Math 259x: Moduli Spaces in Algebraic Geometry

Dori Bejleri

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1 Overview

One of the characterizing features of algebraic geometry is that the set of all geometric objects of a fixed type (e.g. smooth projective curves, subspaces of a fixed vector space, or coherent sheaves on a fixed variety) often itself has the structure of an algebraic variety (or more general notion of algebro-geometric space). Such a space $\mathcal{M}$ is the *moduli space* classifying objects of the given type and in some sense the study of all objects of the given type is reduced to the studying the geometry of the space $\mathcal{M}$. This self-referential nature of algebraic geometry is a crucial aspect of the field.

More precisely, suppose we are interested in studying some class of geometric objects $\mathcal{C}$ with a suitable notion of a family of objects of $\mathcal{C}$

$$\pi : \mathscr{X} := \bigsqcup \{X_b \in \mathcal{C} \ : \ b \in B\} \to B$$

parametrized by some base scheme B. To a first approximation, we may attempt to construct a moduli space for the class $\mathcal{C}$ in two steps. First we find a family $\pi : \mathscr{X} \to B$ such that for each object $X \in \mathcal{C}$, there exists a $b \in B$ with $X_b \cong X$. Next we look for an equivalence relation on B such that $b \sim b'$ if and only if $X_b \cong X_{b'}$ and such that the quotient of B by this equivalence relation inherits the structure of an algebraic variety. If this happens, we may call $\mathcal{M} := B/\sim$ a moduli space and the family of objects $\mathcal{M}$ inherits from π the *universal family* (we will discuss this more carefully soon). In particular, the points of $\mathcal{M}$ are in bijection with isomorphism classes of objects in $\mathcal{C}$.

Moduli spaces give a good answer to the question of classifying algebraic varieties, or more generally objects of some class $\mathcal{C}$. In the best case scenario, we may have that

$$\mathcal{M} = \bigsqcup_{\Gamma} \mathcal{M}_\Gamma$$

where d is some discrete invariant (not necessarily an integer) and each component $\mathcal{M}_\Gamma$ is of finite type. Then classifying the objects of $\mathcal{C}$ reduces to (1) classifying the discrete invariants Γ, and (2) computing the finite type spaces $\mathcal{M}_\Gamma$.

Example 1.1. *The prototypical example which we will discuss at length later in the class is that of smooth projective curves.*[1] *Here there is one discrete invariant, the genus g, and the moduli space is a union*

$$\mathcal{M} = \bigsqcup_{g \in \mathbb{Z}_{\geq 0}} \mathcal{M}_g$$

of smooth $3g-3$-dimensional components. This example was originally studied by Riemann in his 1857 *paper* Theorie der Abel'schen Functionen *where he introduced the word* moduli *to refer to the $3g-3$ parameters that (locally) describe the space $\mathcal{M}_g$.*

1.0.1 Facets of moduli theory

This class will focus on the following three facets of moduli theory.

[1] For us a curve is a finite type k-scheme with pure dimension 1 for k a field.

Existence and construction

Hilbert schemes, algebraic stacks, GIT, Artin algebraization, coarse and good moduli spaces

Compactifications

Semi-stable reduction, Deligne-Mumford-Knudsen-Hassett compactifications, wall-crossing, KSBA stable pairs

Applications

Enumerative geometry and curve counting, computing invariants, constructing representations, combinatorics, arithmetic statistics

Of course we won't have time to cover everything written above (and there are countless more topics that fit under each heading) but I hope to give a feeling of the techniques and tools employed in moduli theory as well as the far reaching applications.

1.0.2 A note on conventions

For most of the class we will be working with finite type (or essentially of finite type) schemes over a field. I will make an effort to make clear when results require assumptions on the field (algebraically closed, characteristic zero) or when we work over a more general base. Not much will be lost if the reader wishes to assume everything is over the complex numbers throughout.

1.1 Motivating examples

Before diving in, I want to give some motivating examples of works that crucially relied on the tools and techniques of moduli theory. Many of the moduli theoretic ideas that come up in these examples will be discussed through the course.

1.1.1 Counting rational curves on K3 surfaces

Recall that a *K3 surface* is a smooth projective surface X with trivial canonical sheaf

$$\omega_X := \Lambda^2 \Omega_X \cong \mathcal{O}_X$$

and $H^1(X, \mathcal{O}_X) = 0$. A *polarized* K3 surface is a pair (X, H) where X is a K3 surface and H is an ample line bundle.

It turns out that for each g, there is a moduli space

$$\mathcal{M}_{2g-2}$$

parametrizing polarized K3 surfaces with $c_1(H)^2 = 2g - 2$.[2] The linear series $|H|$[3] is g-dimensional and the curves in $|H|$ have genus g. In particular one expects finitely many

[2]Recall that $c_1(H)^2$ may be defined as the degree of $\mathcal{O}_X(C)|_C$ where C is the vanishing of a section of H.

[3]Recall the linear series of H is the space of divisors linearly equivalent to H, or equivalently, the projectivization $\mathbb{P}(H^0(X, H))$.

rational curves in $|H|$[4].

Let $n(g)$ denote the number of rational curves in $|H|$ for a generic polarized complex K3 surface $(X, H) \in \mathcal{M}_{2g-2}$. Note that the existence of a moduli space $\mathcal{M}_{2g-2}$ allows us to define generic as "corresponding to a point that lies in some Zariski open and dense subset of $\mathcal{M}_{2g-2}$." Then we have the following formula, conjectured by Yau and Zaslow, and proved by Beauville.

Theorem 1.2 (Beauville-Yau-Zaslow)**.**

$$1 + \sum_{g \geq 1} n(g) q^g = \prod_{n \geq 1} \frac{1}{(1 - q^n)^{24}}$$

In particular, the numbers $n(g)$ are constant for general (X, H).

The proof here uses, among other things, a careful study of the compactified Jacobians of the (necessarily singular!) rational curves in $|H|$ and Hilbert schemes of points on X, two topics we will visit later in the class.

1.1.2 The $n!$-conjecture

A partition of n, denoted $\lambda \vdash n$, is a sequence of integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0$ with

$$\sum \lambda_i = n.$$

We can represent λ by a *Young diagram* of left aligned rows of boxes where the i^{th} row has λ_i boxes. Each box inherits a coordinate $(a, b) \in \mathbb{N}^2$ recording its position. In particular, the diagram has n boxes. For example, the partition $2 + 1 = 3$ corresponds to the following diagram.

$$\begin{array}{|c|c|} \cline{1-1} (0,1) \\ \hline (0,1) & (0,1) \\ \hline \end{array}$$

We can list the n boxes $(a_1, b_1), \ldots, (a_n, b_n)$ of the diagram and then consider the matrix

$$\begin{bmatrix} x_1^{a_1} y_1^{b_1} & x_2^{a_1} y_2^{b_1} & \ldots & x_n^{a_1} y_n^{b_1} \\ \vdots & & \ddots & \vdots \\ x_1^{a_n} y_1^{b_n} & x_2^{a_n} y_2^{b_n} & \ldots & x_n^{a_n} y_n^{a_n} \end{bmatrix}$$

where the x_i and y_i are $2n$ indeterminates. Finally, let Δ_λ be the determinant of the above matrix. Note that Δ_λ is a homogeneous polynomial in both the x_i variables and the y_i variables. Furthermore, S_n acts by permuting the x_i and the y_i, and under this action, S_n acts on Δ_λ by the sign representation. In particular, Δ_λ is well defined up to a sign.

Finally consider the vector space

$$D_\lambda := k[\partial_x, \partial_y] \Delta_\lambda$$

spanned by all partial derivatives of Δ_λ. This space carries a natural action of S_n. The $n!$ *conjecture*, proposed by Haiman and Garsia and later proved by Haiman, states the following.

[4]Recall C is a rational curve if its normalization has genus 0.

Theorem 1.3 (Haiman). *D_λ as an S_n representation is isomorphic to the regular representation. In particular,* $\dim_k D_\lambda = n!$.

In our example partition above, we have the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

with determinant

$$\Delta_\lambda = x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1.$$

The partial derivatives of Δ_λ are itself, constants, and the following linear forms.

$$\begin{array}{ll} x_3 - x_2, & y_2 - y_3 \\ x_1 - x_3, & y_3 - y_1 \\ x_2 - x_1, & y_1 - y_2 \end{array}$$

The two columns above each span a copy of the standard two dimensional representation, Δ_λ spans the sign, and the constants span the trivial representation.

The theorem is proved by a careful study of the geometry of the Hilbert scheme of points on $\mathbb{A}^2$ which we will study in depth later in the class. In fact, if we denote by H_n the Hilbert scheme of n points in $\mathbb{A}^2$, then the $n!$ Theorem is equivalent to a particular moduli space $X_n \to H_n$ lying over H_n being Gorenstein![5]

Remark 1.4. *The motivation for the $n!$ conjecture came from symmetric function theory, and in particular, Macdonald positivity which is a corollary. In fact Macdonald positivity also has an interesting interpretation in terms of Hilbert schemes of points, and more precisely, the McKay correspondence for S_n acting on $\mathbb{A}^{2n}$. We will revisit this later.*

1.1.3 Alterations

Let X be a reduced locally Noetherian scheme over a field k. In many arguments it is useful to be able to replace X with a regular scheme that is "very close" to X. More precisely, a *resolution of singularities* of X is a morphism $f : X' \to X$ such that

- X' is regular,
- f is proper [6], and
- f is birational[7].

Hironaka famously showed that when k has characteristic 0, resolutions of singularities always exist. The characteristic p case remains open. However, we have a positive result if we weaken the notion of being "very close" to X.

We say that $f : X' \to X$ is an *alteration* if it is proper, surjective, and generically finite. Then de Jong proved the following theorem which serves as a suitable replacement of Hironaka's theorem for many applications.

[5]This X_n is called the isospectral Hilbert scheme.
[6]This rules out the trivial operation of taking X' to be the regular locus of X
[7]This is one possible meaning of X' being very close to X.

Theorem 1.5 (de Jong). *Let X be a variety over a field k. Then there exists an alteration $f : X' \to X$ with X' regular.*

The proof of de Jong's theorem crucially uses the existence and properness of the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$ by pointed stable curves. The following diagram gives a very basic sketch of the ideas involved.

$$\begin{tikzcd} X''' \arrow[r, "(5)"'] & X'' \arrow[r] \arrow[d, "(4)"] & X' \arrow[r] \arrow[d, "(2)"] & X \arrow[d, "(0)"] \\ & Y'' \arrow[r, "(3)"'] & Y' \arrow[r, "(1)"'] & Y \end{tikzcd}$$

Here, after possibly replacing X by some blowup, we find a projection (0) of relative dimension 1 with regular generic fiber. Then after taking an alteration (1) of Y, we can construct an alteration $X' \to X$ so that (2) has as fibers curves with at worst nodal singularities. Producing this map (2) with such properties is precisely where the existence of the Deligne-Mumford compactification is used! Then by induction on dimension, we have an alteration (3) with Y'' regular. Now the pullback (4) is a morphism with at worst nodal fibers over a regular base. In this situation X'' has nice singularities that can be explicitly resolved by (5) to obtain a regular X''' with the composition $X''' \to X$ an alteration.

2 Moduli functors and Grassmannians

2.1 Moduli functors and representability

We arrive at the precise definitions that form the backbone of moduli theory.

Let $\mathcal{C}$ be any category. Given an object X of $\mathcal{C}$, we can consider the (contravariant) functor of points associated to X:

$$h_X : \mathcal{C}^{op} \to Set \tag{1}$$
$$T \mapsto \mathrm{Hom}_{\mathcal{C}}(T, X) \tag{2}$$

Note that $h\_$ defines a covariant functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, Set)$: if $a : X \to Y$ is a morphism, then $h_a : \mathrm{Hom}_{\mathcal{C}}(-, X) \to \mathrm{Hom}_{\mathcal{C}}(-, Y)$ is given by composition with a.

We have the following basic but crucial lemma.

Lemma 2.1 (Yoneda). *(a) For any object X of $\mathcal{C}$ and any functor $F : \mathcal{C} \to Set$, there is a natural isomorphism*

$$\mathrm{Nat}(h_X, F) \cong F(X).$$

(b) The functor

$$h\_ : \mathcal{C} \to Fun(\mathcal{C}^{op}, Set)$$

is fully faithful.

In light of this, we will often view $\mathcal{C}$ as a full subcategory of $\mathrm{Fun}(\mathcal{C}^{op}, Set)$ and identify objects X of $\mathcal{C}$ with their functor of points h_X. We often call a functor F a presheaf on $\mathcal{C}$ and refer to $Fun(\mathcal{C}^{op}, Set)$ as the category of presheaves.

Definition 2.2. *We say that a functor $F : \mathcal{C}^{op} \to Set$ is representable if there exists an object X and a natural isomorphism*

$$\xi : F \to h_X.$$

In this case we say F is representable by X.

In this course, the case of interest is when $\mathcal{C} = Sch_S$ is the category of schemes over a base S. In this case, we call F a *moduli problem* or *moduli functor*. In most cases F will be of the form

$$F(T) = \{\text{families of objects over } T\} / \sim$$

where $\sim$ is isomorphism, and F is made into a functor by pulling back families along $T' \to T$.

Definition 2.3. *If F is representable by a scheme M, we say that M is a fine moduli space for the moduli problem F.*

Given a fine moduli space M for F, which is unique if it exists, then we have an element $\xi^{-1}(id_M) \in F(M)$ corresponding to the identity $\mathrm{Hom}_{\mathcal{C}}(M, M)$. In the above picture, this corresponds to a family $U \to M$ over M for the moduli problem F. By (a slightly stronger version of) Yoneda's lemma, this family has the following strong universal property: for any base scheme T and any family $U_T \to T$ in $F(T)$, there exists a morphism $T \to M$ and a pullback square

$$\begin{tikzcd} U_T \arrow[r] \arrow[d] & U \arrow[d] \\ T \arrow[r] & M. \end{tikzcd}$$

When $T = \mathrm{Spec}\, k$, we get a bijection between the set of isomorphism classes of objects over k and the k points $M(k)$ of our moduli space. More generally, we have a bijection between families of objects over T and morphisms $T \to M$ given by pulling back the universal family $U \to M$. In some sense, all the geometry of all families of objects at hand are captured by the geometry of the universal family $U \to M$ over the moduli space M.

Often fine moduli spaces don't exist, but we have the following slightly weaker notion.

Definition 2.4. *A scheme M and a natural transformation $\xi : F \to h_M$ is a coarse moduli space if*

(a) $\xi(k) : F(\mathrm{Spec}\, k) \to h_M(\mathrm{Spec}\, k)$ is a bijection for all algebraically closed fields k, and

(b) for any scheme M' and any natural transformation $\xi' : F \to h_{M'}$, there exists a unique morphism $\alpha : M \to M'$ such that ξ' factors as $h_\alpha \circ \xi$.

We can think of a coarse moduli space as the initial scheme whose closed points correspond to objects of our moduli problem. However, coarse moduli spaces need not have universal families. It is clear from the definition that a fine moduli space for F is a coarse moduli space.

