finite number of steps. More precisely, let $\prec$ be a partial order on W with the following properties ($w' \succ w$ means $w \prec w'$):

(i) $1 \prec w$ for all $w \neq 1$ in W.

(ii) $w \prec w'$ implies that $uwv \prec uw'v$ for all $u, w, w', v \in W$.

(iii) $\prec$ satisfies the descending chain condition: any sequence w_n such that $w_1 \succ w_2 \succ \cdots$ is finite.

(iv) For any $\sigma \in \Sigma$, f_σ is a linear combination of words $\prec w_\sigma$.

The descending chain condition implies that any subset of W has minimal elements. *From now on we shall assume that W has been equipped with such a partial order.* If w is a word and t is a tensor, we shall write $t \prec w$ if t is a linear combination of words $\prec w$. For any linear space L of tensors we write $L_{\prec w}$, the subspace of L consisting of elements that are $\prec w$.

First of all we observe that under this assumption $T_f = T$. For if some word is not reduction finite, there is a minimal such word, say w; w cannot be standard. If R is an elementary reduction operator with $Rw \neq w$, we must have $w = uw_\sigma v$ for some $\sigma \in \Sigma$ and words u, v, and $R = R_{uw_\sigma v}$. But then $Rw = uf_\sigma v \prec w$ so that Rw is in T_f. This implies that w is in T_f. We now consider the ambiguities in the reduction process. These, in their simplest form, are of two kinds. The ambiguity of type O, the *overlap ambiguity*, is a word $w_1w_2w_3$ where the w_i are words and there are $\sigma, \tau \in \Sigma$ such that $w_1w_2 = w_\sigma$, $w_2w_3 = w_\tau$. In reducing such an element we may begin with $w_1w_2 = w_\sigma$ and replace it by f_σ, or we may begin with $w_2w_3 = w_\tau$ and replace it by f_τ. The second type is type I, the *inclusion ambiguity*, which is a word $w_1w_2w_3$ where $w_2 = w_\sigma$, $w_1w_2w_3 = w_\tau$. We shall say that the ambiguities are *resolvable* if there are reduction operators R', R'' such that $R'(f_\sigma w_3) = R''(w_1 f_\tau) \in S$ in the type O case and $R'(w_1 f_\sigma w_3) = R''(f_\tau) \in S$ in the type I case. The basic result is the following:

THEOREM 7.2.3 (Bergman) *Assume that W is equipped with an order as above. Then the basic theorem is true if and only if all ambiguities are resolvable.*

PROOF: Let us assume that all ambiguities are resolvable and prove that the PBW is valid. As we have already observed, every element of T is reduction finite, and so it comes down to showing that every word is reduction unique. This is true for the null word 1, and we shall establish the general case by induction. Let w be any word and let us assume that all words less than w are reduction unique; we shall prove that w is also reduction unique.

Let R_1, R_2 be two elementary reduction operators such that $R_1 w \neq w$, $R_2 w \neq w$. We shall prove that $R_1 w$ and $R_2 w$ are reduction unique and have the same reduced form. We must have $R_1 = R_{u_1 w_\sigma v_1}$ and $R_2 = u_2 w_\tau v_2$ for some $\sigma, \tau \in \Sigma$. We may assume that in w the subword w_σ begins no later than the subword w_τ. Three cases arise. First we consider the case when w_σ and w_τ overlap. Then

$w = uw_1w_2w_3v$ where $w_1w_2 = w_\sigma$ and $w_2w_3 = w_\tau$. By assumption there are reduction operators R', R'' such that $R'(f_\sigma w_3) = R''(w_1 f_\tau)$. On the other hand, for any elementary reduction operator $R_0 = R_{aw_\theta b}$ $(\theta \in \Sigma)$ we have the reduction operator $uR_0v = R_{ua\theta bv}$. So for any reduction operator $R^\sim$, elementary or not, we have a reduction operator $R^\sim_{uv}$ such that for all $t \in T$, $uR^\sim tv = R^\sim_{uv}t$. So if $R_1' = R'_{uv}$, $R_1'' = R''_{uv}$, we have $R_1'(uf_\sigma w_3) = R_1''(uw_1 f_\tau v)$. But since $f_\sigma \prec w_\sigma$, $f_\tau \prec w_\tau$, we see that $uf_\sigma w_3 v \prec uw_\sigma w_3 v = w$, $uw_1 f_\tau v \prec uw_1 w_\tau v = w$ so that $uf_\sigma w_3$ and $uw_1 f_\tau v$ are both in $T_{\prec w}$. Since $\mathcal{R}_{\prec w}$ is well-defined on $T_{\prec w}$ and the above two elements can be reduced to the same element in S, they must have the same image under any reduction operators that takes them to reduced form. In other words, $R_1 w$ and $R_2 w$ have the same reduced form as we wanted to prove. The case when w_σ is a subword of w_τ is similar. The third and remaining case is when w_σ and w_τ do not overlap. This is the easiest of all cases. We can then write $w = uw_\sigma x w_\tau v$. Then $R_1 w = uf_\sigma x w_\tau v$, $R_2 w = uw_\sigma x f_\tau v$. We can reduce w_τ in $R_1 w$ and w_σ in $R_2 w$ to get $uf_\sigma x f_\tau v$ in both cases. This element is in $T_{\prec w}$ and so has a unique reduced form. So $R_1 w$ and $R_2 w$ have the same reduced forms under suitable reductions, and because these are in $T_{\prec w}$, this reduced form is their unique reduced expression. Hence we again conclude that w is reduction unique. Finally, the converse assertion that for PBW to be valid it is necessary that all ambiguities must be resolvable, is obvious. This proves the theorem. □

Proofs of the PBW Theorem for Lie Algebras and Super Lie Algebras. The first application is to the classical PBW theorem for the case of Lie algebras. Let $\mathfrak{g}$ be a Lie algebra over a commutative ring k with unit as above which is free as a k-module. Let $(X_i)_{i\in B}$ be a basis for $\mathfrak{g}$ over k. We assume that B has a total order (this is no restriction) so that for any two indices $i, j \in B$ we have one and only one of the following three possibilities: $i < j$, $j < i$, $i = j$; we write $i > j$ for $j < i$ and $i \le j$ if i is either less than j or $= j$. W is the set of all words with the letters X_i $(i \in B)$. A word $X_{i_1}X_{i_2}\cdots X_{i_m}$ is standard if $i_1 \le i_2 \le \cdots \le i_m$. Let $[\cdot,\cdot]$ be the bracket in $\mathfrak{g}$, so that $[X_i, X_j] = \sum_m c_{ijm}X_m$, $c_{ijm} \in k$. We take Σ to be the set of pairs (i, j) with $i, j \in B$, $i > j$; and for $(i, j) \in \Sigma$, $w_{(i,j)} = X_iX_j$ with $f_{(i,j)} = X_jX_i + \sum_m c_{ijm}X_m$. To define the order in W we proceed as follows: For any word $w = X_{i_1}X_{i_2}\cdots X_{i_m}$ we define its *rank* $rk(w)$ to be m and *index* $i(w)$ to be the number of pairs (a, b) with $a < b$ but $i_a > i_b$. Then a word is standard in our earlier sense if and only if it is standard in the present sense. The ordering of words is defined as follows: $w \prec w'$ if either $rk(w) < rk(w')$ or if $rk(w) = rk(w')$ but $i(w) < i(w')$. All the conditions discussed above are satisfied, and so to prove the PBW theorem we must check that all ambiguities are resolvable. Since all the words in Σ have rank 2, there are only overlap ambiguities, which are words of length 3 of the form $X_rX_jX_i$ where $i < j < r$. We must show that the tensors