Example 2.5. *The global sections functor $Sch_S \to Set$ given by $X \mapsto \mathcal{O}_X(X)$ is representable by $\mathbb{A}^1_S$. The universal global section is $x \in \mathcal{O}_S[x]$ where $\mathbb{A}^1_S = \mathrm{Spec}_S\, \mathcal{O}_S[x]$.*

Example 2.6. *The scheme $\mathbb{P}^n_S$ represents the following functor on the category Sch_S.*

$$X \mapsto \{(L, s_0, \ldots, s_n) \ : \text{satisfying condition (*)}\} / \sim .$$

Here L is a line bundle, $s_i \in H^0(X, L)$ are global sections of L, and condition $(*)$ is that for each $x \in X$, there exists an i such that $s_i(x) \neq 0$. Two such data $(L, s_0, \ldots, s_n)$ and $(L', s'_0, \ldots, s'_n)$ are equivalent if there exists an isomorphism of line bundles

$$\alpha : L \to L'$$

with $\alpha(s_i) = s'_i$. Here the universal line bundle with sections on $\mathbb{P}^n$ is given by $(\mathcal{O}_{\mathbb{P}^n}(1), x_0, \ldots, x_n)$. Another way to write condition $(*)$ is that the map of sheaves

$$\mathcal{O}_X^{n+1} \to L$$

induced by the s_i is surjective.

2.1.1 Criteria for representability

Recall that a presheaf F on Sch_S is a (Zariski) *sheaf* if for any X and any Zariski open cover $\{U_i \to X\}$ the following diagram is an equalizer.

$$F(X) \to \prod_i F(U_i) \rightrightarrows F(U_i \cap U_j)$$

Proposition 2.7. *Representable functors are sheaves for the Zariski topology.*

Proof. We need to check that for any scheme X, $h_X = \mathrm{Hom}_S(-, X)$ is a sheaf. This follows from the fact that we can glue morphisms. □

This gives us our first criterion for ruling out representability of a functor. In particular, given a candidate moduli functor, we had better sheafify it to have any hope of representable.

The following is a useful property of the category of presheaves.

Lemma 2.8. *The category $Fun(\mathcal{C}^{op}, Set)$ is closed under limits and colimits. Furthermore, the Yoneda functor $h\_$ preserves limits.*[8]

Definition 2.9. *(a) We say that a subfunctor F of a functor G is open (respectively closed) if and only if for any scheme T and any morphism $T \to G$, the pullback $T \times_G F$ is representable by an open (respectively closed) subscheme of T.*

(b) We say that a collection of open subfunctors F_i of F is an open cover of F if for any scheme T and any morphism $T \to F$, the pullbacks $\{U_i := T \times_F F_i \to T\}$ form an open cover of T.

We can rephrase the above definitions using the moduli functor language as follows. An open (resp. closed) subfunctor $F \subset G$ is one such that for any family $\xi \in G(T)$, there is an open set $U \subset T$ (resp. closed subset $Z \subset T$) such that a morphism $f : T' \to T$ factors through U (resp. Z) if and only if $f^*\xi \in F(T')$. Similarly, a collection of open subfunctors $\{F_i \subset F\}$ form an open cover if for any $\xi \in F(T)$, there exists an open cover $\{U_i \to T\}$ such that $\xi|_{U_i} \in F_i(U_i)$.

Proposition 2.10. *Let $F \in Fun(Sch_S^{op}, Set)$ be a functor such that*

[8] Note it does not in general preserve colimits.

(a) F is a Zariski sheaf, and

(b) F has an open cover $\{F_i\}$ by representable open subfunctors.

Then F is representable by a scheme.

Proof. Let X_i be the scheme representing F_i with universal object $\xi_i \in F_i(X_i)$. We can consider the pullback

$$\begin{tikzcd} U_{ij} \arrow[r] \arrow[d] & X_i \arrow[d] \\ F_j \arrow[r] & F \end{tikzcd}$$

where $U_{ij} \subset X_i$ is an open immersion since F_j is an open subfunctor. Furthermore, we have an equality

$$\xi_i|_{U_{ij}} = \xi_j|_{U_{ij}}$$

by commutativity of the pullback diagram. This induces an isomorphism $\varphi_{ij} : U_{ij} \to U_{ji}$ such that

$$\varphi_{ij}^* \xi_j = \xi_i|_{U_{ij}}.$$

Now we want to construct a scheme X along with an object $\xi \in F(X)$ by gluing the schemes X_i along the open subsets U_{ij} using the isomorphisms φ_{ij}. We need to check that the isomorphisms $\varphi_{ij} : U_{ij} \cong U_{ji}$ satisfy the cocycle condition. To make sense of this, we first want to know that φ_{ij} identifies

$$U_{ij} \cap U_{ik} \cong U_{ji} \cap U_{jk}.$$

This follows since the left side (resp. the right side) is characterized by the fact that $\xi_i|_{U_{ij} \cap U_{ik}} \in F_k(U_{ij} \cap U_{ik})$ (resp. $\xi_j|_{U_{ji} \cap U_{jk}} \in F_k(U_{ji} \cap U_{jk})$) and $\varphi_{ij}^* \xi_j = \xi_i$.

Now it makes sense to require that

$$\varphi_{jk}|_{U_{ji} \cap U_{jk}} \circ \varphi_{ij}|_{U_{ij} \cap U_{ik}} = \varphi_{ik}|_{U_{ik} \cap U_{ij}}$$

as maps $U_{ij} \cap U_{ik} \to U_{ki} \cap U_{kj}$. This follows since both maps pullback ξ_k to ξ_i.

Now we can glue the X_i along the open subsets U_{ij} using the isomorphisms φ_{ij} to obtain a scheme X. Moreover, the universal objects ξ_i over X_i are identified on the overlaps U_{ij} and so since F is a Zariski sheaf, the ξ_i glue to form a $\xi \in F(X)$ induced by a morphism $X \to F$.

Now we need to show that (X, ξ) represents F. Let T be a scheme with a morphism $T \to F$ induced by an object $\zeta \in F(T)$. Since F_i form an open cover, there exists an open cover U_i of T such that $\zeta|_{U_i} =: \zeta_i$ defines a morphism $U_i \to X_i$. Moreover,

$$\zeta_i|_{U_i \cap U_j} = \zeta_j|_{U_i \cap U_j}$$

so the morphisms $U_i \to X_i$ glue to give a morphism $f : T \to X$ with $f^* \xi = \zeta$.

$\square$

2.2 Grassmannians

Definition 2.11. *For any k, n, let $Gr(k,n)$ denote the functor $Sch \to Set$ given by*

$$S \mapsto \{\alpha : \mathcal{O}_S^{\oplus n} \twoheadrightarrow \mathcal{V}\}/\sim$$

where α is a surjection, $\mathcal{V}$ is a rank k locally free sheaf, and $\sim$ is given by isomorphism $\mathcal{E} \cong \mathcal{E}'$ commuting with the surjections α and α'.

We will use the above representability criteria to construct a scheme representing $Gr(k,n)$. Note that when $k = 1$, we recover the functor represented by $\mathbb{P}^{n-1}$ as above.

Remark 2.12. *Let $\mathcal{E} = \ker(\alpha : \mathcal{O}_S^{\oplus n} \to \mathcal{V})$. Since $\mathcal{V}$ is locally free and the sequence*

$$0 \to \mathcal{E} \to \mathcal{O}_S^{\oplus n} \to 0$$

is exact, then for each $x \in S$, we have

$$0 \to \mathcal{E}|_x \to k(x)^n \to \mathcal{V}_x \to 0$$

is exact. Thus $\mathcal{E}|_x$ is an $n-k$-dimensional subspace of $k(x)^n$ for each $x \in S$. In particular $\mathcal{E}$ is a rank $n-k$ locally free sheaf on S and we can think of the inclusion of $\mathcal{E} \to \mathcal{O}_S \otimes V$ as a family of rank $n-k$ subspaces of an n-dimensional vector space parametrized by the scheme S. More precisely, this identifies the Grassmannian functor with the functor

$$S \mapsto \{\textit{rank } n-k \textit{ sub-bundles of } \mathcal{O}_S^n\}.$$

Let us give some a sketch of the construction over a field that we will make more precise later. When S is the spectrum of an algebraically closed field, $\mathcal{V}$ is just the trivial bundle and so a map $\alpha : \mathcal{O}_S^{\oplus n} \to \mathcal{O}_S^{\oplus k}$ is given by a $k \times n$ matrix. The condition that α is surjective is that the $k \times k$ minors don't all vanish. Finally, isomorphism is given by the action of GL_k on the left.

Thus, set theoretically, the set of closed points of the Grassmannian is the quotient set U/GL_k where $U \subset \mathbb{A}^{nk}$ is the open subset of the space of $k \times n$ matrices of full rank.

To give it the structure of a variety over a field, we note that for each subset $i \subset \{1, \ldots, n\}$ of size k, we can consider the set of full rank $k \times n$ matrices where the i^{th} minor doesn't vanish. Then using the GL_k action, put such a matrix into a form where the i^{th} minor is the identity matrix. E.g. if $i = \{1, \ldots, k\}$ then we act by GL_k so our matrix looks like

$$\begin{bmatrix} 1 & & & & a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n} \\ & 1 & & & \vdots & \ddots & & \vdots \\ & & \ddots & & \vdots & & \ddots & \vdots \\ & & & 1 & a_{k,k+1} & a_{k,k+2} & \cdots & a_{k,n} \end{bmatrix}$$

This identifies GL_k orbits of such matrices with an affine space $\mathbb{A}^{k(n-k)}$ and we can glue these affine spaces together by changing basis. This gives $Gr(k,n)$ the structure of an affine variety.

We will upgrade the above construction to obtain a proof over a general base scheme using the above representability criterion.

Theorem 2.13. *$Gr(k, n)$ is representable by a finite type scheme over* Spec $\mathbb{Z}$.

We will prove this next time using the representability criterion above. For each subset $i \subset \{1, \dots, n\}$ of size k, we will define a subfunctor F_i of the Grassmannian functor as follows. First, let

$$s_i : \mathcal{O}_S^k \to \mathcal{O}_S^n$$

denote the the inclusion where the j^{th} direct summand is mapped by the identity to the i_j^{th} direct summand. Now let F_i be defined as the subfunctor

$$F_i(S) = \{\alpha : \mathcal{O}_S^n \to \mathcal{V} \mid \alpha \circ s_i \text{ is surjective } \} \subset Gr(k, n)(S).$$

3 Grassmannians (cont.) and flat morphisms

3.1 Constructing Grassmannians

Recall we aim to prove the following:

Theorem 3.1. *$Gr(k, n)$ is representable by a finite type scheme over* Spec $\mathbb{Z}$.

Proof. We will use the representability criterion from Lecture 2. It is clear $Gr(k, n)$ is a sheaf by gluing locally free sheaves.

For each subset $i \subset \{1, \dots, n\}$ of size k, we will define a subfunctor F_i of the Grassmannian functor. First, let

$$s_i : \mathcal{O}_S^k \to \mathcal{O}_S^n$$

denote the the inclusion where the j^{th} direct summand is mapped by the identity to the i_j^{th} direct summand. Now let F_i be defined as the subfunctor

$$F_i(S) = \{\alpha : \mathcal{O}_S^n \to \mathcal{V} \mid \alpha \circ s_i \text{ is surjective } \} \subset Gr(k, n)(S).$$

Note this is a functor since for any $f : T \to S$, f^* is right-exact.

We need to show that each F_i is representable and that the collection $\{F_i\}$ is an open cover of the functor $Gr(k, n)$.

For any scheme S and any map $S \to Gr(k, n)$ corresponding to the object $(\alpha : \mathcal{O}_S^n \to V) \in F(S)$, we have a natural morphism of finite type quasi-coherent sheaves $\alpha \circ s_i : \mathcal{O}_S^n \to \mathcal{V}$. Now let $\mathcal{K}$ be the cokernel of $\alpha \circ s_i$. Then $\alpha \circ s_i$ is surjective at a point $x \in S$ if and only if $\mathcal{K}_x = 0$ if and only if $x \notin \text{Supp}(\mathcal{K}_x)$. Since $\text{Supp}(\mathcal{K}_x)$ is closed, the set U_i where $\alpha \circ s_i$ is surjective is open.

We need to show that for any other scheme T and a morphism $f : T \to S$, f factors through U_i if and only if

$$f^*(\alpha : \mathcal{O}_S^n \to \mathcal{V}) = (f^*\alpha : \mathcal{O}_T^n \to f^*\mathcal{V}) \in F_i(T)$$

. Suppose $t \in T$ maps to $x \in S$. By Nakayama's lemma, $x \in U_i$ if and only if

$$(\alpha \circ s_i)|_x : k(x)^k \to \mathcal{V}_x / \mathfrak{m}_x \mathcal{V}_x$$

is surjective. Now the stalk $(f^*\mathcal{V})_t$ is given by the pullback $\mathcal{V}_x \otimes_{\mathcal{O}_{S,x}} \mathcal{O}_{T,t}$ along the local ring homomorphism $f^* : \mathcal{O}_{S,x} \to \mathcal{O}_{T,t}$. Thus the map on fibers

$$(f^*\alpha \circ f^* s_i)|_t : k(t)^k \to (f^*\mathcal{V}_t) / \mathfrak{m}_t \mathcal{V}_t$$

is the pullback of $(\alpha \circ s_i)|_x$ by the residue field extension $k(x) \subset k(t)$. In particular, one is surjective if and only if the other is so $f : T \to S$ factors through U_i if and only if $(f^*\alpha \circ f^*s_i)|_t$ is surjective for all $t \in T$ if and only if $f^*(\alpha : \mathcal{O}_S^k \to \mathcal{V}) \in F_i(T)$. This proves that the F_i are open subfunctors.

Next, we need to know that the collection $\{F_i\}$ covers F. This amounts to showing that for any $S \to Gr(k,n)$ as above and any $s \in S$, there exists an i such that $s \in U_i$. As above, by Nakayama's lemma we may check surjectivity on fibers, thus we need to show that for each $s \in S$, there exists an i such that the composition

$$k(s)^k \xrightarrow{s_i} k(s)^n \xrightarrow{\alpha} \mathcal{V}_s/\mathfrak{m}_s\mathcal{V}_s$$

but this is clear from linear algebra.

Finally, we need to show that the F_i are representable. Given $(\alpha : \mathcal{O}_S^n \to \mathcal{V}) \in F_i(S)$, the composition $\alpha \circ s_i : \mathcal{O}_S^k \to \mathcal{V}$ is a surjection between finite locally free modules of the same rank. Then we apply the following lemma from commutative algebra.

Lemma 3.2. *Let R be a ring and M be a finite R-module. Let $\varphi : M \to M$ be a surjective R-module map. Then φ is an isomorphism.*

By applying the lemma to an open cover of S where $\mathcal{V}$ is trivial, we get that $\alpha \circ s_i$ is actually an isomorphism. In particular, we obtain a splitting of α so that α is determined by its restriction to the complimentary $n-k$ components of $\mathcal{O}_S^n$. The restriction of α to each component is a map $\mathcal{O}_S \to \mathcal{O}_S^k$ which is the same as a k-tuple of sections. Thus, the functor F_i is isomorphic to the functor of a $(n-k)$ many k-tuples of sections, i.e. to $\mathbb{A}_{\mathbb{Z}}^{k(n-k)}$.

□

3.1.1 Some properties of the Grassmannian

Next we want to use the functorial point of view to show that $Gr(k,n)$ is in fact a projective variety. First we review the valuative criterion of properness.

Theorem 3.3. *Suppose $f : X \to Y$ is a locally of finite type morphism of schemes with Y locally Noetherian. Then the following are equivalent:*

(a) f is separated (resp. universally closed, resp. proper),

(b) for any solid commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where R is a DVR with fraction field K, any dashed arrow is unique (resp. there exists a dashed arrow, resp. there exists a unique dashed arrow).

The nice thing about this is that condition (b) can be phrased completely in terms of the functor of points: for any DVR Spec $R \to Y$ over Y with fraction field K, the map of sets

$$\mathrm{Hom}_Y(\operatorname{Spec} R, X) \to \mathrm{Hom}_Y(\operatorname{Spec} K, X)$$

is injective (resp. surjective, resp. bijective). This is useful to prove properness of moduli spaces directly from the moduli functor. It amounts to saying that any family over Spec K can be uniquely "filled in" over the the closed of point of Spec R to a family over Spec R

Proposition 3.4. *$Gr(k,n)$ is proper over* Spec $\mathbb{Z}$*.*

Proof. By construction, $Gr(k,n)$ has a finite cover by finite dimensional affine spaces over Spec $\mathbb{Z}$ so it is of finite type of over Spec $\mathbb{Z}$. By the valuative criterion we need to check that we can uniquely fill in the dashed arrow in the following diagram.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Gr(k,n) \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

The top morphism is a surjection of K-vector spaces

$$K^n \to V$$

where V has rank k. We have a natural inclusion of R-modules $R^n \to K^n$. We need to find a locally free rank k module M over R with a surjection $R^n \to M$ such that the following diagram commutes.

$$\begin{tikzcd} K^n \arrow[r] & V \\ R^n \arrow[r] \arrow[u, hook] & M \arrow[u, hook] \end{tikzcd}$$

Then we can just take M to be the image of $R^n \subset K^n \to V$. Since V is a vector space over K, it is torsion free as an R-module so the finitely generated M is free. By construction $M \otimes_R K = V$ so M has rank k and M is clearly unique. $\square$

Next we will show that $Gr(k,n)$ is in fact projective over Spec $\mathbb{Z}$!. To do this we will use the following criterion:

Proposition 3.5. *Suppose $f : X \to Y$ is a proper monomorphism of schemes. Then f is a closed embedding.*

Proposition 3.6. *$Gr(k,n)$ is smooth and projective over* Spec $\mathbb{Z}$*.*

Proof. We will show that $Gr(k,n)$ is a subfunctor of projective space. Then combining the previous two propositions, it follows that $Gr(k,n)$ is a closed subscheme of projective space. For smoothness, we saw in the construction that $Gr(k,n)$ is covered by open subsets isomorphic to $\mathbb{A}^{k(n-k)}$, so it is smooth.

We define a natural transformation of functors $Gr(k,n) \to \mathbb{P}^N_{\mathbb{Z}}$ where $N = \binom{n}{k} - 1$. For any scheme S and any S-point $(\alpha : \mathcal{O}_S^n \twoheadrightarrow \mathcal{V}) \in Gr(k,n)(S)$, we consider the induced map on the k^{th} alternating power.

$$\Lambda^k \alpha : \Lambda^k \mathcal{O}_S^n = \mathcal{O}_S^{\binom{n}{k}} \to \Lambda^k \mathcal{V}$$

Since α is surjective, so is $\Lambda^k \alpha$. Moreover, $\Lambda^k \mathcal{V}$ is a line bundle since the rank $\mathcal{V}$ is k. Finally, Λ^k commutes with pullbacks. Thus $(\Lambda^k \alpha : \mathcal{O}_S^{\binom{n}{k}} \to \Lambda^k \mathcal{V}) \in \mathbb{P}^N_{\mathbb{Z}}(S)$ is an S point of $\mathbb{P}^N_{\mathbb{Z}}$.

We need to show that the natural transformation $Gr(k,n) \to \mathbb{P}^N_{\mathbb{Z}}$ is a subfunctor. We will do this by restricting to the open subfunctors $F_i \subset Gr(k,n)$ described in the proof that $Gr(k,n)$ is representable. Let G_i be the corresponding open subfunctors of $\mathbb{P}^N$ (i.e. the subfunctor where the i^{th} section is nonvanishing, or equivalently where the composition $\mathcal{O}_S \to \mathcal{O}_S^{N+1} \to L$ with the i^{th} copy is surjective).

It is clear that the natural transformation above maps F_i to G_i. Thus it suffices to show that F_i is a subfunctor of G_i. Now F_i is the functor corresponding to $k \times n$ matrices of global sections of $\mathcal{O}_S$ where the columns indexed by the subset i form the identity matrix, and the natural transforation $F_i \to G_i$ is given by taking $k \times k$ minors of this matrix. Now it is an exercise in linear algebra to see that the minors of such a matrix uniquely determine the matrix. Thus $F_i \to G_i$ is a subfunctor so $Gr(k,n) \to \mathbb{P}^N$ is a monomorphism. $\square$

Over the Grassmannian we have the universal quotient

$$\mathcal{O}^n_{Gr(k,n)} \to \mathcal{Q}.$$

The above proof shows in fact that $\Lambda^k \mathcal{Q} =: \det \mathcal{Q}$ is a very ample line bundle on $Gr(k,n)$ which induces the closed embedding into projective space.

3.1.2 Relative Grassmannians

For any scheme S, we can basechange from Spec $\mathbb{Z}$ to S to get the scheme representing the Grassmannian functor $Sch_S \to Set$. More generally, if $\mathcal{E}$ is any rank n vector bundle on a scheme S, we can define for any $k < n$, the Grassmannian of $\mathcal{E}$ as the functor $Sch_S \to Set$ given by

$$Gr_S(k,\mathcal{E})(f : T \to S) = \{\alpha : f^*\mathcal{E} \twoheadrightarrow \mathcal{V} \mid \mathcal{V} \text{ is a rank } k \text{ locally free sheaf}\}$$

Theorem 3.7. *$Gr_S(k,\mathcal{E})$ is representable by a smooth projective scheme over S.*

Proof. We leave the details to the reader, but the idea is to cover S by open subsets where $\mathcal{E}$ is locally free. Over these subsets the Grassmannian functor is representable by the argument above. Then we may glue these schemes together. The properties of being proper and being a monomorphism are both local on the base and compatible with base change. Putting this all together we get a scheme representing $Gr_S(k,\mathcal{E})$ as well as a closed embedding into $\mathbb{P}_S(\Lambda^k\mathcal{E})$. $\square$

Remark 3.8. *(A note on projectivity) Let $f : X \to S$ be a morphism of schemes. There are several notions of projectivity for f.*

(a) there is a closed embedding of X into $\mathrm{Proj}_S \mathrm{Sym}^*_{\mathcal{O}_S} \mathcal{F}$ for some coherent sheaf $\mathcal{F}$ on S;

(b) there is a closed embedding of X into $\mathbb{P}_S(\mathcal{E})$ for some finite rank locally free sheaf $\mathcal{E}$ on S;

(c) there is a closed embedding of X into $\mathbb{P}^n_S$.

For each notion of projectivity we can define quasi-projective morphisms as those which factor through an open embedding into a projective one. The above theorem shows that relative Grassmannians $Gr_S(k,\mathcal{E})$ are projective over S in the sense of (b). We have implications $(c) \implies (b) \implies (a)$ but these notions are not equivalent in general. If S is Noetherian and satisfies the resolution property[9] then $(a) \implies (b)$ and if furthermore S admits an ample line bundle, then $(b) \implies (c)$. For simplicity we will usually use projective to mean (c).

[9] That is, if every coherent sheaf admits a surjection from a finite rank locally free sheaf.

3.2 Recollections on flaness

We want to move on to moduli spaces of general varieties. To do this we need a good notion of continuously varying family of varieties (or schemes, or sheaves) parametrized by a base scheme S. It turns out the right notion is that of *flatness*.

Definition 3.9. *Let $f : X \to S$ be a morphism of schemes and $\mathcal{F}$ a quasi-coherent sheaf on X. We say that $\mathcal{F}$ is flat over S at $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module. We say that $\mathcal{F}$ is flat over S if it is flat at every point $x \in X$. We say that the morphism f is flat if $\mathcal{O}_X$ is flat over S.*

We recall some basic facts about flatness.

Proposition 3.10.

1. *The property of being flat is stable under base change and compositions;*
2. *localizations of flat modules are flat so in particular open embeddings are flat;*
3. *if R is a PID, then an R-module M is flat if and only if it is torsion free;*
4. *if $t \in \mathcal{O}_{S,f(x)}$ is a non-zero divisor, then $f^*t \in \mathcal{O}_{X,x}$ is a non-zero divisor;*
5. *if S is Noetherian and f is finite then $\mathcal{F}$ is flat if and only if $f_*\mathcal{F}$ is locally free of finite rank.*

Recall that an associated point of a scheme X is a point $x \in X$ so that the corresponding prime ideal $\mathfrak{m}_x$ is generated by zero divisors. If X is reduced, then these are just the generic points of irreducible components of X.

Proposition 3.11. *Let $f : X \to C$ be a morphism of schemes with C an integral regular scheme of dimension 1. Then f is flat if and only if it maps all associated points of X to the generic point of C.*

4 Flat morphisms and Hilbert polynomials

4.1 More on flat morphisms

Last time we left off with the following statement.

Proposition 4.1. *Let $f : X \to Y$ be a morphism of schemes with Y an integral regular scheme of dimension 1. Then f is flat if and only if it maps all associated points of X to the generic point of Y.*

Proof. Suppose f is flat and take $x \in Y$ with $f(x) = y$ a closed point. Then $\mathcal{O}_{Y,y}$ is a DVR with uniformizing parameter $t_y \in \mathfrak{m}_y$. Since t_y is a non-zero divisor, $f^*t_y \in \mathfrak{m}_x$ is a non-zero divisor so x is not associated.

Conversely, if f is not flat, there is some $x \in X$ with $y = f(x)$ a closed point and $\mathcal{O}_{X,x}$ is not a flat $\mathcal{O}_{Y,y}$ module. Since $\mathcal{O}_{Y,y}$ is a DVR, this means $\mathcal{O}_{X,x}$ is not torsion free so f^*t_y is a zero divisor which must be contained in some associated prime mapping to y. $\square$

Corollary 4.2. *Let $X \to Y$ as above. Then f is flat if and only if for each $y \in Y$, the scheme theoretic closure of $X \setminus X_y$ inside X is equal to X.*

The slogan to take away from the above corollary is that flat morphisms over a smooth curve are continuous in the following sense:

$$\lim_{y \to y_0} X_y = X_{y_0}$$

for each point $y_0 \in C$.

Corollary 4.3. *Let Y be as above and $y \in Y$. Suppose $X \subset \mathbb{P}^n_{Y \setminus y}$ is flat. Then there exists a unique subscheme $\overline{X} \subset \mathbb{P}^n_Y$ such that $\overline{X} \to Y$ is flat.*

In particular, the functor of flat subschemes of a projective scheme satisfies the valuative criterion of properness!

Example 4.4. *Consider the subscheme $X \subset \mathbb{P}^3_{\mathbb{A}^1_a \setminus 0}$ defined by the ideal*

$$I = (a^2(xw + w^2) - z^2, ax(x+w) - yzw, xz - ayw).$$

For each $a \neq 0$, this is the ideal of the twisted cubic which is the image of the morphism

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^3 \\ [s,t] &\mapsto [t^2 s - s^3, t^3 - ts^2, ats^2, s^3]. \end{aligned}$$

By the above Corollary, we can compute the flat limit

$$\lim_{a \to 0} X_a = \overline{X}_0$$

by computing the closure $\overline{X}$ of X in $\mathbb{P}^3_{\mathbb{A}^1}$. We can do this by taking $a \to 0$ in the ideal I but we have to be careful! Note that the polynomial

$$y^2 w - x^2(x+w)$$

is contained in the ideal I. In fact

$$I/aI = (z^2, yz, xz, y^2 w - x^2(x+w))$$

which gives the flat limit of this family of twisted cubics. Note that set theoretically this is a nodal cubic curve in the $z = 0$ plane but at $[0,0,0,1]$ it has an embedded point that "sticks out" of the plane.

The following is an interesting characterization of flatness over a reduced base.

Theorem 4.5 (somewhere in ega)**.** *(Valuative criterion for flatness) Let $f : X \to S$ be a locally of finite presentation morphism over a reduced Noetherian scheme S. Then f is flat at $x \in X$ if and only if for each DVR R and morphism* Spec $R \to S$ *sending the closed point of* Spec R *to $f(s)$, the pullback of f to* Spec R *is flat at all points lying over x.*

We will see a proof of this in the projective case soon.

Proposition 4.6. *Let $f : X \to Y$ be a flat morphism of finite type and suppose Y is locally Noetherian and locally finite-dimensional. Then for each $x \in X$ an $y = f(x)$,*

$$\dim_x(X_y) = \dim_x(X) - \dim_y(Y).$$

Proof. It suffices to check after base change to Spec $\mathcal{O}_{Y,y}$ so suppose Y is the spectrum of a finite dimensional local ring. We will induct on the dimension of Y. If $\dim(Y) = 0$, then $X_y = X_{red}$ so there is nothing to check. If $\dim(Y) > 0$, then there is some non-zero divisor $t \in \mathfrak{m}_y \subset \mathcal{O}_{Y,y}$ so that $f^*t \in \mathfrak{m}_x$ is a non-zero divisor. Then the induced map $X' = \text{Spec } \mathcal{O}_{X,x}/f^*t \to Y' = \text{Spec } \mathcal{O}_{Y,y}/t$ is flat, $\dim(X') = \dim_x(X) - 1$, $\dim(Y') = \dim(Y) - 1$, and the result follows by induction. □

Corollary 4.7. *If X and Y are integral k-schemes, then $n = \dim(X_y)$ is constant for $y \in \text{im}(f)$ and $\dim(X) = n + \dim(Y)$.*

4.2 Hilbert polynomials

Let $X \subset \mathrm{P}^n_k$ be a projective variety over a field k. Recall that the *Hilbert polynomial* of a coherent sheaf $\mathcal{F}$ on X may be defined as

$$P_{\mathcal{F}}(d) := \chi(X, \mathcal{F}(d)) := \sum_{i=0}^{n} (-1)^i h^i(X, \mathcal{F}(d))^{10}$$

where $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes d}$. By the Serre vanishing theorem,

$$\chi(X, \mathcal{F}(d)) = \dim H^0(X, \mathcal{F}(d))$$

for $n \gg 0$. When $\mathcal{F} = \mathcal{O}_X$, then we call $P_X(d) := P_{\mathcal{O}_X}(d)$ the Hilbert polynomial of X.

We have the following important theorem.