$$X_jX_rX_i + [X_r, X_j]X_i\,, \qquad X_rX_iX_j + X_r[X_j, X_i]\,,$$

have identical reductions to standard forms under suitable reduction operators. The first expression can be reduced to

$$X_iX_jX_r + X_j[X_r, X_i] + [X_j, X_i]X_r + [X_r, X_j]X_i$$

while the second reduces to

$$X_iX_jX_r + [X_r, X_i]X_j + X_r[X_j, X_i] + X_i[X_r, X_j].$$

The quadratic terms in these expressions admit further reduction. For a commutator $[X, Y]$ with $X, Y \in \mathfrak{g}$ and any index $m \in B$, let us write $[X, Y]_{>m}$ to be the part of the expression for $[X, Y]$ in terms of the basis containing only the X_a with $a > m$, and similarly when greater than m is replaced by $<, \leq, \geq$. Notice now that the quadratic terms in the above two expressions differ by the reversal of the multiplications. Now, for any index c the reduction to standard form of $[X, Y]X_c$ and $X_c[X, Y]$ $(X, Y \in \mathfrak{g})$ is given by

$$[X, Y]X_c = [X, Y]_{\leq c}X_c + X_c[X, Y]_{>c} + [[X, Y]_{>c}, X_c],$$
$$X_c[X, Y] = X_c[X, Y]_{>c} + [X, Y]_{\leq c}X_c + [X_c, [X, Y]_{\leq c}].$$

Hence the difference between these two reduced forms is

$$[X_c, [X, Y]].$$

It follows from this calculation that the two reduced expressions for the word $X_rX_jX_i$ differ by

$$[X_r, [X_j, X_i]] + [X_j, [X_i, X_r]] + [X_i, [X_r, X_j]],$$

which is 0 precisely because of the Jacobi identity.

The second application is when $\mathfrak{g}$ is a Lie superalgebra. Recall that $\mathfrak{g}$ is $\mathbf{Z}_2$ graded with a bracket $[\cdot, \cdot]$ satisfying the skew symmetry condition

$$[X, Y] = -(-1)^{p(X)p(Y)}[Y, X]$$

and the super Jacobi identity, which encodes the fact that the adjoint map is a representation; writing as usual $\operatorname{ad} X : Y \longmapsto [X, Y]$, the Jacobi identity is the statement that $\operatorname{ad}[X, Y] = \operatorname{ad} X \operatorname{ad} Y - (-1)^{p(X)p(Y)} \operatorname{ad} Y \operatorname{ad} X$; i.e., for all $X, Y, Z \in \mathfrak{g}$ we have

$$[[X, Y], Z] = [X, [Y, Z]] - (-1)^{p(X)p(Y)}[Y, [X, Z]].$$

If U is the universal enveloping algebra of $\mathfrak{g}$, the skew symmetry becomes, when X is odd, the relation $2X^2 = [X, X]$. For this to be an effective condition, we assume that 2 is invertible in the ring k and rewrite this relation as

$$X^2 = \left(\frac{1}{2}\right)[X, X], \quad p(X) = 1.$$

Furthermore, when we take $X = Y = Z$ all odd in the Jacobi identity, we get $3[X, X] = 0$, and so we shall assume 3 is invertible in the ring k and rewrite this as

$$[[X, X], X] = 0.$$

For the PBW theorem we choose the basis (X_i) to be homogeneous; i.e., the X_i are either even or odd. Let $p(i)$ be the parity of X_i. The set Σ is now the set of pairs (i, j) with either $i > j$ or (i, i) with i odd. The corresponding $w_{(i,j)}$ are

$$w_{(i,j)} = X_iX_j, \ \ i > j, \quad w_{(i,i)} = X_i^2, \ \ p(i) = 1,$$

and the $f_{(i,j)}$ are given by

$$f_{(i,j)} = (-1)^{p(i)p(j)} X_j X_i + [X_j, X_i], \quad i > j,$$
$$f_{(i,i)} = \left(\frac{1}{2}\right)[X_i, X_i], \quad p(i) = 1.$$

Once again the only ambiguities are of the overlap type. These are the words $X_r X_j X_i$ where now we have to consider $i \leq j \leq r$. We have to consider various cases where there may be equalities. The first case is, of course, when $i < j < r$.

$i < j < r$. We want to show that the reduction to standard form of $X_r X_j X_i$ is the same whether we start with $X_r X_j$ or $X_j X_i$. Starting with $X_r X_j$ we find the expression, with $q = p(i)p(j) + p(j)p(r) + p(r)p(i)$,

$$(7.4) \quad (-1)^q X_i X_j X_r + [X_r, X_j]X_i + (-1)^{p(r)p(j)} X_j [X_r, X_i] + (-1)^{p(r)p(j)+p(r)p(i)} [X_j, X_i] X_r.$$

For the expression starting from $X_j X_i$ we find

$$(7.5) \quad (-1)^q X_i X_j X_r + X_r [X_j, X_i] + (-1)^{p(i)p(j)} [X_r, X_i] X_j + (-1)^{p(i)p(j)+p(r)p(i)} X_i [X_r, X_j].$$

Apart from the cubic term, which is standard, these expressions contain only quadratic terms and these need further reduction. For any three indices a, b, c we have, writing $t = p(c)p(a) + p(c)p(b)$,

$$[X_a, X_b]X_c = [X_a, X_b]_{\leq c} X_c + (-1)^t X_c [X_a, X_b]_{>c} + [[X_a, X_b]_{>c}, X_c],$$
$$X_c [X_a, X_b] = X_c [X_a, X_b]_{>c} + (-1)^t [X_a, X_b]_{\leq c} X_c + [X_c, [X_a, X_b]_{\leq c}].$$

If c is even, the two expressions on the right side above are already standard because the term $[X_a, X_b]_{\leq c} X_c$ is standard since there is no need to reduce X_c^2; if c is odd, we have to replace X_c^2 by $(1/2)[X_c, X_c]$ to reach the standard form. If E_1, E_2 are the two standard reduced expressions, it follows by a simple calculation that

$$E_1 - (-1)^t E_2 = [[X_a, X_b]_{>c}, X_c] - (-1)^t [X_c, [X_a, X_b]_{\leq c}].$$

Using the skew symmetry on the second term, we finally get

$$(7.6) \qquad E_1 - (-1)^{p(c)p(a)+p(c)p(b)} E_2 = [[X_a, X_b], X_c].$$

We now apply this result to the reductions of the two expressions in (7.4) and (7.5). Let S_1 and S_2 be the corresponding standard reductions. Using (7.6), we find for $S_1 - S_2$ the expression

$$[[X_r, X_j], X_i] - (-1)^{p(i)p(j)} [[X_r, X_i], X_j] + (-1)^{p(r)p(j)+p(r)p(i)} [[X_j, X_i], X_r].$$

By using skew symmetry this becomes

$$[[X_r, X_j], X_i] - [X_r, [X_j, X_i]] + (-1)^{p(r)p(j)} [X_j, [X_r, X_i]],$$

which is 0 by the super Jacobi identity.