Theorem 4.8. *Let $f : X \to Y$ be a projective morphism over a locally Noetherian scheme Y. If $\mathcal{F}$ is a coherent sheaf on X which is flat over Y, then the Hilbert polynomial $P_{\mathcal{F}|_{X_y}}(d)$ is locally constant for $y \in Y$. If Y is reduced, then the converse holds.*

Proof. By pulling back along the inclusion Spec $\mathcal{O}_{Y,y} \to Y$, we may assume that $Y = \text{Spec } A$ is the spectrum of a Noetherian local ring. Moreover, by considering the pushforward $i_*\mathcal{F}$ under the map $i : X \hookrightarrow \mathbb{P}^n_Y$, we may assume that $X = \mathbb{P}^n_Y$. We have the following lemma:

Lemma 4.9. *$\mathcal{F}$ is flat over Y if and only if $H^0(X, \mathcal{F}(d))$ is a finite free A-module for $d \gg 0$.*

Proof. $\Longrightarrow$: Let $\mathcal{U} = \{U_i\}$ be an affine open covering of X and consider the Čech complex

$$0 \to H^0(X, \mathcal{F}(d)) \to C^0(\mathcal{U}, \mathcal{F}(d)) \to C^1(\mathcal{U}, \mathcal{F}(d)) \to \ldots \to C^n(\mathcal{U}, \mathcal{F}(d)) \to 0.$$

By Serre vanishing, this sequence is exact for $d \gg 0$. Since $\mathcal{F}$ is flat, each term $C^i(\mathcal{U}, \mathcal{F}(d))$ is a flat finitely generated A-module. We repeatedly apply the following fact: if $0 \to A \to B \to C \to 0$ is exact and B and C are flat, then A is flat. It follows that $H^0(X, \mathcal{F}(d))$ is a finitely generated flat module over the local ring A, and in particular, is free.

$\Longleftarrow$: Suppose that d_0 is such that $H^0(X, \mathcal{F}(d))$ is finite and free for $d \geq d_0$ and consider the $S = A[x_0, \ldots, x_n]$ module

$$M = \bigoplus_{d \geq d_0} H^0(X, \mathcal{F}(d)).$$

[10] It is not a priori clear that this is a polynomial n. To prove this, one can induct on the dimension of X and use the additivity of Euler characteristics under short exact sequences.

Now M is A-flat since it's a direct sum of flat modules. Furthermore, M defines a quasicoherent sheaf $\tilde{M}$ on X which is just $\mathcal{F}$ itself. Explicitly, $\tilde{M}$ is obtained by gluing together the degree 0 parts of the localizations of M by each x_i. Since flatness is preserved by localization and direct summands of flat modules are flat, we conclude that $\tilde{M} = \mathcal{F}$ is flat. □

Now the first part of the theorem would follow if we know that the rank of the A-module $H^0(X, \mathcal{F}(d))$ equals $P_{\mathcal{F}|_{X_y}}(d)$. This is implied by the following equality base change statement.

$$H^0(X, \mathcal{F}(d)) \otimes_A k(y) = H^0(X_y, \mathcal{F}(d)|_{X_y}) \tag{3}$$

Remark 4.10. *One can rewrite equality 3 as saying the natural map*

$$u^* f_*(\mathcal{F}(d)) \to f'_* u'^*(\mathcal{F}(d))$$

where

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd} \tag{4}$$

is the Cartesian diagram with $i : Y' = \text{Spec } k(y) \to Y$ the inclusion. More generally, given any Cartesian diagram as above and any quasicoherent sheaf $\mathcal{F}$ on X, there are natural maps

$$u^* R^i f_*(\mathcal{F}) \to R^i f'_*(u'^* \mathcal{F}).$$

One can ask more generally if this map is an isomorphism, and if it is we say that base change holds (for this diagram, this sheaf, and this i), or that the i^{th} cohomology of $\mathcal{F}$ commutes with base change by u. We highlight this here since this situation will come up again.

Suppose first that $y \in Y$ is a closed point. Then consider a resolution of $k(y)$ of the form

$$A^m \to A \to k(y) \to 0. \tag{5}$$

Pulling back and tensoring with $\mathcal{F}$ we get a resolution

$$\mathcal{F}^m \to \mathcal{F} \to \mathcal{F}|_{X_y} \to 0.$$

For $d \gg 0$ and by Serre vanishing, the sequence

$$H^0(X, \mathcal{F}(d)^{\oplus m}) \to H^0(X, \mathcal{F}(d)) \to H^0(X_y, \mathcal{F}(d)|_{X_y}) \to 0$$

is exact. On the other hand, we can tensor sequence 5 by the A-module $H^0(X, \mathcal{F}(d))$ to get an exact sequence

$$H^0(X, \mathcal{F}(d))^{\oplus m} \to H^0(X, \mathcal{F}(d)) \to H^0(X, \mathcal{F}(d)) \otimes_A k(y) \to 0.$$

Comparing the two yields the required base change isomorphism. Now if y is not a closed point of Y, we can consider the Cartesian diagram as in 4 where $Y' = \text{Spec } \mathcal{O}_{Y,y}$. Then u is flat and y is a closed point of Y' and we can reduce to this case by applying the following.

Lemma 4.11. *(Flat base change) Consider the diagram*

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd} \tag{6}$$

where f is qcqs [11] *and u is flat and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then the base change morphism*

$$u^* R^i f_*(\mathcal{F}) \to R^i f'_*(u'^* \mathcal{F}).$$

is an isomorphism for all $i \geq 0$.

Now when Y is a reduced local ring, a module M is free if and only if $\dim M_y$ is independent of y for each $y \in Y$ so using the now proven base change isomorphism

$$H^0(X, \mathcal{F}(d)) \otimes_A k(y) = H^0(X_y, \mathcal{F}(d)|_{X_y})$$

we obtain that $H^0(X, \mathcal{F}(d))$ is a finite free A-module if and only if $P_{\mathcal{F}|_{X_y}}(d)$ is independent of $y \in Y$.

□

Remark 4.12. *As a corollary, we obtain the valuative criterion for flatness in the case of a projective morphism since the constancy of the Hilbert polynomial can be checked after pulling back to a regular curve.*

Remark 4.13. *The Hilbert polynomial encodes a lot of geometric information about a projective variety X such as the dimension, degree of projective dimension, and arithmetic genus. In particular, these invariants are constant in projective flat families.*

5 Base change, the Hilbert functor

5.1 Remarks on base change

Last time we proved the constancy of Hilbert polynomials in projective flat families:

Theorem 5.1. *Let $f : X \to Y$ be a projective morphism over a locally Noetherian scheme Y. If $\mathcal{F}$ is a coherent sheaf on X which is flat over Y, then the Hilbert polynomial $P_{\mathcal{F}|_{X_y}}(d)$ is locally constant for $y \in Y$. If Y is reduced, then the converse holds.*

In the process we proved the lemma that when $Y = \operatorname{Spec} A$ is the spectrum of a Noetherian local ring, then $\mathcal{F}$ is flat if and only if $H^0(X, \mathcal{F}(d))$ is a finite free A-module for $d \gg 0$. Note that this statement immediately globalizes:

Corollary 5.2. *Let $f : X \to Y$ and $\mathcal{F}$ be as above with Y Noetherian. Then $\mathcal{F}$ is flat over Y if and only if $f_*\mathcal{F}(d)$ is a finite rank locally free sheaf for all $d \gg 0$.*

[11] quasi-compact quasi-separated

Then we had to use two base change results. Namely we needed to show the following isomorphism (still in the local case $Y = \text{Spec } A$):

$$H^0(X, \mathcal{F}(d)) \otimes_A k(y) \cong H^0(X_y, \mathcal{F}(d)_y) \tag{7}$$

for all $y \in Y$ and $d \gg 0$. In proving (7) we needed the following flat base change.

Lemma 5.3. *(Flat base change) Consider the diagram*

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd} \tag{8}$$

where f is qcqs [12] *and u is flat and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then the base change morphism*

$$u^* R^i f_*(\mathcal{F}) \to R^i f'_*(u'^* \mathcal{F}).$$

is an isomorphism for all $i \geq 0$.

Proof. (Sketch) The question is local on Y and Y' so we can assume that $Y = \text{Spec } A$ and $Y' = \text{Spec } B$ where B is a flat A-algebra. Then the higher direct image functors are just taking cohomology so the statement becomes that the natural map

$$H^i(X, \mathcal{F}) \otimes_A B \to H^i(X', u'^* \mathcal{F})$$

is an isomorphism of B-modules. When f is separated we can cover X by affines and compute $H^i(X, \mathcal{F})$ using Čech cohomology. Furthermore, the pullback of this open cover to X' is a cover of X' by affines from which we can compute $H^i(X', u'^* \mathcal{F})$. Now we use that tensoring by B preserves the cohomology of the Čech complex since B is flat. In the more general qcqs setting, one must use the Čech-to-derived spectral sequence.

□

We also noted that the proof of (7) did not actually use flatness of $\mathcal{F}$ over Y since it dealt with only global sections. Indeed we have the following more general base change without flatness.

Proposition 5.4. *(Base change without flatness) Suppose we have a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd} \tag{9}$$

where f is projective, Y' and Y are Noetherian, and suppose $\mathcal{F}$ is a coherent sheaf on X. Then the base change morphism

$$u^* f_*(\mathcal{F}(d)) \to f'_* u'^*(\mathcal{F}(d))$$

is an isomorphism for $d \gg 0$.

[12] quasi-compact quasi-separated, though for our use separated suffices

Proof. (Sketch) The strategy is the same as in the proof of (7). First the question is local on Y so we can suppose $Y = \operatorname{Spec} A$ is affine. Then we reduce to the case $Y' = \operatorname{Spec} A'$ is affine using flat base change. Furthermore, we can suppose $X = \mathbb{P}^n_A$. Then we take a resolution $P_1 \to P_0 \to \mathcal{F} \to 0$ by direct sums of twisting sheaves $\mathcal{O}_X(a)$. Pulling back by u' gives us a resolution of $P_1' \to P_0' \to u'^*\mathcal{F} \to 0$ by direct sums of the corresponding twisting sheaves on X'. After twisting by $\mathcal{O}_X(d)$ (resp. $\mathcal{O}_{X'}(d)$) for $d \gg 0$, higher cohomologies vanish and so applying H^0 gives us a resolution $H^0(X, \mathcal{F}(d))$ as an A-module and a resolution of $H^0(X', u'^*\mathcal{F}(d))$ as an A' module by direct sums of $H^0(X, \mathcal{O}_X(a))$ (resp. $H^0(X', \mathcal{O}_{X'}(a))$. By identifying the spaces of sections of $\mathcal{O}_X(a)$ with degree a polynomials over A, it is clear that base change holds for this module:

$$H^0(X, \mathcal{O}_X(a)) \otimes_A A' \cong H^0(X', \mathcal{O}_{X'}(a)). \tag{10}$$

Applying $- \otimes_A A'$ to the resolution of $H^0(X, \mathcal{F}(d))$ yields a resolution of $H^0(X, \mathcal{F}(d)) \otimes_A A'$ and we see by 10 this is the same as the resolution of $H^0(X', u'^*\mathcal{F}(d))$. Since the base change morphisms for $\mathcal{O}_X(a)$ commute with those for $\mathcal{F}(d)$ we conclude the base change morphism for $\mathcal{F}(d)$ is an isomorphism. □

Here is an example to show that in general, even flatness of $\mathcal{F}$ is not enough to ensure that the base change morphism is an isomorphism.

Example 5.5. *Let $X = E \times_k E$ where E is an elliptic curve over a field k with origin $e \in E(k)$. Let $\Delta \subset X$ denote the diagonal and consider the line bundle*

$$L = \mathcal{O}_X(\Delta - p_2^*e)$$

where $p_i : X \to E$ are the projections. Now consider the base change diagram

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

where $Y = E$, $f = p_1$, $Y' = \operatorname{Spec} k$, and $u = e : \operatorname{Spec} k \to E$ is the origin. Then $X' = X_e \cong E$ and $f' \to \operatorname{Spec} k$ is just the structure map. The pullback $u'^*L = L|_{E_e} \cong \mathcal{O}_E$ so

$$f'_*u'^*L = H^0(E, \mathcal{O}_X) = k.$$

On the other hand, f_*L is a torsion free sheaf on the integral regular curve Y so it is locally free. We may compute its stalk at the generic point of Y by flat base change. We get that $f_*L_\eta = H^0(E_\eta, \mathcal{O}_{E_\eta}(\Delta_\eta - e_\eta)) = 0$ since Δ_η and e_η are distinct points of the genus one curve E_η. Thus $f_*L = 0$ so $u^*f_*L = 0$ and we see that the base change map is not an isomorphism. Of course in this case, the projection f is flat and L is a line bundle so L is flat over Y.

What goes wrong here is that the cohomology of the fibers jumps at $e \in E$. This situation is completely understood by the following two theorems.

Theorem 5.6. *(Semi-continuity) Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes. Let $\mathcal{F}$ be a coherent sheaf on X flat over Y. Then the function*

$$y \mapsto \dim H^i(X_y, \mathcal{F}_y)$$

is upper semi-continuous. Moreover, the function

$$y \mapsto \chi(X_y, \mathcal{F}_y) = \sum (-1)^i \dim H^i(X_y, \mathcal{F}_y)$$

is locally constant.

Theorem 5.7. *(Cohomology and base change) Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X flat over Y. Suppose for some $y \in Y$, the base change map*

$$\varphi^i_y : (R^i f_* \mathcal{F})_y \to H^i(X_y, \mathcal{F}_y)$$

is surjective. Then

1. *there exists an open neighborhood U of y such that for all $y' \in U$, $\varphi^i_{y'}$ is an isomorphism, and*
2. *φ^{i-1}_y is surjective if and only if $R^i f_* \mathcal{F}$ is locally free in a neighborhood of y.*

Often times in moduli theory, one needs to show that various constructions on families are functorial so that they induce a construction on the moduli space. Functoriality usually means compatibility with base change. As such, the following generalization (and direct corollary) of the cohomology and base change theorem is very useful.

Proposition 5.8. *Let $f : X \to Y$ and $\mathcal{F}$ be as above. Suppose that φ^i_y is an isomorphism and $R^i f_* \mathcal{F}$ is locally free (or equivalently φ^{i-1}_y is an isomorphism) for all $y \in Y$. Then for any locally Noetherian scheme Y' and cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

the base change map

$$\varphi^i_u : u^* R^i f_* \mathcal{F} \to R^i f'_* (u'^* \mathcal{F})$$

is an isomorphism. In particular, if $H^1(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$, then $f_* \mathcal{F}$ is locally free and $u^* f_* \mathcal{F} \cong f'_* u'^* \mathcal{F}$.

When the conclusion of the proposition holds, we often say the formation of $R^i f_* \mathcal{F}$ commutes with arbitrary base change.

We won't prove semi-continuity and cohomology and base change here but let us say a few words about the proof. First, the statements are all local on Y so we may suppose $Y = \operatorname{Spec} A$ where A is local and Noetherian. The proofs then are based on the idea of Grothendieck to consider the functor on the category of A-modules given by

$$M \mapsto H^i(X, \mathcal{F} \otimes_A M).$$

Then one proves a sort of "representability" result for this functor. There exists a complex $K^\bullet$, the Grothendieck complex of $\mathcal{F}$, such that each term K^i is a finite free module, and such that there are isomorphisms

$$H^i(X, \mathcal{F} \otimes_A M) \cong H^i(K^\bullet \otimes_A M)$$

functorial in M. This reduces base change and semi-continuity problems to linear algebra of this complex $K^\bullet$ and the theorems follow from a careful study of the properties of complexes of flat modules under base change using Nakayama's lemma.

5.2 The Hilbert and Quot functors

Now we can define Hilbert functor of a projective morphism $f : X \to S$. Note that implicit in this is a fixed embedding of X into $\mathbb{P}^n_S$ and thus a fixed very ample line bundle $\mathcal{O}_X(1)$ that we can take the Hilbert polynomial with respect to.

Definition 5.9. *Let $f : X \to S$ be a projective morphism. The Hilbert functor $H_{X/S} : Sch_S \to Set$ is the functor*

$$T \mapsto \{\text{closed subschemes } Z \subset X_T := X \times_S T \mid Z \to T \text{ is flat and proper}\}.$$

This is a functor by pulling back Z along $T' \to T$. An element $(Z \subset X_T) \in H_{X/S}(T)$ is called a flat family of subschemes of X parametrized by T. Let P be any polynomial. We define the subfunctor $H^P_{X/S} \subset H_{X/S}$ by

$$H^P_{X/S}(T) = \{\text{flat families of subschemes } Z \subset X_T \mid P_{Z_t}(n) = P(n) \text{ for all } t \in T\}.$$

By the local constancy of Hilbert polynomials in flat projective families, we see that

$$H_{X/S} = \bigsqcup_P H^P_{X/S}$$

Our goal for the next few classes is to prove that for each $f : X \to S$ and P as above, $H^P_{X/S}$ is representable by a projective scheme, the *Hilbert scheme* $\mathrm{Hilb}^P_{X/S}$, over S.

6 The Hilbert and Quot schemes

Recall last time we defined the Hilbert functor $H_{X/S}$ parametrizing flat families of closed subschemes $Z \subset X_T$ over any base scheme T. Moreover, we noted that

$$H_{X/S} = \bigsqcup H^P_{X/S}$$

where P runs over all numerical polynomials and $H^P_{X/S}$ is the subfunctor parametrizing those flat families of closed subschemes for which the Hilbert polynomial $P_{Z_t}(d) = P$ for all $t \in T$.[13]

Giving a subscheme $i : Z \subset X_T$ is the same as giving an ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_{X_T}$ with quotient $i_*\mathcal{O}_Z$. We have that $Z \to T$ is flat if and only if $i_*\mathcal{O}_Z$ is flat over T, and $i_*\mathcal{O}_Z$ is a quotient of $\mathcal{O}_{X_T}$ with kernel $\mathcal{I}_Z$ so the Hilbert functor is the same as the functor for equivalence classes of quotients $q : \mathcal{O}_{X_T} \to \mathcal{F} \to 0$ where $\mathcal{F}$ is flat over T with Hilbert polynomial P. Two such quotients $(q, \mathcal{F})$ and $(q', \mathcal{F}')$ give the same ideal sheaf (and thus

[13]We had a question as to why this disjoint union decomposition holds in general since we only showed the local constancy of Hilbert polynomials over a locally Noetherian base. Note however that for any $T \to S$, the morphism $X_T \to T$ is locally of finite presentation. For the local constancy we may restrict to $T = \operatorname{Spec} A$ being local in which case $X_T \to T$ is a morphism of finite presentation. Thus for any closed subscheme $Z \subset X_T$ flat and proper over T, $Z \to T$ is a morphism of finite presentation. Then a usual trick shows that $Z \to T$ is pulled back from a morphism $Z' \to T'$ where T' is finitely presented over S. Then we have constancy of the Hilbert polynomial for $Z' \to T'$ which implies constancy for $Z \to T$.

the same subscheme of X_T) if and only if there is an isomorphism $\alpha : \mathcal{F} \to \mathcal{F}'$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{O}_{X_T} \arrow[r, "q"] \arrow[d, equal] & \mathcal{F} \arrow[d, "\alpha"] \\ \mathcal{O}_{X_T} \arrow[r, "q'"'] & \mathcal{F}' \end{tikzcd}$$

Thus we have the equality of functors

$$H^P_{X/S}(T) = \{q : \mathcal{O}_{X_T} \twoheadrightarrow \mathcal{F} \mid \mathcal{F} \text{ flat over } T \text{ with proper support, } P_{\mathcal{F}_t}(n) = P(n)\}/\sim$$

where $\sim$ is the equivalence relation of pairs $(q, \mathcal{F})$ given by diagrams as above.

More generally, we can consider a fixed coherent sheaf $\mathcal{E}$ on X. For any $\varphi : T \to S$, let us denote by $\mathcal{E}_T$ the pullback of $\mathcal{E}$ to X_T or $\varphi^*\mathcal{E}$.

Definition 6.1. *The Quot functor $Q_{\mathcal{E},X/S} : Sch_S \to Set$ is the functor*

$$T \mapsto \{q : \mathcal{E}_T \twoheadrightarrow \mathcal{F} \mid \mathcal{F} \textit{ flat over } T,\ \operatorname{Supp}(\mathcal{F}) \to T \textit{ is proper}\}/\sim$$

where $(q, \mathcal{F}) \sim (q', \mathcal{F}')$ if and only if there exists an isomorphism $\alpha : \mathcal{F} \to \mathcal{F}'$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{E}_T \arrow[r, "q"] \arrow[d, equal] & \mathcal{F} \arrow[d, "\alpha"] \\ \mathcal{E}_T \arrow[r, "q'"'] & \mathcal{F}' \end{tikzcd}$$

Given a polynomial P, we have the subfunctor $Q^P_{\mathcal{E},X/S}$ of those quotients $(q, \mathcal{F})$ such that for each $t \in T$, $P_{\mathcal{F}_t}(n) = P(n)$. This is a functor by pullback, where we note that for any $\varphi : T' \to T$, $\varphi^*q : \varphi^*\mathcal{E}_T = \mathcal{E}_{T'} \to \varphi^*\mathcal{F}$ is surjective.

Remark 6.2. *Note that $H_{X/S} = Q_{\mathcal{O}_X,X/S}$ by the above discussion.*

As before we have that

$$Q_{\mathcal{E},X/S} = \bigsqcup_P Q^P_{\mathcal{E},X/S}.$$

We have the following main representability result.

Theorem 6.3. *Let $f : X \to S$ be a projective morphism over a Notherian scheme S and let P be a polynomial. Then there exists a projective S-scheme $\operatorname{Hilb}^P_{X/S}$ as well as a closed subscheme*

$$\mathcal{Z}^P_{X/S} \subset X \times_S \operatorname{Hilb}^P_{X/S}$$

such that $\mathcal{Z}^P_{X/S} \to \operatorname{Hilb}^P_{X/S}$ is flat and proper with Hilbert polynomial P and the pair $(\operatorname{Hilb}^P_{X/S}, Z^P_{X/S})$ represents the Hilbert functor $H^P_{X/S}$. More generally, if $\mathcal{E}$ is a coherent sheaf on X, then there exists a projective S-scheme $\operatorname{Quot}^P_{\mathcal{E},X/S}$ as well as a quotient sheaf

$$q^P_{\mathcal{E},X/S} : \mathcal{E}_{\operatorname{Quot}^P_{\mathcal{E},X/S}} \to \mathcal{F}^P_{\mathcal{E},X/S} \to 0$$

on $\operatorname{Quot}^P_{\mathcal{E},X/S} \times_S X$ which is flat with proper support over $\operatorname{Quot}^P_{\mathcal{E},X/S}$ and has Hilbert polynomial P such that the pair

$$(\operatorname{Quot}^P_{\mathcal{E},X/S}, q^P_{\mathcal{E},X/S})$$

represents the Quot functor $Q^P_{\mathcal{E},X/S}$.

Thus we have projective fine moduli spaces $\mathrm{Hilb}^P_{X/S}$ and $\mathrm{Quot}^P_{\mathcal{E},X/S}$ for closed subschemes and quotients of a coherent sheaf respectively!

The basic idea of the construction is simple. To illustrate it, let us consider the Hilbert functor for $X = \mathbb{P}^n_k$ over a base field $S = \mathrm{Spec}\, k$. Now a subscheme $Z \subset X$ is determined by its equations which form an ideal $I \subset k[x_0, \ldots, x_n]$ which we can view as a linear subspace of the infinite dimensional vector space $k[x_0, \ldots, x_n]$. This gives, at least set theoretically, an inclusion

$$\{\text{Subschemes of } \mathbb{P}^n_k\} \hookrightarrow Gr(k[x_0, \ldots, x_n])$$

to some infinite dimensional Grassmannian of the vector space $k[x_0, \ldots, x_n]$. Now to proceed we need to do two things:

(a) cut down the dimension of the target to a finite dimensional Grassmannian which we proved already is representable by a projective scheme;

(b) show that the image of this set theoretic map is actually an algebraic subscheme which represents the functor of flat families.

These two steps respectively require the following two technical results.

Theorem 6.4. *(Uniform Castelnuovo-Mumford regularity) For any polynomial P and integers m, n, there exists an integer $N = N(P, m, n)$ such that for any field k and any coherent subsheaf of $\mathcal{F} \subset \mathcal{O}^{\oplus m}_{\mathbb{P}^n_k}$ with Hilbert polynomial P we have the following. For any $d \geq N$,*

1. *$H^i(\mathbb{P}^n_k, \mathcal{F}(d)) = 0$ for all $i \geq 1$,*
2. *$\mathcal{F}(d)$ is generated by global sections, and*
3. *$H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(d+1))$ is surjective.*

Theorem 6.5. *(Flattening stratification) Let $f : X \to S$ be a projective morphism over a Noetherian scheme S and let $\mathcal{F}$ be a coherent sheaf on X. For every polynomial P there exists a locally closed subscheme $i_P : S_P \subset S$ such that a morphism $\varphi : T \to S$ factors through S_P if and only if $\varphi^*\mathcal{F}$ on $T \times_S X$ is flat over T with Hilbert polynomial P. Moreover, S_P is nonempty for finitely many P and the disjoint union of inclusions*

$$i : S' = \bigsqcup_P S_P \to S$$

induces a bijection on the underlying set of points. That is, $\{S_P\}$ is a locally closed stratification of S.

In the second theorem, we can think of S_P as well as the pullback $i_P^*\mathcal{F}$ which is necessarily flat over S_P as the fine moduli space for the functor that takes a scheme T to the set of morphisms to S which pull back $\mathcal{F}$ to be flat with Hilbert polynomial P.

6.1 Proof of the representability of $H^P_{X/S}$ and $Q^P_{\mathcal{E},X/S}$

We are now going to prove the representability of $H^P_{X/S}$ and $Q^P_{\mathcal{E},X/S}$ assuming uniform CM regularity and flattening stratifications. We will return to the proofs of these statements later. Note that $H^P_{X/S} = Q^P_{\mathcal{O}_X,X/S}$ so we will prove representability of $Q^P_{\mathcal{O}_X,X/S}$ for any coherent sheaf $\mathcal{F}$ on X.

Proof. **Step 1:** First we reduce to the case $X = \mathbb{P}^n_S$ and $\mathcal{E} = \mathcal{O}_X^{\oplus k}$. This is a consequence of the following lemmas.

Lemma 6.6. *For any integer r, tensoring by $\mathcal{O}_{X_T}(r)$ induces an isomorphism of functors*

$$Q^{P(d)}_{\mathcal{E},X/S} \cong Q^{P(d+r)}_{\mathcal{E}(r),X/S}.$$

Proof. Tensoring by a line bundle is an equivalence of categories since there is an inverse given by tensoring by the dual. Thus for each, T, tensoring by $\mathcal{O}_{X_T}(r)$ induces a bijection

$$Q^{P(d)}_{\mathcal{E},X/S}(T) \cong Q^{P(d+r)}_{\mathcal{E}(r),X/S}(T).$$

This is a natural transformation since for any $\varphi : T' \to T$, $\varphi^*\mathcal{O}_{X_T}(r) = \mathcal{O}_{X_{T'}}(r)$. $\square$

Lemma 6.7. *Suppose $\alpha : \mathcal{E}' \twoheadrightarrow \mathcal{E}$ is a quotient of coherent sheaves on X. Then the induced map $Q^P_{\mathcal{E},X/S} \to Q^P_{\mathcal{E}',X/S}$ is a closed subfunctor.*

Proof. This map is given by noting that a quotient $q : \mathcal{E} \twoheadrightarrow \mathcal{F}$ induces a quotient $q' = q \circ \alpha : \mathcal{E}' \twoheadrightarrow \mathcal{F}$ by composition. We need to show that for any scheme T' over S and object $(q', \mathcal{F}) \in Q^P_{\mathcal{E}',X/S}(T')$, there exists a closed subscheme $T \subset T'$ satisfying the following universal property. For any other S-scheme T'', a morphism $\varphi : T'' \to T'$ factors through T if and only if $\varphi^* q' : \mathcal{E}'_{T''} \twoheadrightarrow \varphi^*\mathcal{F}$ factors through a map $q : \mathcal{E}_{T''} \to \varphi^*\mathcal{F}$. Since the morphism $X_{T'} \to T'$ is of locally of finite presentation, and the condition of being closed can be checked locally, we may assume T' is affine in which case $X_{T'} \to T'$ is of finite presentation and then we can use the finite presentation trick to reduce to the case that T', T'' are Noetherian.

Let $\mathcal{K} = \ker(\alpha) \subset \mathcal{E}'$ be the kernel of α. Then the morphism $q' : \mathcal{E}'_{T'} \to \mathcal{F}$ factors through $\mathcal{E}_{T'}$ if and only if the composition $\mathcal{K}_{T'} \to \mathcal{F}$ is the zero map. Indeed $\mathcal{K}_{T'}$ surjects onto the kernel of $\mathcal{E}'_{T'} \to \mathcal{E}_{T'}$ by right exactness of pullback and a morphism factors through a surjection if and only if it is 0 on the kernel. Let us denote the composition $\mathcal{K}_{T'} \to \mathcal{F}$ by r. Thus we want to show that there exists a closed subscheme $T \subset T'$ such that a morphism $\varphi : T'' \to T'$ factors through T if and only if the composition $\varphi^* r : \mathcal{K}_{T''} \to \varphi^*\mathcal{F}$ of coherent sheaves on $X_{T''}$ is zero. The result now follows by applying the following lemma.