$i = j < r$, $p(i) = 1$, or $i < j = r$, $p(j) = 1$. These two cases are similar, and so we consider only the first of these two alternatives, namely, the reductions of

$X_jX_iX_i$ with $i < j$ and i odd (we have changed r to j). The two ways of reducing are to start with X_jX_i or X_iX_i. The first leads to

$$X_iX_iX_j + (-1)^{p(j)}X_i[X_j, X_i] + [X_j, X_i]X_i .$$

The second leads to

$$\left(\frac{1}{2}\right)X_j[X_i, X_i] .$$

We proceed exactly as before. The reduction of the first expression is

$$\frac{1}{2}\big([X_i, X_i]_{\leq j}X_j + X_j[X_i, X_i]_{>j} + [[X_i, X_i]_{>j}, X_j]\big) + (-1)^{p(j)}[X_i, [X_j, X_i]] .$$

The second expression reduces to

$$\left(\frac{1}{2}\right)[X_i, X_i]_{\leq j}X_j + \left(\frac{1}{2}\right)X_j[X_i, X_i]_{>j} + \left(\frac{1}{2}\right)[X_j, [X_i, X_i]_{\leq j}] .$$

The difference between these two is

$$\left(\frac{1}{2}\right)[[X_i, X_i], X_j] + (-1)^{p(j)}[X_i, [X_j, X_i]] ,$$

which is 0 by the super Jacobi identity.

$i = j = r$, i odd. We can start with either the first X_iX_i or the second one. The difference between the two reduced expressions is

$$\left(\frac{1}{2}\right)[[X_i, X_i], X_i] ,$$

which is 0 by the super Jacobi identity.

If we order the indices such that all even induced come before the odd ones and we use Latin for the even and Greek for the odd indices, we have a basis (X_i, X_α), and the PBW theorem asserts that the monomials

$$X_{i_1}X_{i_2}\cdots X_{i_r}X_{\alpha_1}X_{\alpha_2}\cdots X_{\alpha_s} , \quad i_1 \leq i_2 \leq \cdots \leq i_r,\ \alpha_1 < \alpha_2 < \cdots < \alpha_s ,$$

form a basis for the universal enveloping algebra of the super Lie algebra $\mathfrak{g}$. We have thus finished the proof of Theorem 7.2.1.

We note that the ring k has been assumed to have the property that 2 and 3 are invertible in it. In particular, this is true if k is a $\mathbf{Q}$-algebra, for instance, if k is a field of characteristic 0, or if its characteristic is different from 2 and 3.

It is possible to give a formulation of the PBW theorem that does not mention bases. Let $\mathcal{U}^{(n)}$ be the image of $T^{\otimes n}$ under the map $T \longrightarrow \mathcal{U}$; write $\mathcal{U}^{(0)} = k1$. Then

$$\mathcal{U}^{(0)} \subset \mathcal{U}^{(1)} \subset \mathcal{U}^{(2)} \subset \cdots , \qquad \mathcal{U}^{(m)}\mathcal{U}^{(n)} \subset \mathcal{U}^{(m+n)} ,$$

so that $(\mathcal{U}^{(n)})$ is an ascending filtration of $\mathcal{U}$. We define $\mathrm{Gr}\,\mathcal{U}$ as the associated graded algebra as usual. $\mathrm{Gr}\,\mathcal{U}$ is graded by $\mathbf{Z}$, $\mathrm{Gr}_n\,\mathcal{U} = \mathcal{U}^{(n)}/\mathcal{U}^{(n-1)}$, and $\mathcal{U}^{(n)} = 0$ or $n < 0$. If $x_n \in \mathrm{Gr}_n\mathcal{U}$ and $x'_n \in \mathcal{U}^{(n)}$ above x_n, then $x_rx_s =$ image of $x'_rx'_s$ lies in $\mathrm{Gr}_{r+s}\,\mathcal{U}$. If $a, b \in \mathfrak{g}$, we have $ab = (-1)^{p(a)p(b)}ba + [a, b]$, from which it follows that for $x = a_1\cdots a_m$, $y = b_1\cdots b_n$ with $a_i, b_j \in \mathfrak{g}$ and homogeneous,

$$xy - (-1)^{p(x)p(y)}yx \in \mathcal{U}^{(m+n-1)} .$$

This shows that Gr $\mathcal{U}$ is a *supercommutative algebra* generated by the image of $\mathfrak{g}$. On the other hand, we have the *universal* supercommutative algebra generated by $\mathfrak{g}$, namely,

$$\mathrm{Sym}(\mathfrak{g}) = T/J\,,$$
$$J = \text{two-sided ideal generated by } x\otimes y - (-1)^{p(x)p(y)}yx\,, \quad x, y \in \mathfrak{g}\,.$$

It is quite easy to see that Sym($\mathfrak{g}$) is $\mathbf{Z}$-graded and

$$\mathrm{Sym}(\mathfrak{g}) \simeq \mathrm{Sym}(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1)\,.$$

Indeed, it is easy to show that the right side has the universal property. It is clear that Sym($\mathfrak{g}$) is the universal enveloping algebra of the super Lie algebra whose underlying super vector space is $\mathfrak{g}$ but all of whose brackets are 0. From the universal property it is immediate that we have a morphism

$$\mathrm{Sym}(\mathfrak{g}) \longrightarrow \mathrm{Gr}\,\mathcal{U}$$

that preserves the $\mathbf{Z}$-gradings. The reformulation of the PBW theorem is that this map is an *isomorphism*.

THEOREM 7.2.4 *We have the isomorphism*

$$\mathrm{Sym}(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Gr}\,\mathcal{U}$$

of supercommutative algebras that is the identity on $\mathfrak{g}$ *and preserves the* $\mathbf{Z}$*-gradings.*

Exactly as in the classical case we can lift the above isomorphism to a *linear* isomorphism of super vector spaces

$$\lambda : \mathrm{Sym}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{U}$$

by defining

$$\lambda(a_1 \cdots a_n) = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{p(\sigma)}a_{\sigma^{-1}}\cdots a_{\sigma^{-1}(n)}\,,$$

where (see Section 3.1)

$$p(\sigma) = \#\{(k,\ell) \mid k<\ell, a_k, a_\ell \in \mathfrak{g}_1, \sigma(k) > \sigma(\ell)\}\,.$$

Deligne and Morgan have given an entirely different perspective on the PBW theorem as reformulated in Theorem 7.2.4 by proving it for a *Lie algebra in an arbitrary tensor category.*[1] To understand what they do, notice that we can use the isomorphism λ of Theorem 7.2.4 to transfer the superalgebra structure on $\mathcal{U}$ to one on Sym($\mathfrak{g}$). Let us write $u * v$ for the product of u and v in this structure, $u, v \in$ Sym($\mathfrak{g}$). The Deligne-Morgan point of view is that this $*$-product can be defined directly on Sym($\mathfrak{g}$) and that one can then establish the universal property for (Sym($\mathfrak{g}$), $*$). The definition of the $*$-product is *sufficiently explicit that it makes sense for the Lie algebra in an arbitrary tensor category*. See 1.3.7 in Deligne and Morgan for details.