$\square$

Lemma 6.8. *Let $f : X \to S$ be a projective morphism over a Noetherian scheme and let $r : \mathcal{K} \to \mathcal{F}$ be a map of coherent sheaves on X with $\mathcal{F}$ flat over S. Then there exists a closed subscheme $Z \subset S$ such that for any T Noetherian and any $\alpha : T \to S$, α factors through Z if and only if $\alpha^* r$ is the zero map.*

Proof. For any d, r is zero if and only if the twist $r(d) : \mathcal{K}(d) \to \mathcal{F}(d)$ is zero. For large enough $d \gg 0$, the pushforward $f_*\mathcal{F}(d)$ is locally free since $\mathcal{F}$ is flat over S and $\mathcal{K}(d)$ is gloablly generated over S so that

$$f^* f_* \mathcal{K}(d) \to \mathcal{K}(d)$$

is surjective. Thus $r(d)$ is 0 if and only if $f^* f_* \mathcal{K}(d) \to \mathcal{F}(d)$ is 0 if and only if $f_* r(d) : f_*\mathcal{K}(d) \to f_*\mathcal{F}(d)$ is 0. By the hom-tensor adjunction, using that $f_*\mathcal{F}(d)$ is locally free, this is the same as the cosection $t_d : f_*\mathcal{K}(d) \otimes (f_*\mathcal{F}(d))^\vee \to \mathcal{O}_S$ vanishing. Now the cosection t_d defines an ideal sheaf $I_d \subset \mathcal{O}_S$ by its image and it is clear that t_d vanishes at a point $s \in S$ if and only if $s \in V(I_d) = Z_d$.

Now consider the chain

$$I_{d_0} \subset I_{d_0} + I_{d_0+1} \subset \dots$$

where d_0 is some large enough number so that $\mathcal{K}(d)$ is globally generated and $f_*\mathcal{F}(d)$ is locally free. By the Noetherian condition, this chain terminates in some ideal I with vanishing subscheme $V(I) = Z$ the scheme theoretic intersection of the Z_d. Now for large enough $d \gg 0$, the formation of $f_*\mathcal{K}(d)$ and $f_*\mathcal{F}(d)$ commute with base change so $s \in Z$ if and only if $(t_d)_s = 0$ for all $d \gg 0$ if and only if $r(d)_s = 0$ for all $d \gg 0$ if and only if $r_s = 0$.[14]

We will check this Z satisfies the universal property. Suppose $\alpha : T \to S$ satisfies that $\alpha^* r$ is the zero map, then $\alpha^* r(d)$ is the zero map for $d \gg 0$ and $f_*\mathcal{K}(d)$, $f_*\mathcal{F}(d)$ commute with base change by α for d large enough (depending on α) so $\alpha^* f_* r(d) : \alpha^* f_*\mathcal{K}(d) \to \alpha^* f_*\mathcal{F}(d)$ is the zero map so α factors through Z_d for all $d \gg 0$ so α factors through Z. On the other hand, if α factors through Z, then for all $d \gg 0$, $\alpha^* f_*\mathcal{K}(d) \to \alpha^* f_*\mathcal{F}(d)$ is the zero map but by base change without flatness, for large enough d, the formation of these pushforwards commutes with basechange by α so we have that

$$(f_T)_*\alpha^*\mathcal{K}(d) \to (f_T)_*\alpha^*\mathcal{F}(d)$$

is the zero map. Thus $f_T^*(f_T)_*\alpha^*\mathcal{K}(d) \to \alpha^*\mathcal{F}(d)$ is the zero map. Since $\alpha^*\mathcal{K}(d)$ is globally generated for $d \gg 0$, then $\alpha^* r(d) : \alpha^*\mathcal{K}(d) \to \alpha * \mathcal{F}(d)$ is the zero map for d large enough so $\alpha^* r$ is the zero map.

□

□

7 The Hilbert and Quot schemes (cont.)

7.1 Proof of representability (cont.)

We are continuing the proof of representability of the Quot functor $Q^P_{\mathcal{E},X/S}$ (and thus Hilbert functors) for $\mathcal{E}$ a coherent sheaf on X with $X \to S$ projective over S Noetherian.

Proof. **Step 2:** Recall we are reducing to the case $X = \mathbb{P}^n$ and $\mathcal{E} = \mathcal{O}_X^{\oplus k}$. We have completed the proof of the following two lemmas.

Lemma 7.1. *For any integer r, tensoring by $\mathcal{O}_{X_T}(r)$ induces an isomorphism of functors*

$$Q^{P(d)}_{\mathcal{E},X/S} \cong Q^{P(d+r)}_{\mathcal{E}(r),X/S}.$$

Lemma 7.2. *Suppose $\alpha : \mathcal{E}' \to \mathcal{E}$ is a quotient of coherent sheaves on X. Then the induced map $Q^P_{\mathcal{E},X/S} \to Q^P_{\mathcal{E}',X/S}$ is a closed subfunctor.*

Now given any sheaf $\mathcal{E}$ on X and $i : X \to \mathbb{P}^n_S$ the projective embedding, a quotient $q : \mathcal{E} \to \mathcal{F}$ is the same as a quotient $i_*\mathcal{E} \to i_*\mathcal{F}$ since i_* is an equivalence of categories between sheaves on X and sheaves on $\mathbb{P}^n_S$ supported on X which preserves Hilbert polynomials. Thus suppose $X = \mathbb{P}^n_S$. Then for $a \gg 0$, $\mathcal{E}(a)$ is globally generated so there is a surjection

$$\mathcal{O}_X(a)^{\oplus k} \to \mathcal{E}(a)$$

[14] In particular, all the Z_d have the same set theoretic support as Z and we only need to worry about the right scheme structure.

for some k. Thus by the second lemma above,

$$Q^P_{\mathcal{E}(a),X/S} \hookrightarrow Q^P_{\mathcal{O}_X(a)^{\oplus k},X/S}$$

is a closed subfunctor so it suffices to prove $Q^P_{\mathcal{O}_X(a)^{\oplus k},X/S}$ is representable by a projective scheme over S. Then by the first lemma, there is an isomorphism of functors

$$Q^{P(d-a)}_{\mathcal{O}_X^{\oplus k},X/S} \cong Q^{P(d)}_{\mathcal{O}_X^{\oplus k}(a),X/S}.$$

Step 3: Now we are in the situation $X = \mathbb{P}^n_S$ and $\mathcal{E} = \mathcal{O}_X^{\oplus k}$. Let $q : \mathcal{E}_T \to \mathcal{F}$ be an element of $Q^P_{\mathcal{E},X/S}(T)$ and let $\mathcal{K}$ be the kernel of q. By flatness of $\mathcal{F}$ over T, for any $t \in T$ we have that

$$0 \to \mathcal{K}_t \to \mathcal{E}_t \to \mathcal{F}_t \to 0$$

is exact. Then by additivity of Euler characteristics, the Hilbert polynomial $P_{\mathcal{K}_t}$ is given by

$$P_{\mathcal{K}_t}(d) = k\binom{n+d}{d} - P(d).$$

In particular it is independent of $t \in T$ or even of $(q, \mathcal{F})$. Thus by uniform CM regularity applied to $\mathcal{K}_t$ and $\mathcal{E}_t$ [15], there exists an N depending only on $P(d)$, k and n such that for all T, all $(q, F) \in Q^P_{\mathcal{E},X/S}(T)$, and all $t \in T$, we have that for all $a \geq N$,

- $H^i(X_t, \mathcal{K}_t(a)) = H^i(X_t, \mathcal{E}_t(a)) = H^i(X_t, \mathcal{F}_t(a)) = 0$ for $i \geq 1$, and
- $\mathcal{K}_t(a), \mathcal{E}_t(a)$ and $\mathcal{F}_t(a)$ are globally generated.

Since $\mathcal{E}$ and $\mathcal{F}$ are both flat, then $\mathcal{K}$ is also flat. Then we can apply cohomology and base change to see that for all $d \geq N$,

$$0 \to (f_T)_*\mathcal{K}(a) \to (f_T)_*\mathcal{E}(a) \to (f_T)_*\mathcal{F}(a) \to 0 \tag{11}$$

is an exact sequence of locally free sheaves of rank

$$k\binom{n+a}{a} - P(a),\ k\binom{n+a}{a},\ \text{and } P(a)$$

respectively. Moreover, the formation of all three of these locally free sheaves is compatible with base change by any $T' \to T$ with T' locally Noetherian.[16] The sequence (11) gives an element of $Gr(P(a), f_*\mathcal{E}(a))(T)$ and since (11) is compatible with base change, the induced set theoretic map

$$Q^P_{\mathcal{E},X/S}(T) \to Gr(P(a), f_*\mathcal{E}(a))(T)$$

is a natural transformation of functors.

Moreover, one can compute explicitly that $f_*\mathcal{E}(a)$ is actually the free sheaf of rank $k\binom{n+a}{a}$. Indeed, $\mathcal{E}(a) = \mathcal{O}_X(a)^{\oplus k}$ so it suffices to check for $k = 1$ in which case $f_*\mathcal{O}_X(a)$ restricted to any affine open Spec $A \subset S$ is simply

$$A[x_0, \ldots, x_n]_a$$

[15] as well as a diagram chase to conclude the result for $\mathcal{F}_t$

[16] I'll leave it to the reader to use the finite presentation trick and convince themselves that this is good enough!

so globally $f_*\mathcal{O}_X(a) = \mathcal{O}_S[x_0, \dots, x_n]_a$ where $\mathbb{P}^n_S = \text{Proj}_S \mathcal{O}_S[x_0, \dots, x_n]$. Thus

$$Gr(P(a), f_*\mathcal{E}(a)) = Gr_S\left(P(a), k\binom{n+a}{a}\right)$$

which we showed previously is representable by a projective S-scheme. More canonically, $f_*\mathcal{O}_X(a)^{\oplus k} = V_a \otimes_{\mathbb{Z}} \mathcal{O}_S$ where $V_a = \mathbb{Z}[x_0, \dots, x_n]_a^{\oplus k}$ so we can write the Grassmannian as $G_S = G \times_{\mathbb{Z}} S$ where $G = Gr(P(a), V_a)$.

Step 4: Our strategy now is to show that the natural transformation of functors $Q^P_{\mathcal{E},X/S} \to G_S$ is an inclusion of a subfunctor and then identify the subfunctor $Q^P_{\mathcal{E},X/S}$ with the functor of points of some locally closed subvariety of G_S.

Toward that end, we need to show that for T and any $(q, \mathcal{F}) \in Q^P_{\mathcal{E},X/S}(T)$, $(q, \mathcal{F})$ is determined by the sequence (11):

$$0 \to (f_T)_*\mathcal{K}(a) \to (f_T)_*\mathcal{E}(a) \to (f_T)_*\mathcal{F}(a) \to 0.$$

By global generation and the fact that these sheaves are all locally free, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & f_T^*(f_T)_*\mathcal{K}(a) \arrow[r] \arrow[d] & f_T^*(f_T)_*\mathcal{E}(a) \arrow[r] \arrow[d] & f_T^*(f_T)_*\mathcal{F}(a) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K}(a) \arrow[r] & \mathcal{E}(a) \arrow[r, "q(a)"] & \mathcal{F}(a) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are surjections and the horizontal sequences are exact. Let

$$h : f_T^*(f_T)_*\mathcal{K}(a) \to \mathcal{E}(a)$$

be the composition. It suffices to show that q may be determined from h, but indeed by exactness, the cokernel of h is natural identified with $q(a) : \mathcal{E}(a) \to \mathcal{F}(a)$ and so by twisting by $\mathcal{O}_X(-a)$ we recover q from the sequence (11) and conclude that

$$Q^P_{\mathcal{E},X/S}(T) \to G_S(T)$$

is injective.

Step 5: Now we will use flattening stratifications to show that $Q^P_{\mathcal{E},X/S}$ as a subfunctor of the Grassmannian is representable by a locally closed subscheme. Over G_S we have the universal quotient sequence

$$0 \to \mathcal{K} \to V_a \otimes \mathcal{O}_{G_S} \to \mathcal{Q} \to 0$$

and $V_a \otimes \mathcal{O}_{G_S} = (f_{G_S})_*\mathcal{E}_{G_S}(a)$ where $f_{G_S} : \mathbb{P}^n_{G_S} \to G_S$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n_{G_S}}(d)^{\oplus k}$.

Now we can pullback this sequence to get

$$0 \to f_{G_S}^*\mathcal{K} \to f_{G_S}^*(f_{G_S})_*\mathcal{E}_{G_S}(a) \to f_{G_S}^*\mathcal{Q} \to 0.$$

The middle sheaf comes with a surjective map to $\mathcal{E}_{G_S}(a)$ since $\mathcal{E}_{G_S}(a)$ is gloablly generated on any projective space ($a > 0$). Let us denote by $h : f_{G_S}^*\mathcal{K} \to \mathcal{E}_{G_S}(a)$ the composition and let

$\mathcal{F}$ be the cokernel of h. Then $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_{G_S}$ and we can consider the flattening stratification for $\mathcal{F}$ over G_S.

Let $G_S^{P(d+a)} \subset G_S$ be the stratum over which $\mathcal{F}$ is flat with Hilbert polynomial $P(d+a)$. Then $G_S^{P(d+a)}$ is universal for maps to $T \to G_S$ such that $\mathcal{F}_T$ is flat over T with Hilbert polynomial $P(d+a)$, but that exactly means that the quotient map $\mathcal{E}_T \to \mathcal{F}_T(-a)$ is an element of the subfunctor $Q^P_{\mathcal{E},X/S}$ so the locally closed subscheme $G_S^{P(d+a)}$ with the restriction of the quotient map $\mathcal{E}_{G_S} \to \mathcal{F}$ represents the subfunctor $Q^P_{\mathcal{E},X/S}$.

Step 6: Finally we show that $Q^P_{\mathcal{E},X/S}$ satisfies the valuative criterion of properness. This implies the stratum $G_S^{P(d+a)}$ is actually a closed subscheme of G_S. We showed earlier that G_S is projective over S so we conclude that $Q^P_{\mathcal{E},X/S}$ is representable by a projective scheme over S.

We already showed that the special case of the Hilbert functor is proper. The proof for $Q^P_{\mathcal{E},X/S}$ is similar. Let $T = \text{Spec } R$ the spectrum of a DVR and let $T^0 = \text{Spec } K$ the spectrum of its function field. We need to show that for any solid diagram as below, there exists a unique dashed arrow.

$$\begin{tikzcd} T^0 \arrow[r] \arrow[d] & Q^P_{\mathcal{E},X/S} \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

That is, given $(q^0, \mathcal{F}^0) \in Q^P_{\mathcal{E},X/S}(T^0)$, there is a unique extension to a flat quotient $(q, \mathcal{F})$. For this we can compose $\mathcal{E}_T \to \mathcal{E}_{T^0} \to \mathcal{F}^0$ and let $\mathcal{F}$ be the image of $\mathcal{E}_T$ with q the composition. Then $\mathcal{F}$ is flat and by the criterion for flatness over a DVR, it is the unique flat extension so this gives the required lift.

$\square$

7.2 Some applications

Here we discuss some applications that follow from the representability and projectivity of Hilbert and Quot schemes.

7.2.1 Grassmannians of coherent sheaves

We consider the case that $f = id_S$ is the identity. In this case $\mathcal{E}$ is a coherent sheaf on S, flatness is equivalent to local freeness, and the Hilbert polynomial of $\mathcal{F}_s$ is just the dimension $\dim_{k(s)} \mathcal{F}_s$. Thus, given a k, we have the Quot scheme $Q^k_{\mathcal{E},S/S}$ for constant Hilbert polynomial $P(d) = k$ so that a map

$$T \to Q^k_{\mathcal{E},S/S}$$

is a locally free quotient $q : \mathcal{E}_T \to \mathcal{V}$ of rank k on T. When $\mathcal{E}$ is itself a locally free sheaf, this is just the Grassmannian $Gr_S(k, \mathcal{E})$ so we have Grassmannians for any coherent sheaf $\mathcal{E}$ which we also denote $Gr_S(k, \mathcal{E})$. In particular, when $k = 1$, we denote $Gr_S(1, \mathcal{E})$ by $\mathbb{P}(\mathcal{E})$ the projective "bundle" of $\mathcal{E}$ whose fiber over $s \in S$ is the projectivization of the vector space $\mathcal{E}_s$.

7.2.2 Lifting rational curves

Let X/k be a projective variety over a field. A degree r rational curve $C \subset X$ is a genus 0 curve with $\deg(\mathcal{O}_X(1)|_C) = r$. The Hilbert polynomial of such a curve is always $P(d) = rd + 1$. Indeed, $\chi(O_C) = 1$ since C is rational and $H^0(C, \mathcal{O}_C(d)) = rd + 1$ by Riemann-Roch for $d \gg 0$.

Now let $\mathcal{X}$ be a projective scheme over $\mathbb{Z}$ with geometric generic fiber $\mathcal{X}_{\overline{\mathbb{Q}}} =: X$ and $\mathcal{X}_{\mathbb{F}_p} =: X_p$. Suppose X_p contains a degree d rational curve for infinitely many d. Then X also contains a degree d rational curve. Indeed, consider $H^{rd+1}_{\mathcal{X}/\operatorname{Spec}\mathbb{Z}}$. This is a projective scheme over Spec $\mathbb{Z}$ so it has finitely many irreducible components. Since the fiber of $H^{rd+1}_{X/\operatorname{Spec}\mathbb{Z}}$ is nonempty over infinitely many Spec $\mathbb{F}_p$, there must be some component which dominates Spec $\mathbb{Z}$. Since the morphism to Spec $\mathbb{Z}$ is proper then the fiber over Spec $\mathbb{Q}$ must be nonempty. But a Spec $\overline{\mathbb{Q}}$ point of the fiber over Spec $\mathbb{Q}$ exactly corresponds to a rational curve of degree d on X.

8 Hom schemes, CM regularity

8.1 Hom schemes

Let X and Y be two schemes over S. The hom functor $\mathcal{H}om_S(X, Y) : Sch_S \to Set$ is given by

$$T \mapsto \{\text{morphisms } X_T \to Y_T \text{ over } T\}.$$

Theorem 8.1. *Suppose X and Y are projective over S with $X \to S$ flat. Then $\mathcal{H}om_S(X, Y)$ is representable by a quasi-projective scheme* $\operatorname{Hom}_S(X, Y)$ *over S.*

Proof. Given $f : X_T \to Y_T$, we have the graph $\Gamma_f : X_T \to X_T \times_T Y_T = (X \times_S Y)_T$ which is a closed embedding. Now $\operatorname{im}(\Gamma_f) \cong X_T$ is flat over T by assumption so it defines a map $T \to \operatorname{Hilb}_{(X\times_S Y)/S}$. This construction is compatible with basechange so we obtain a natural transformation of functors

$$\mathcal{H}om_S(X, Y) \to H_{(X\times_S Y)/S}.$$

Since a morphism is determined by its graph, this is a subfunctor. Moreover, we can characterize the graphs of morphisms as exactly those closed subschemes $Z \subset X_T \times_T Y_T$ such that the projection $Z \to X_T$ is an isomorphism. This identifies $\mathcal{H}om_S(X, Y)$ with the subfunctor of $H_{(X\times_S Y)/S}$ given by

$$T \mapsto \{\text{closed subsets } Z \subset X_T \times_T Y_T \mid Z \to T \text{ flat and proper, } Z \to X_T \text{ is an isomorphism}\}.$$

We will prove this is representable by an open subscheme of $\operatorname{Hilb}_{(X\times_S Y)/S}$.

We can consider the universal family $\mathcal{Z} \to \operatorname{Hilb}_{(X\times_S Y)/S}$ which is a closed subscheme of $X \times_S Y \times_S \operatorname{Hilb}_{(X\times_S Y)/S}$. Then $\mathcal{Z}$ comes with a projection $\pi : \mathcal{Z} \to X \times_S \operatorname{Hilb}_{(X\times_S Y)/S}$. Now we consider the diagram

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "\pi"] \arrow[d, "q"'] & X \times_S \operatorname{Hilb}_{(X\times_S Y)/S} \arrow[d, "p"] \\
\operatorname{Hilb}_{(X\times_S Y)/S} \arrow[r, equal] & \operatorname{Hilb}_{(X\times_S Y)/S}.
\end{tikzcd}$$

Then the required open subscheme is given by the following lemma.

Proposition 8.2. *Let $T = \text{Spec } R$ be the spectrum of a Noetherian local ring and let $0 \in T$ be the closed point. Let $f : X \to T$ be flat and proper and $g : Y \to T$ proper. Let $p : X \to Y$ be a morphism such that $p_0 : X_0 \to Y_0$ is an isomorphism. Then $p : X \to Y$ is an isomorphism.*

Proof. Since X is proper and Y is separated over T, the morphism $p : X \to Y$ must be proper. Moreover, since g is proper, every closed point of Y lies in Y_0. Furthermore, since p_0 is an isomorphism, then p has finite fibers over closed points of Y so p is quasi-finite. Indeed since p is proper, the fiber dimension is upper-semicontinuous on Y and it is 0 on closed points. Therefore r is finite, and in particular, affine. This implies that $R^i p_* \mathcal{F} = 0$ for any coherent sheaf $\mathcal{F}$ and $i \geq 1$. Now the result follows if we know that f is flat. Indeed in this case, $p_* \mathcal{O}_X$ is locally free of rank one by cohomology and base change. On the other hand, the natural map $\mathcal{O}_Y \to p_* \mathcal{O}_X$ is an isomorphism at all closed points $y \in Y_0 \subset Y$ and since both source and target are line bundles, it must be an isomorphism. Then, since p is affine, we have

$$X = \text{Spec}_Y f_* \mathcal{O}_X = \text{Spec}_Y \mathcal{O}_Y = Y.$$

Thus, it suffices to prove the following that p is flat. We will use the following lemma.

Lemma 8.3. *Let $p : X \to Y$ be a morphism of locally Noetherian T-schemes over a locally Noetherian scheme T. Let $x \in X$ a point in the fiber X_t for $t \in T$ and set $y = p(x)$ its image in the fiber Y_t. Then the following are equivalent:*

1. *X is flat over T at x and $p_t : X_t \to Y_t$ is flat at $x \in X_t$;*
2. *Y is flat over T at y and p is flat at $x \in X$.*

Proof. Consider the sequence local ring homomorphisms

$$\mathcal{O}_{T,t} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}.$$

Let $I = \mathfrak{m}_t \mathcal{O}_{Y,y}$. Suppose (1) holds. Then $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{T,t}$ module and $\mathcal{O}_{X,x}/I\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}/I$-module. Consider the composition

$$\mathfrak{m}_t \otimes \mathcal{O}_{X,x} \to I \otimes \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}.$$

The first map is surjective by right exactness of tensor products and the composition is injective by since $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{T,t}$ so both maps are in fact injections. Thus

$$\text{Tor}_1^{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}/I, \mathcal{O}_{X,x}) = 0. \tag{12}$$

Since $I \subset \mathfrak{m}_y$, then I annihilates the residue field $k(y)$ and one can check that Equation (12) and the assumptions imply that $\text{Tor}_1^{\mathcal{O}_{Y,y}}(k(y), \mathcal{O}_{X,x}) = 0$ by the following lemma we will leave as an exercise.

Lemma 8.4. *Suppose R is a Noetherian ring and $I \subset R$ is a proper ideal. Let M be an R-module such that M/IM is a flat R/I-module and such that*

$$\text{Tor}_1^R(R/I, M) = 0.$$

Then for any I-torsion R-module N,

$$\text{Tor}_1^R(N, M) = 0.$$

Then $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$-module by the local criterion for flatness.

Since everything is local, $\mathcal{O}_{X,x}$ is in fact faithfully flat over $\mathcal{O}_{Y,y}$. Now we want to show that $\mathrm{Tor}_1^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) = 0$. Pulling back the sequence

$$0 \to \mathfrak{m}_t \to \mathcal{O}_{T,t} \to k(t) \to 0.$$

to $\mathcal{O}_{Y,y}$ gives us

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) \to \mathfrak{m}_t \otimes \mathcal{O}_{Y,y} \to I \to 0.$$

Since $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$, then

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) \otimes \mathcal{O}_{X,x} \to \mathfrak{m}_t \otimes \mathcal{O}_{X,x} \to I \otimes \mathcal{O}_{X,x} \to 0.$$

We saw above that the second map is injective so

$$\mathrm{Tor}_1^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) \otimes \mathcal{O}_{X,x} = 0$$

but $\mathcal{O}_{X,x}$ is faithfully flat over $\mathcal{O}_{Y,y}$ so $\mathrm{Tor}_1^{\mathcal{O}_{T,t}}(k(t), \mathcal{O}_{Y,y}) = 0$ and $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{T,t}$.

For the converse, suppose (2) holds. Then $Y \to T$ is flat at $y \in Y$ and $p : X \to Y$ is flat at $x \in X$ so the composition $X \to T$ is flat at $x \in X$. Moreover, p_t is the pullback p to Y_t and flatness is stable under basechange so p_t is flat at $x \in X$.

□

□

Corollary 8.5. *Let $f : X \to T$ be flat and proper and $g : Y \to T$ proper over a Noetherian scheme T. Let $p : X \to Y$ be a morphism. Then there exists an open subscheme $U \subset T$ such that for any T' and $\varphi : T' \to T$, φ factors through U if and only if $\varphi^* p : X_{T'} \to Y_{T'}$ is an isomorphism.*

Proof. The locus where $p : X \to Y$ is an isomorphism is open on the target Y so let $Z \subset Y$ be the closed subset over which p is not an isomorphism. Since g is proper, $g(Z) \subset T$ is closed. Let $U \subset T$ be its complement. By the proposition, a point $t \in T$ is contained in U if and only if the the map on the fibers $p_t : X_t \to Y_t$ is an isomorphism. Since this is a fiberwise condition on $t \in T$, it is clear that U satisfies the required universal property. □

□

8.2 Castelnuovo-Mumford regularity

We will now discuss the first main ingredient in the proof of representability of Hilbert and Quot functors.

Theorem 8.6. *(Uniform Castelnuovo-Mumford regularity) For any polynomial P and integers m, n, there exists an integer $N = N(P, m, n)$ such that for any field k and any coherent subsheaf of $\mathcal{F} \subset \mathcal{O}_{\mathbb{P}^n_k}^{\oplus m}$ with Hilbert polynomial P we have the following. For any $d \geq N$,*

1. *$H^i(\mathbb{P}^n_k, \mathcal{F}(d)) = 0$ for all $i \geq 1$,*
2. *$\mathcal{F}(d)$ is generated by global sections, and*

3. $H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(d+1))$ *is surjective.*

To prove this we will define a more general notion of the Castelnuovo-Mumford regularity of a sheaf $\mathcal{F}$ on projective space.

Definition 8.7. *(CM regularity) A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$ is said to be m-regular if*

$$H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) = 0$$

for all $i > 0$.

The notion of CM regularity is well adapted to running inductive arguments by taking a hyperplane section.

Proposition 8.8. *Let $\mathcal{F}$ be m-regular. Then*

1. $H^i(\mathbb{P}^n_k, \mathcal{F}(d)) = 0$ *for all $d \geq m - i$ and $i > 0$, that is, $\mathcal{F}$ is m' regular for all $m' \geq m$,*
2. $H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(d+1))$ *is surjective for all $d \geq m$.*
3. $\mathcal{F}(d)$ *is globally generated for all $d \geq m$, and*

Proof. The definition of m-regularity and the conclusions of the proposition can all be checked after passing to a field extension since field extensions are faithfully flat so we may suppose the field k is infinite. Now we will induct on the dimension n.

If $n = 0$ the results trivially hold since all higher cohomology vanishes, all sheaves are globally generated and $\mathcal{O}(1) = \mathcal{O}$. Suppose $n > 0$ and let $H \subset \mathbb{P}^n_k$ be a general hyperplane.[17] Now consider the short exact sequence

$$0 \to \mathcal{F}(m-i-1) \to \mathcal{F}(m-i) \to \mathcal{F}_H(m-i) \to 0$$

where $\mathcal{F}_H = \mathcal{F}|_H$ is the restriction. Taking the long exact sequence of cohomology yields

$$\ldots \to H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) \to H^i(\mathbb{P}^n_k, \mathcal{F}_H(m-i)) \to H^{i+1}(\mathbb{P}^n_k, \mathcal{F}(m-i-1)) \to \ldots,$$

The first and last terms are 0 for all $i > 0$ by assumption so $H^i(H, \mathcal{F}_H(m-i)) = 0$ for all $i > 0$. That is, $\mathcal{F}_H$ is m-regular.

We will continue next time.

□

9 CM regularity, flattening stratifications

9.1 CM regularity (cont.)

Recall that a sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$ is m-regular is $H^i(\mathcal{F}(m-i)) = 0$ for all $i \geq 1$. We are proving the following.

[17] Here general means that H avoids all associated points of $\mathcal{F}$. This is where we use the infinite field assumption.

Proposition 9.1. *Let $\mathcal{F}$ be m-regular. Then*

1. $H^i(\mathbb{P}^n_k, \mathcal{F}(d)) = 0$ *for all* $d \geq m - i$ *and* $i > 0$, *that is,* $\mathcal{F}$ *is* m' *regular for all* $m' \geq m$,
2. $H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(d+1))$ *is surjective for all* $d \geq m$.
3. $\mathcal{F}(d)$ *is globally generated for all* $d \geq m$, *and*

Proof. Last time we started the proof by showing that for a general hyperplane H, the restriction $\mathcal{F}_H$ of $\mathcal{F}$ to H is an m-regular sheaf on the projective space H. Now by induction, the conclusions of the proposition hold for $\mathcal{F}_H$ since it is supported on the one dimension lower projective space H. Now we twist to obtain an exact sequence

$$\ldots \to H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) \to H^i(\mathbb{P}^n_k, \mathcal{F}(m+1-i)) \to H^i(\mathbb{P}^n_k, \mathcal{F}(m+1-i)) \to \ldots.$$

Now the last term is zero by conclusion (1) applied to $\mathcal{F}_H$ and the first term is zero by assumption so the middle term is zero, i.e., $\mathcal{F}$ is $(m+1)$-regular. Now we induct on m to see it is m'-regular for all $m' \geq m$.[18] This proves (1).