7.3. The Classical Series of Super Lie Algebras and Groups

Over an algebraically closed field k one can carry out a classification of simple super Lie algebras similar to what is done in the classical theory. A super Lie algebra $\mathfrak{g}$ is *simple* if it has no proper nonzero ideals and $\mathfrak{g} \neq k^{1|0}$. A super Lie algebra $\mathfrak{g}$ is called *classical* if it is simple and $\mathfrak{g}_0$ acts completely reducibly on $\mathfrak{g}_1$; i.e., $\mathfrak{g}_1$ is a direct sum of irreducible $\mathfrak{g}_0$-modules. Then one can obtain a complete list of these. Let us introduce, for any field k, the following super Lie algebras:

$\mathfrak{gl}(p|q)$. This is the super Lie algebra $M_L^{p|q}$.

$\mathfrak{sl}(p|q)$. This is given by

$$\mathfrak{sl}(p|q) = \{X \in \mathfrak{gl}(p|q) \mid \mathrm{str}(X) = 0\}.$$

We write

$$A(p|q) = \begin{cases} \mathfrak{sl}(p+1|q+1) & \text{if } p \neq q,\ p,q \geq 0 \\ \mathfrak{sl}(p+1|p+1)/kI & \text{if } p = q \geq 1. \end{cases}$$

For $A(p|q)$ the even parts and the odd modules are as follows:

$$\mathfrak{g} = A(p|q) : \mathfrak{g}_0 = A(p) \oplus A(q) \oplus k,\ \mathfrak{g}_1 = f_p \otimes f_q',$$
$$\mathfrak{g} = A(p|p) : \mathfrak{g}_0 = A(p) \oplus A(p),\ \mathfrak{g}_1 = f_p \otimes f_p',$$

where the f's are the defining representations, the primes denote duals, and $A(p) = \mathfrak{sl}(p+1)$.

$\mathfrak{osp}(\Phi)$. Let $V = V_0 \oplus V_1$ be a super vector space and let Φ be a symmetric, nondegenerate, even bilinear form $V \times V \longrightarrow k$. Then Φ is symmetric nondegenerate on $V_0 \times V_0$, symplectic on $V_1 \times V_1$, and is zero on $V_i \otimes V_j$ where $i \neq j$. Then

$$\mathfrak{osp}(\Phi) =$$
$$\left\{L \in \mathbf{End}(V) \mid \Phi(Lx, y) + (-1)^{p(L)p(x)}\Phi(x, Ly) = 0 \text{ for all } x, y \in V\right\}.$$

This is called the *orthosymplectic super Lie algebra associated with* Φ. It is an easy check that this is a super Lie algebra. If k is algebraically closed, Φ has a unique standard form, and then the corresponding super Lie algebra takes a standard appearance. The series osp(Φ) splits into several subseries:

$$\begin{aligned} B(m,n) &= \mathfrak{osp}(2m+1|2n), & m \geq 0,\ n \geq 1, \\ D(m,n) &= \mathfrak{osp}(2m|2n), & m \geq 2,\ n \geq 1, \\ C(n) &= \mathfrak{osp}(2|2n-2), & n \geq 2. \end{aligned}$$

The even parts of these and the corresponding odd parts as modules for the even parts are given as follows:

$$\mathfrak{g} = B(m,n) : \mathfrak{g}_0 = B(m) \oplus C(n),\ \mathfrak{g}_1 = f_{2m+1} \otimes f_{2n}',$$
$$\mathfrak{g} = D(m,n) : \mathfrak{g}_1 = f_{2m} \otimes f_{2n}',$$
$$\mathfrak{g} = C(n) : \mathfrak{g}_1 = k \otimes f_{2n-2}.$$

$P(n) (n \geq 2)$. This is the super Lie algebra defined by

$$P(n) = \left\{ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} \middle| \operatorname{tr}(a) = 0,\ b \text{ symmetric},\ c \text{ skew-symmetric} \right\}.$$

The Q-series is a little more involved in its definition. Let us consider the super Lie algebra $\mathfrak{gl}(n+1|n+1)$ of all matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and let us define the *odd trace* $\operatorname{otr}(g) = \operatorname{tr}(b)$. Let

$$Q^{\sim}(n) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| \operatorname{tr}(b) = 0 \right\}$$

and let

$$Q(n) = Q^{\sim}(n)/kI_{2n+2}\,.$$

For the even parts and the odd modules we have

$$\mathfrak{g} = P(n) : \mathfrak{g}_0 = \mathfrak{sl}(n+1|n+1),\ \mathfrak{g}_1 = \operatorname{Symm}^2(n+1) \oplus \Lambda^2(n+1)\,,$$
$$\mathfrak{g} = Q(n) : \mathfrak{g}_0 = A(n),\ \mathfrak{g}_1 = \operatorname{ad} A(n)\,.$$

THEOREM 7.3.1 (Kac) *Let k be algebraically closed. Then the simple and classical super Lie algebras are precisely*

$$A(m|n),\quad B(m|n),\quad D(m|n),\quad C(n),\quad P(n),\quad Q(n),$$

and the following exceptional series:

$$F(4),\quad G(3),\quad D(2|1,\alpha),\ \alpha \in k \setminus (0, \pm 1)\,.$$

REMARK. For all of this see Kac.[5] Here is some additional information regarding the exceptional series:

$$\mathfrak{g} = F(4) : \mathfrak{g}_0 = B(3) \oplus A(1), \mathfrak{g}_1 = \operatorname{spin}(7) \otimes f_2, \dim = 24|16\,,$$
$$\mathfrak{g} = G(3) : \mathfrak{g}_0 = G(2) \oplus A(1), \mathfrak{g}_1 = \mathbf{7} \otimes \mathbf{2}, \dim = 17|14\,,$$
$$\mathfrak{g} = D(2|1,\alpha) : \mathfrak{g}_1 = A(1) \oplus A(1) \oplus A(1), \mathfrak{g}_1 = \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}, \dim = 9|8\,.$$

The interesting fact is that the $D(2|1,\alpha)$ depend on a *continuous parameter.*

Classical Super Lie Groups. We restrict ourselves only to the linear and orthosymplectic series.

$\mathrm{GL}(p|q)$. The functor is $S \longmapsto \mathrm{GL}(p|q)(S)$ where S is any supermanifold and $\mathrm{GL}(p|q)(S)$ consists of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $a \in \mathrm{GL}(p)(\mathcal{O}(S)_0)$, $b \in \mathrm{GL}(q)(\mathcal{O}(S)_0)$, while b, c are matrices with entries from $\mathcal{O}(S)_1$. The representing supermanifold is the open submanifold of the affine space of dimension $p^2 + q^2|2pq$ defined by $\mathrm{GL}(p) \times \mathrm{GL}(q)$.

$\mathrm{SL}(p|q)$. The functor is $S \longmapsto \mathrm{SL}(p|q)(S)$ where $\mathrm{SL}(p|q)(S)$ is the kernel of the Berezinian. The representing supermanifold is the submanifold of $\mathrm{GL}(p|q)$ defined by the condition that the Berezinian is 1. One can also view it as the kernel of the morphism Ber from $\mathrm{GL}(p|q)$ to $\mathrm{GL}(1|0)$.

$\mathrm{OSp}(p|2q)$. The functor is $S \longmapsto \mathrm{OSp}(p|2q)(S)$ where $\mathrm{OSp}(p|2q)(S)$ is the subgroup of $\mathrm{GL}(p|2q)(S)$ fixing the appropriate even symmetric bilinear form Φ. The representability criterion mentioned earlier applies.

It is possible to describe the super Lie groups for the P and Q series also along similar lines. See Deligne and Morgan.[1]

For $\mathrm{SL}(p|q)$ and $\mathrm{OSp}(m|2n)$ we have appealed to the representability criterion mentioned earlier to justify the fact that they are super Lie groups. This can also be shown directly. We sketch below the calculations that will allow us to view the respective functors as super Lie groups.

We need a simple lemma for this purpose. Let $M \subset \mathbf{R}^{p|q}$ (or $\mathbf{C}^{p|q}$) be a super domain, i.e., an open subsupermanifold. Let $f_i (1 \le i \le r+s)$ be global homogeneous sections of $\mathcal{O}^{p|q}$ with f_i even for $i \le r$ and odd for $i > r$. We assume that the differentials $df_1, \dots, df_{r+s}$ are linearly independent at all points of the reduced manifold

$$N = \left\{ f_1^{\sim} = 0, \dots, f_r^{\sim} = 0 \right\}.$$

Then N is a smooth submanifold of $\mathbf{R}^p$ (or $\mathbf{C}^p$), and at each point n of N we can find a subset of the coordinates on the ambient space, $x^i (i \in I), \theta^\alpha (\alpha \in J)$, such that

$$f_1, \dots, f_r, x^i (i \in I), f_{r+1}, \dots f_{r+s}, \theta^\alpha (\alpha \in J)$$

form a coordinate system for the ambient space at n. It follows from this that if $\mathcal{K}$ is the subsheaf of $\mathcal{O}^{p|q}$ of ideals generated by the f^i, and $\mathcal{O}_N := \mathcal{O}^{p|q}/\mathcal{K}$, then $(N, \mathcal{O}_N)$ is a supermanifold of dimension $p - r|q - s$.

LEMMA 7.3.2 *Let S be a supermanifold and let $\psi (S \longrightarrow M)$ be a morphism. Let F^i $(1 \le i \le p)$, G^α $(1 \le \alpha \le q)$ be global sections of $\mathcal{O}_S$ that correspond to ψ, i.e., $F^i = \psi^*(x^i)$, $G^\alpha = \psi^*(\theta^\alpha)$. The ψ defines a morphism $S \longrightarrow N$ if and only if*

$$f^i(F^1, \dots, F^p, G^1, \dots, G^q) = 0, \quad 1 \le i \le r+s.$$

PROOF: Clearly ψ defines a morphism $S \longrightarrow N$ if and only if ψ^* vanishes on the ideal sheaf $\mathcal{K}$. Since $\mathcal{K}$ is generated by the f^i the lemma follows immediately. □

We shall apply this lemma to $\mathrm{SL}(p|q)$ and $\mathrm{OSp}(p|2q)$. In both cases the reduced manifold is a classical Lie group while the set of relations defining the super Lie group is invariant under the left translations of points of this classical group. It is therefore sufficient to verify the linear independence of the differentials just at the identity element e.

$\mathrm{SL}(p|q)$: Here $M = \mathrm{GL}(p|q)$ and we have a single even function f whose vanishing defines the subgroup in question. If

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is the coordinate matrix of M, then

$$F = \det(A - BD^{-1}C)\det(D)^{-1} - 1.$$

We must check that $(df)_e \neq 0$, i.e., some partial of f with respect to a coordinate is nonzero. This is immediate since

$$\left(\frac{\partial f}{\partial a_{11}}\right)_e = 1\,.$$

The lemma shows that the functor of points or $\mathrm{SL}(p|q)$ is

$$S \longmapsto \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \mathrm{Ber}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = 1 \right\}.$$

$\mathrm{OSp}(p|2q)$: The first step is to exhibit this as an affine superalgebraic subgroup of $\mathrm{GL}(p|2q)$. Let Φ be an even symmetric nondegenerate bilinear form on $\mathbf{R}^{p|2q}$ (or $\mathbf{C}^{p|2q}$). If A is any supercommutative ring, Φ defines a corresponding bilinear map into A as follows: Let e_i $(1 \le i \le p+2q)$ be the standard basis vectors of $\mathbf{R}^{p|2q}$ (or $\mathbf{C}^{p|2q}$) and let $p(i)$ be the parity of e_i, so that $p(i) = 0$ for $i \le p$ and $p(i) = 1$ for $i > p$. Let

$$\Phi_{km} = \Phi(e_k, e_m)\,.$$

Then

$$\Phi_A\left(\sum_k e_k x^k, \sum_m e_m y^m\right) = \sum_{km} (-1)^{p(m)p(x^k)} \Phi_{km} x^k y^m\,. \tag{7.7}$$

We are interested in determining the subgroup of elements $g \in \mathrm{GL}(p|2q, A)$ that fix Φ_A. Let $g = (g_{ab})$ where $p(g_{ab}) = p(a) + p(b)$. The condition on g is

$$\Phi_A(g\cdot e_k x, g\cdot e_m y) = \Phi_A(e_k x, e_m y) \tag{7.8}$$

where $x, y \in A$ are homogeneous. The left side of (7.8) becomes

$$\begin{aligned} \Phi_A\left(\sum_r e_r g_{rk} x, \sum_s e_s g_{sm} y\right) &= \sum_{rs} (-1)^{p(s)[p(x)+p(r)+p(k)]} \Phi_{rs} g_{rk} x g_{sm} y \\ &= (-1)^{p(m)p(x)} \sum_{rs} (-1)^{p(s)[p(r)+p(k)]} \Phi_{rs} g_{rk} g_{sm} x y\,. \end{aligned}$$

The right side of (7.8) is

$$(-1)^{p(m)p(x)} \Phi_{km} x y\,.$$

Hence we get

$$\Phi_{km} = \sum_{rs} (-1)^{p(s)[p(r)+p(k)]} g_{rk} \Phi_{rs} g_{sm}\,. \tag{7.9}$$

Let us define the supertranspose g^{ST} of g by

$$g^{\mathrm{ST}} = \begin{pmatrix} A^{\mathrm{T}} & -C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{pmatrix}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

so that

$$g^{\mathrm{ST}}_{kr} = (-1)^{p(r)[p(k)+1]} g_{rk}\,.$$

Of course, $(gh)^{\mathrm{ST}} = h^{\mathrm{ST}} g^{\mathrm{ST}}$; in verifying this we must keep in mind that $(LM)^{\mathrm{T}} = -M^{\mathrm{T}} L^{\mathrm{T}}$ for matrices L, M of odd elements. The map $g \longmapsto g^{\mathrm{ST}}$ has period 4. The condition on g becomes, since $\Phi_{rs} = 0$ if r, s are of opposite parity,

$$\Phi_{km} = \sum_{rs} g^{\mathrm{ST}}_{kr} \Phi_{rs} g_{sm}$$

or

$$\Phi = g^{\mathrm{ST}} \Phi g \,. \tag{7.10}$$

In symbolic notation, with Φ in standard form,

$$\mathrm{OSp}(p|2q) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, g^{\mathrm{ST}} \begin{pmatrix} I_p & 0 \\ 0 & J_{2q} \end{pmatrix} g = \begin{pmatrix} I_p & 0 \\ 0 & J_{2q} \end{pmatrix} \right\}$$

where

$$J_{2q} = \begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix} .$$

Let

$$F = g^{\mathrm{ST}} \Phi g - \Phi \,. \tag{7.11}$$

Then

$$F = \begin{pmatrix} A^{\mathrm{T}} A - C^{\mathrm{T}} J C - I & A^{\mathrm{T}} B - C^{\mathrm{T}} J D \\ B^{\mathrm{T}} A + D^{\mathrm{T}} J C & B^{\mathrm{T}} B + D^{\mathrm{T}} J D - J \end{pmatrix} .$$

The two diagonal blocks are symmetric and skew-symmetric, respectively, while the off diagonal blocks are transposes of each other. Hence the number of relations described by F is

$$\frac{p(p+1)}{2} + q(2q-1) \mid 2pq \quad \text{(even | odd)}.$$

We must show that the differentials of these are linearly independent at the identity element e of $\mathrm{O}(p) \times \mathrm{Sp}(2q)$. It is a question of proving that the space of matrices L such that $\mathrm{Tr}(L(dF)_e) = 0$ has dimension

$$\frac{p(p-1)}{2} + q(2q+1) + 2pq \,.$$

We now calculate $(dF)_e$. If matrices X, Y contain both even and odd elements, $d(XY)$ is in general not equal to $(dX)Y + X(dY)$, but *it is true* at any point of the reduced manifold. In particular, we have

$$(d(XY))_e = (dX)_e Y(e) + X(e)(dY)_e \,. \tag{7.12}$$

To see this, let d_0 and d_1 be, respectively, the parts of the d-operator corresponding to the even and odd coordinates. Then $d_0(fg) = (d_0 f)g + f(d_0 g)$ and $d_1(fg) = (d_1 f)g + (-1)^{p(f)} f(d_1 g)$. Hence, evaluating at a *classical point* x we have, as $f(x) = 0$ for f odd,

$$(d_1(fg))_x = (d_1 f)_x g(x) + f(x)(d_1 g)_x \,.$$

This leads to (7.12).

Returning to F, we have, by (7.11),

$$(dF)_e = (d(g^{\mathrm{ST}}\Phi g - \Phi))_e = (dg^{\mathrm{ST}})_e\Phi + \Phi(dg)_e$$
$$= \begin{pmatrix} dA^{\mathrm{T}} + dA & -dC^{\mathrm{T}}J + dB \\ dB^{\mathrm{T}} + JdC & dD^{\mathrm{T}}J + JdD \end{pmatrix}_e .$$

Hence for a *constant* matrix

$$L = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

we find for $\mathrm{Tr}(L(dF)_e)$ the expression

$$\mathrm{Tr}\left(P(dA^{\mathrm{T}} + dA)_e + Q(dB^{\mathrm{T}} + JdC)_e + R(dB - dC^{\mathrm{T}}J)_e + S(dD^{\mathrm{T}}J + JdD)_e\right).$$

Using $\mathrm{Tr}(XY) = \mathrm{Tr}(X^{\mathrm{T}}Y^{\mathrm{T}})$ and $\mathrm{Tr}(XYJ) = \mathrm{Tr}(JXY) = -\mathrm{Tr}(X^{\mathrm{T}}JY^{\mathrm{T}})$, we get for $\mathrm{Tr}(L(dF)_e)$ the expression

$$\mathrm{Tr}\left\{(P + P^{\mathrm{T}})(dA)_e + (Q^{\mathrm{T}} + R)(dB)_e + ((Q + R)J)(dC)_e + ((S - S^{\mathrm{T}})J)(dD)_e\right\}.$$

Because the differentials $(dA)_e$, $(dB)_e$, ... are linearly independent, we have

$$\mathrm{Tr}(L(dF)_e) = 0 \iff P^{\mathrm{T}} = -P, \quad S^{\mathrm{T}} = S, \ R = -Q^{\mathrm{T}},$$

and so the space of L for which this trace is 0 has the dimension

$$\frac{p(p-1)}{2} + q(2q+1) + 2pq .$$

We leave it to the reader to verify that

$$\mathrm{Lie}(\mathrm{SL}(p|q) = \mathfrak{sl}(p|q), \quad \mathrm{Lie}(\mathrm{OSp}(p|2q) = \mathfrak{osp}(p|2q).$$

7.4. Superspacetimes

Superspacetimes are supermanifolds M such that M_{red} is a classical spacetime. They are constructed so that they are homogeneous spaces for super Poincaré groups, which are super Lie groups acting on them. The superspacetimes are themselves super Lie groups, just like classical spacetime, which is a translation group. They are thus called *supertranslation groups*. They are *abelian*. However, they are two-step nilpotent, as we shall see later on.

Super Poincaré Algebras. We have seen an example of this, namely, the super Lie algebra of Gol′fand-Likhtman and Volkov-Akulov. Here we shall construct them in arbitrary dimension and Minkowski signature. Let V be a real quadratic vector space of signature $(1, D-1)$. The usual case is when $D = 4$, but other values of D are also of interest. For conformal theories V is taken to be of signature $(2, D-2)$. We shall not discuss the conformal theories here.

The Poincaré Lie algebra is the semidirect product

$$\mathfrak{g}_0 = V \times' \mathfrak{so}(V).$$

We shall denote by S a real spinorial representation of $\mathrm{Spin}(V)$. We know that there is a symmetric nonzero map

$$\Gamma : S \times S \longmapsto V \tag{7.13}$$

equivariant with respect to Spin(V); Γ is projectively unique if S is irreducible. Let

$$\mathfrak{g} = \mathfrak{g}_0 \oplus S\,.$$

We regard S as a $\mathfrak{g}_0$-module by requiring that V act as 0 on S. Then if we define

$$[s_1, s_2] = \Gamma(s_1, s_2)\,, \quad s_i \in S\,,$$

then with the $\mathfrak{g}_0$-action on $\mathfrak{g}_1$ we have a super Lie algebra, because the condition

$$[s, [s, s]] = -[\Gamma(s, s), s] = 0\,, \quad s \in S\,,$$

is automatically satisfied since $\Gamma(s, s) \in V$ and V acts as 0 on S. $\mathfrak{g}$ is a supersymmetric extension of the Poincaré algebra and is an example of a super Poincaré algebra. The Gol'fand-Likhtman-Volkov-Akulov algebra is a special case when $D = 4$ and S is the Majorana spinor. We shall normalize Γ so that it is *positive* in the sense of Section 6.7, i.e., $\Gamma(s, s)$ lies in the interior of the forward light cone in V for all nonzero $s \in S$. Let $S = NS_0$ where S_0 is irreducible, and select a basis (Q_a) for S_0 and the standard basis (P_μ) for $V \simeq \mathbf{R}^{1,D-1}$. Then, in the notation of Section 6.7,

$$\Gamma_0(Q_a, Q_b) = \sum_\mu \Gamma^\mu_{ab} P_\mu\,, \quad \Gamma^\mu_{ab} = \Gamma^\mu_{ba}\,.$$

We can then choose a basis (Q^i_a) for S $(1 \le i \le N)$ so that

$$[Q^i_a, Q^j_b] = \delta_{ij} \sum_\mu \Gamma^\mu_{ab} P_\mu\,.$$

The fact that Γ takes values in V means that

$$\mathfrak{l} = V \oplus S$$

is also a super Lie algebra. It is a supersymmetric extension of the abelian spacetime translation algebra V, but $\mathfrak{l}$ is *not* abelian since $\Gamma \neq 0$. It is often called a *super translation algebra.* However, it is two-step nilpotent, namely,

$$[a, [b, c]] = 0\,, \quad a, b, c \in \mathfrak{l}\,.$$

The corresponding super Lie groups will be the *superspacetimes.*

The super Lie group L corresponding to $\mathfrak{l}$ will be constructed by the exponential map. We have not discussed this, but we can proceed informally and reach a definition that can then be rigorously checked. Using the Baker-Campbell-Hausdorff formula informally and remembering that triple brackets are zero in $\mathfrak{l}$, we have

$$\exp A \exp B = \exp\left(A + B + \left(\frac{1}{2}\right)[A, B]\right), \quad A, B \in \mathfrak{l}\,.$$

This suggests that we identify L with $\mathfrak{l}$ and *define* the group law by

$$A \circ B = A + B + \left(\frac{1}{2}\right)[A, B]\,, \quad A, B \in \mathfrak{l}\,.$$

More precisely, let us first view the super vector space $\mathfrak{l}$ as a supermanifold. If (B_μ), (F_a) are bases for V and S, respectively, then for any supermanifold T, Hom(T, $\mathfrak{l}$) can be identified with (β_μ, τ_a) where β_μ, τ_a are elements of $\mathcal{O}(T)$ that are even and odd, respectively. In a basis-independent manner we can identify this with

$$\mathfrak{t}(T) := (\mathfrak{t} \otimes \mathcal{O}(T))_0 = V \otimes \mathcal{O}(T)_0 \oplus S \otimes \mathcal{O}(T)_1 .$$

It is clear that $\mathfrak{t}(T)$ is a *Lie algebra*. Indeed, all brackets are zero except for pairs of elements of $S \otimes \mathcal{O}_1$, and for these the bracket is defined by

$$[s_1 \otimes \tau_1, s_2 \otimes \tau_2] = -\Gamma(s_1, s_2)\tau_1\tau_2 , \quad \tau_1, \tau_2 \in \mathcal{O}(T)_1 .$$

Notice that the sign rule has been used since the s_j and τ_j are odd; the super Lie algebra structure of $\mathfrak{t}$ is necessary to conclude that this definition converts $\mathfrak{t}(T)$ into a *Lie algebra*. We now take

$$L(T) = \mathfrak{t}(T)$$

and define a binary operation on $L(T)$ by

$$A \circ B = A + B + \left(\frac{1}{2}\right)[A, B] , \quad A, B \in \mathfrak{t}(T) .$$

The Lie algebra structure on $\mathfrak{t}(T)$ implies that this is a group law. In the bases (B_μ), (F_a) defined above,

$$(\beta^\mu, \tau^a) \circ (\beta'^\mu, \tau'^a) = (\beta''_\mu, \tau''^a)$$

where

$$\beta''_\mu = \beta_\mu + \beta'_\mu - \left(\frac{1}{2}\right)\Gamma^\mu_{ab}\tau_a\tau'_b , \quad \tau''^a = \tau'^\mu + \tau'^\mu .$$

Symbolically this is the same as saying that L has coordinates (x^μ), (θ^a) with the group law

$$(x, \theta)(x', \theta') = (x'', \theta'')$$

where

$$x''^\mu = x^\mu + x'^\mu - \left(\frac{1}{2}\right)\Gamma^\mu_{ab}\theta^a\theta'^b , \quad \theta''^a = \theta^a + \theta'^a \tag{7.14}$$

(with summation convention). The supermanifold L thus defined by the data V, S, Γ has dimension $\dim(V)|\dim(S)$. It is the underlying manifold of a super Lie group L with $L_{\text{red}} = V$. This is the supertranslation group associated to V, S, Γ.

Because L is a super Lie group, one can introduce the left and right invariant differential operators on L that make differential calculus on L very elegant, just as in the classical case. Recall that the left invariant vector fields are obtained by differentiating the group law at $x'^\mu = \theta'^a = 0$, and for the right invariant vector fields we differentiate the group law with respect to the unprimed variables at 0. Let D_μ, D_a (D^r_μ, D^r_a) be the left (right) invariant vector fields with tangent vector $\partial/\partial x^\mu$, $\partial/\partial\theta^a$ at the identity element. Let ∂_μ, ∂_a be the global vector fields on L

defined by $\partial_\mu = \partial/\partial x^\mu$, $\partial_a = \partial/\partial\theta^a$. Then, using (7.1$\ell$) and (7.1$r$) we get

$$D_\mu = D^r_\mu = \partial_\mu ,$$
$$D_a = \left(\frac{1}{2}\right)\Gamma^\mu_{ab}\theta^b\partial_\mu + \partial_a ,$$
$$D^r_a = -\left(\frac{1}{2}\right)\Gamma^\mu_{ab}\theta^b\partial_\mu + \partial_a .$$

It is an easy verification that

$$[D_a, D_b] = \Gamma^\mu_{ab}\partial_\mu .$$

This gives the super Lie algebra structure associated with V, S, Γ described earlier if we make the identification

$$\partial_\mu \longleftrightarrow P_\mu , \quad \partial_a \longleftrightarrow Q_a .$$

When $S = NS_0$, we use odd coordinates (θ^{ai}) and (7.14) becomes

$$x''^\mu = x^\mu + x'^\mu - \frac{1}{2}\sum_i \Gamma^\mu_{ab}\theta^{ai}\theta^{bi} , \quad \theta''^{ai} = \theta^{ai} + \theta'^{ai} . \tag{7.15}$$

Complex and Chiral Superspacetimes. The superspacetime constructed when $D = 4$ with S as the Majorana spinor is denoted by $M^{4|4}$. We shall now discuss a variant of the construction above that yields what are called *chiral superspacetimes*.

We take (F_a) and $(\overline{F}_{\dot a})$ as bases for S^+ and S^- so that if $g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, then g acts on $S^\pm$ by

$$g^+ \sim \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \quad g^- \sim \begin{pmatrix}\overline{\alpha} & \overline{\beta}\\ \overline{\gamma} & \overline{\delta}\end{pmatrix}.$$

If $v = \sum u^a F_a$, $\overline{v} = \sum_{\dot a} \overline{u^{\dot a}}\,\overline{F}_{\dot a}$, then

$$\overline{g^+ v} = g^-\overline{v} .$$

On $S = S^+ \oplus S^-$ we define the conjugation σ by

$$\sigma(u, \overline{v}) = (v, \overline{u}) .$$

Let

$$V_{\mathbf{C}} = S^+ \otimes S^- , \quad B_{a\dot b} = F_a F_{\dot b},\ B_{\dot a b} = F_{\dot a} F_b \ \text{(tensor multiplication).}$$

The symmetric nonzero map

$$\Gamma : (S^+ \oplus S^-) \otimes (S^+ \oplus S^-) \longrightarrow V_{\mathbf{C}}$$

is nonzero only on $S^\pm \otimes S^\mp$ and is uniquely determined by this and the relations $\Gamma(F_a, \overline{F_{\dot b}}) = B_{a\dot b}$. So

$$\mathfrak{t}_{\mathbf{C}} = V_{\mathbf{C}} \oplus (S^+ \oplus S^-)$$

is a complex super Lie algebra and defines a complex Lie group $L_{\mathbf{C}}$ exactly as before; the group law defined earlier extends to $L_{\mathbf{C}}$. But now, *because we are operating over* $\mathbf{C}$, *the subspaces*

$$\mathfrak{l}_{\mathbf{C}}^{\pm} = V_{\mathbf{C}} \oplus S^{\pm}$$

are super Lie algebras over $\mathbf{C}$ *and determine corresponding complex super Lie groups* $L_{\mathbf{C}}^{\pm}$. Moreover, because Γ vanishes on $S^{\pm} \otimes S^{\pm}$, these are *abelian* and the super Lie algebras $\mathfrak{l}_{\mathbf{C}}^{\pm}$ are actually *abelian ideals of* $\mathfrak{l}_{\mathbf{C}}$. The $L_{\mathbf{C}}^{\pm}$ are the *chiral superspacetimes*; actually, we define $L_{\mathbf{C}}^{+}$ as the chiral and $L_{\mathbf{C}}^{-}$ as the *antichiral* superspacetime. Moreover,

$$L_{\mathbf{C}} = L_{\mathbf{C}}^{+} \times_{V_{\mathbf{C}}} L_{\mathbf{C}}^{-}$$

where the suffix denotes the fiber product.

We have conjugations on $V_{\mathbf{C}}$ and on $S^{+} \oplus S^{-}$. On $V_{\mathbf{C}}$ the conjugation is given by

$$\sigma : u \otimes \overline{v} \longmapsto v \otimes \overline{u} \,,$$

while the one on $S^{+} \oplus S^{-}$, also denoted by σ, is

$$(u, \overline{v}) \longmapsto (v, \overline{u}) \,.$$

The map Γ is compatible with these two conjugations. Hence we have a conjugation σ on $\mathfrak{l}_{\mathbf{C}}$ and hence on $L_{\mathbf{C}}$. We have

$$L = L_{\mathbf{C}}^{\sigma} \,.$$

In other words, L may be viewed as the real supermanifold defined inside $L_{\mathbf{C}}$ as the fixed-point manifold of σ. If

$$y^{a\dot{b}}, \theta^{a}, \overline{\theta}^{\dot{b}},$$

are the coordinates on $L_{\mathbf{C}}$, then L is defined by the reality constraint

$$y^{a\dot{b}} = \overline{y^{b\dot{a}}} \,, \quad \overline{\theta}^{\dot{a}} = \overline{\theta^{a}} \,.$$

The left invariant vector fields on $L_{\mathbf{C}}$ are the complex derivations ∂_{μ} and the D_a, $\overline{D}_{\dot{a}}$ with

$$D_a = \partial_a + \left(\frac{1}{2}\right)\overline{\theta}^{\dot{b}}\partial_{a\dot{b}} \,, \quad \overline{D}_{\dot{a}} = \partial_{\dot{a}} + \left(\frac{1}{2}\right)\theta^{b}\partial_{b\dot{a}} \,,$$

where repeated indices are summed over.

Let us now go over to new coordinates

$$z^{a\dot{b}}, \varphi^{a}, \overline{\varphi}^{\dot{b}}$$

defined by

$$y^{a\dot{b}} = z^{a\dot{b}} - \left(\frac{1}{2}\right)\varphi^{b}\overline{\varphi}^{\dot{a}} \,, \quad \theta^{a} = \varphi^{a} \,, \quad \overline{\theta}^{\dot{a}} = \overline{\varphi}^{\dot{a}} \,.$$

Chiral (antichiral) superfields are those sections of the structure sheaf of $L_{\mathbf{C}}$ that depend only on z, φ (z, $\overline{\varphi}$). A simple calculation shows that

$$D_a = \frac{\partial}{\partial \varphi^{\dot{a}}} \,, \quad \overline{D}_{\dot{a}} = \frac{\partial}{\partial \overline{\varphi}^{\dot{a}}} \,.$$

So it is convenient to use these coordinates, which we can rename $y, \theta, \overline{\theta}$. These constructions generalize immediately to the case when V has dimension $D \equiv 0, 4 \pmod 8$. In this case S_0 is of type $\mathbf{C}$ and we have chiral superspacetimes. We leave it to the reader to work out the details.

7.5. Super Poincaré Groups

The super Poincaré algebra is $\mathfrak{g} = \mathfrak{g}_0 \oplus S$ where $\mathfrak{g}_0 = V \oplus \mathfrak{h}$; here $\mathfrak{h}$ is the Lie algebra $\mathfrak{so}(V)$. The Lie algebra of superspacetime is $\mathfrak{l} = V \oplus S$. Let $H = \mathrm{Spin}(V)$. Then H acts on $\mathfrak{l}$ as a group of super Lie algebra automorphisms of $\mathfrak{l}$. This action lifts to an action of L on the supermanifold L by automorphisms of the super Lie group L. The semidirect product

$$G = L \times' H$$

is the super Poincaré group. The corresponding functor is

$$S \longmapsto G(S)$$

where

$$G(S) = L(S) \times' H(S)\,.$$

This description also works for $L_{\mathbf{C}}, L_{\mathbf{C}}^{\pm}$ with H replaced by the complex spin group.

Superfield Equations. Once superspacetimes are defined, one can ask for the analogue of the Poincaré invariant field equations in the super context. This is a special case of the following more general problem: if M is a supermanifold and G is a super Lie group acting on M, find the invariant super differential operators D and the spaces of the solutions of the equations $D\Psi = 0$ where Ψ is a global section of the structure sheaf. In the case of superspacetimes this means the construction of the differential operators that extend the Klein-Gordon and Dirac operators. The superfields are the sections of the structure sheaf, and it is clear that in terms of the components of the superfield we will obtain several ordinary field equations. This leads to the notion of a *multiplet* and the idea that a superparticle defines a multiplet of ordinary particles. We do not go into this aspect at this time.

7.6. References

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