Next, consider the diagram

$$\begin{tikzcd}
H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(1)) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & H^0(H, \mathcal{F}_H(d)) \otimes H^0(H, \mathcal{O}_H(1)) \arrow[d, "\delta"] \\
H^0(\mathbb{P}^n_k, \mathcal{F}(d+1)) \arrow[r, "\beta"'] & H^0(H, \mathcal{F}_H(d+1))
\end{tikzcd}$$

where the horizontal maps are restriction to H and suppose $d \geq m$. Now the restriction $H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \to H^0(H, \mathcal{F}_H(d))$ is surjective since $H^1(\mathbb{P}^n_k, \mathcal{F}(d)) = 0$ by conclusion (1) thus α is surjective. For the same reason, β is surjective. Moreover, δ is surjective by conclusion (2) for $\mathcal{F}_H$. Thus $\beta \circ \gamma$ is surjective but the kernel of β is exactly the image of $\gamma(- \otimes h)$: $H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(d+1))$ where $h \in H^0(\mathbb{P}^n_k, \mathcal{O}(1))$ is the defining equation of H by the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{F}(d) \to \mathcal{F}(d+1) \to \mathcal{F}_H(d+1) \to 0$$

induced by multiplication by h. Thus $\ker(\beta)$ is contained in the image of γ so γ must be surjective and $\mathcal{F}$ satisfies (2).

Finally, the global generation of $\mathcal{F}(d)$ is equivalent to the fact that for each point $x \in \mathbb{P}^n_k$, there exists a collection of section $s_i \in H^0(\mathbb{P}^n_k, \mathcal{F}(d))$ such that $\bar{s}_i = s_i(x) \in \mathcal{F}(d) \otimes k(x)$ span the fiber $\mathcal{F}(d) \otimes k(x)$. By (2), we have a surjection

$$H^0(\mathbb{P}^n_k, \mathcal{F}(d)) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}(a)) \to H^0(\mathbb{P}^n_k, \mathcal{F}(d+a))$$

for all $a \geq 1$. For large enough $a \gg 0$, $\mathcal{F}(d+a)$ is globally generated by Serre vanishing so for each $x \in X$, there exists such sections $s_i \in H^0(\mathbb{P}^n_k, \mathcal{F}(d+a))$ whose values at x span the fiber, but since every such section comes from multiplying a section of $\mathcal{F}(d)$ by a homogeneous polynomial, there must be sections of $\mathcal{F}(d)$ spanning the fiber at x so $\mathcal{F}(d)$ is globally generated.

□

[18] Note here we have used extensively that coherent cohomology is preserved by closed embeddings so that the cohomology of $\mathcal{F}_H$ on $\mathbb{P}^n_k$ is the same as that on H.

Corollary 9.2. *Suppose*

$$0 \to \mathcal{F}'' \to \mathcal{F} \to \mathcal{F}' \to 0$$

is a short exact sequence of coherent sheaves. Suppose $\mathcal{F}''$ is $(m+1)$-regular and $\mathcal{F}$ is m-regular. Then $\mathcal{F}'$ is m-regular. In particular, all the three sheaves are in fact $(m+1)$-regular.

Proof. Consider the long exact sequence

$$\ldots \to H^i(\mathcal{F}(m-i)) \to H^i(\mathcal{F}'(m-i)) \to H^{i+1}(\mathcal{F}''(m-i)) \to \ldots.$$

The first term is vanishes since $\mathcal{F}$ is m-regular and the last term vanishes since $\mathcal{F}''$ is $(m+1)$-regular so the middle term vanishes. The final conclusion follows from the proposition. $\square$

Now that we have the language of CM regularity, we can state the uniform CM regularity theorem in its usual form and sketch the proof.

Theorem 9.3. *(Uniform Castelnuovo-Mumford regularity) For any polynomial P and integers m, n, there exists an integer $N = N(P, m, n)$ such that for any field k and any coherent subsheaf $\mathcal{F} \subset \mathcal{O}_{\mathbb{P}^n_k}^{\oplus m}$ with Hilbert polynomial P, $\mathcal{F}$ is N-regular.*

Proof. We will induct on n. Let H be a general hypreplane section as before and consider the sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0.$$

Now $\mathcal{F}_H \subset \mathcal{O}_H^{\oplus m}$ and the Hilbert polynomial of $\mathcal{F}_H$ is given by $P(d) - P(d-1)$ which depends only on P so by induction, there exists an N_1 depending only on P, m, and n such that $\mathcal{F}_H$ is N_1-regular.

Now consider the long exact sequence

$$\ldots \to H^{i-1}(\mathcal{F}_H(d+1)) \to H^i(\mathcal{F}(d)) \to H^i(\mathcal{F}(d+1)) \to H^i(\mathcal{F}_H(d+1)) \to \ldots.$$

For all $i \geq 2$ and $d \geq N_1 - i$, the terms with H vanish by conclusion (1) of the proposition. Thus $H^i(\mathcal{F}(d)) \to H^i(\mathcal{F}(d+1))$ is an isomorphism in this range. By Serre vanishing, these cohomology groups also vanish for d large enough so we get that $H^i(\mathcal{F}(d)) = 0$ for $i \geq 2$ and $d \geq N_1 - i$.

We need to control the groups $H^1(\mathcal{F}(d))$. Consider the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} = \mathcal{O}_{\mathbb{P}^n_k}^{\oplus m} \to \mathcal{Q} \to 0.$$

Then $\mathcal{Q}$ has Hilbert polynomial $P'(d) = m\binom{n+d}{d} - P(d)$. By the long exact sequence of cohomology and the fact that $H^i(\mathcal{E}(a)) = 0$ for all $i > 0$ and $a > 0$, the vanishing of $H^i(\mathcal{F}(d))$ for $i \geq 2$ and $d \geq N_1 - i$ implies the vanishing $H^i(\mathcal{Q}(d)) = 0$ for all $i \geq 1$ and $d \geq N_1 - i$. In particular $\mathcal{Q}$ is N_1-regular. Then $H^0(\mathcal{Q}(d))$ surjects onto $H^1(\mathcal{F}(d))$ and has rank given by $P'(d)$ for all $d \geq N_1 - 1$. Thus

$$\dim H^1(\mathcal{F}(d)) \leq P'(d)$$

so we have uniform control on $H^1(\mathcal{F}(d))$. We conclude by the following lemma.

Lemma 9.4. *The sequence $\{\dim H^1(\mathcal{F}(d))\}$ for $d \geq N_1 - 1$ is strictly decreasing to 0.*

Proof. Consider the long exact sequence associated to

$$0 \to \mathcal{F}(d) \to \mathcal{F}(d+1) \to \mathcal{F}_H(d+1) \to 0.$$

Since $\mathcal{F}_H$ is N_1-regular, we have $H^1(\mathcal{F}_H(d)) = 0$ for all $d \geq N_1 - 1$ and so $H^1(\mathcal{F}(d)) \to H^1(\mathcal{F}(d+1))$ is surjective. Thus the sequence is weakly decreasing. Suppose that for some d_0, $H^1(\mathcal{F}(d_0)) \cong H^1(\mathcal{F}(d_0+1))$. The previous map

$$\varphi_{d_0} : H^0(\mathcal{F}(d_0+1)) \to H^0(\mathcal{F}_H(d_0+1))$$

is surjective. Since $\mathcal{F}_H$ is N_1-regular, then the map

$$H^0(\mathcal{F}_H(d_0+1)) \otimes H^0(\mathcal{O}_H(1)) \to H^0(\mathcal{F}_H(d_0+2))$$

is surjective and by commutativity of the diagram

$$\begin{tikzcd} H^0(\mathcal{F}(d_0+1)) \otimes H^0(\mathcal{O}(1)) \arrow[r] \arrow[d] & H^0(\mathcal{F}_H(d_0+1)) \otimes H^0(\mathcal{O}_H(1)) \arrow[d] \\ H^0(\mathcal{F}(d_0+2)) \arrow[r, "\varphi_{d_0+1}"] & H^0(\mathcal{F}_H(d_0+2)) \end{tikzcd}$$

we conclude that φ_{d_0+1} is surjective. Thus $H^1(\mathcal{F}(d_0+1)) \cong H^1(\mathcal{F}(d_0+2))$ by the long exact sequence and so on. It follows that if $\dim H^1(\mathcal{F}(d_0)) = \dim H^1(\mathcal{F}(d_0+1))$ for some d_0, then $\dim H^1(\mathcal{F}(d_0)) = H^1(\mathcal{F}(d))$ for all $d \geq d_0$. On the other hand, this vanishes for $d \gg 0$ and so $H^1(\mathcal{F}(d_0)) = 0$. Thus the sequence must strictly decrease until it hits zero.

□

Now by the monotonicity of the sequence above we see that if $N_2 := \dim H^1(\mathcal{F}(N_1 - 1))$, then $H^1(\mathcal{F}(d)) = 0$ for all $d \geq N_1 - 1 + N_2$. Now $N_2 \leq P'(N_1 - 1)$ by the previous discussion and so $H^1(\mathcal{F}(d)) = 0$ for all $d \geq N_1 - 1 + P'(N_1 - 1)$ and so $\mathcal{F}$ is $N_1 - P'(N_1 - 1)$ regular. This quantity depends only on P, m and n and so we are done.

□

9.2 Flattening stratifications

We will now address the existence of flattening stratifications. Recall the statement.

Theorem 9.5. *(Flattening stratification) Let $f : X \to S$ be a projective morphism over a Noetherian scheme S and let $\mathcal{F}$ be a coherent sheaf on X. For every polynomial P there exists a locally closed subscheme $i_P : S_P \subset S$ such that a morphism $\varphi : T \to S$ factors through S_P if and only if $\varphi^*\mathcal{F}$ on $T \times_S X$ is flat over T with Hilbert polynomial P. Moreover, S_P is nonempty for finitely many P and the disjoint union of inclusions*

$$i : S' = \bigsqcup_P S_P \to S$$

induces a bijection on the underlying set of points. That is, $\{S_P\}$ is a locally closed stratification of S.

Let us first consider the special case where f is the identity map $S \to S$ so that $\mathcal{F}$ is a coherent sheaf on S. Then $\mathcal{F}$ is flat if and only if it is locally free and the Hilbert polynomial of the fiber $\mathcal{F}_s$ is simply its dimension $\dim_{k(s)} \mathcal{F}_s$ over the residue field.

Proposition 9.6. *Let $\mathcal{F}$ be a coherent sheaf on S Noetherian. Then there exists a finite locally closed stratification $\{S_d\}$ of S such that $\mathcal{F}|_{S_d}$ is locally free of rank d. Moreover, for any locally Noetherian scheme T, a morphism $\varphi : T \to S$ factors as $T \to S_d \subset S$ if and only if $\varphi^*\mathcal{F}$ is locally free of rank d.*

Proof. First, note that by the universal property of the strata S_d, they are unique. In particular, if $I \subset S$ is an open subset, then the stratum U_d for $\mathcal{F}|_U$ is the pullback of S_d, if it exists, to U. Thus, if we prove the proposition for an open affine cover of S, it will follow for S. Therefore, without loss of generality, we may replace S by an affine open Spec $A \subset S$ and suppose that $\mathcal{F}$ is the coherent sheaf associated to a finitely generated module M.

Let $s \in S$ and suppose that the rank of the fiber $\mathcal{F}_s = M \otimes k(s)$ is d. By Nakayama's lemma, we may lift the d generators of $M \otimes k(s)$ to d sections $A^{\oplus d} \to M$ which, after shrinking to a smaller open subset of Spec A, we may suppose is a surjective map. Thus we get a resolution

$$A^{\oplus e} \to A^{\oplus d} \to M \to 0.$$

By construction, the last map is an isomorphism after tensoring with $k(s)$, thus we have $\psi_{ij}(s) = 0$ for all (i,j), where the first map is given by the matrix (ψ_{ij}). Now M is locally free if and only if it has constant fiber dimension d if and only if the functions ψ_{ij} vanish, $\psi_{ij} = 0$ for all (i,j). Thus we can consider the subscheme $S_d \subset S$ given by the vanishing of all these ψ_{ij}. It is a closed subscheme containing $s \in S$.

By right exactness of pullbacks, for any $\varphi : T \to S$, pullback of the above resolution gives us a resolution

$$\mathcal{O}_T^{\oplus e} \xrightarrow{(\varphi^*\psi_{ij})} \mathcal{O}_T^{\oplus d} \longrightarrow \varphi^*\mathcal{F} \longrightarrow 0.$$

It is clear that $\varphi^*\psi_{ij}(t) = 0$ if and only if $\psi_{ij}(s) = 0$ where $s = \varphi(t)$. On the other hand, $\varphi^*\mathcal{F}$ is locally free of rank d if and only if $\varphi^*\psi_{ij} = 0$ if and only if φ factors through S_d.

By construction, each $s \in S$ is in some stratum S_d, namely for $d = \dim M \otimes k(s)$. Finally, by Noetherian induction, the locally closed stratification $\{S_d\}$ is finite since the set of ranks of fibers of the coherent sheaf $\mathcal{F}$ on the noetherian S is finite.

□

10 Flattening stratifications, functoriality properties of Hilb and Quot

10.1 Flattening stratifications (cont.)

Theorem 10.1. *(Flattening stratification) Let $f : X \to S$ be a projective morphism over a Noetherian scheme S and let $\mathcal{F}$ be a coherent sheaf on X. For every polynomial P there exists a locally closed subscheme $i_P : S_P \subset S$ such that a morphism $\varphi : T \to S$ factors through S_P if and only if $\varphi^*\mathcal{F}$ on $T \times_S X$ is flat over T with Hilbert polynomial P. Moreover, S_P is nonempty for finitely many P and the disjoint union of inclusions*

$$i : S' = \bigsqcup_P S_P \to S$$

induces a bijection on the underlying set of points. That is, $\{S_P\}$ is a locally closed stratification of S.

We began last time by proving the following special case corresponding to $X = S$.

Proposition 10.2. *Let $\mathcal{F}$ be a coherent sheaf on S Noetherian. Then there exists a finite locally closed stratification $\{S_d\}$ of S such that $\mathcal{F}|_{S_d}$ is locally free of rank d. Moreover, for any locally Noetherian scheme T, a morphism $\varphi : T \to S$ factors as $T \to S_d \subset S$ if and only if $\varphi^*\mathcal{F}$ is locally free of rank d.*

Now the idea of the proof in general is to use the following result from lecture 5.

Corollary 10.3. *Let $f : X \to S$ be a projective morphism with S noetherian. If $\mathcal{F}$ is a coherent sheaf on X then $\mathcal{F}$ is flat over S if and only if $f_*\mathcal{F}(d)$ is a locally free sheaf of finite rank for all $d \gg 0$.*

The idea of the proof then is to apply the proposition to the sheaves $f_*\mathcal{F}(d)$ and find a stratification that is universal for these pushforwards being locally free. Then by the corollary and some base-change arguments, this stratification will be universal for $\mathcal{F}$ being flat.

Proof. (flattening stratifications in general)

The first step of the proof is to bound the Hilbert polynomials of the fibers of $\mathcal{F}$ using the generic freeness theorem (from problem set 1).

Theorem 10.4. *Let A be noetherian integral domain, B a finitely generated A-algebra, and M a finite B-module. Then there exists an $f \in A$ such that M_f is a free A_f-module.*

Now we will use this to produce a finite stratification of S into reduced locally closed subschemes $V_i \subset S$ for $i = 1, \dots, m$ such that $\mathcal{F}|_{V_i}$ is flat over V_i (via the pullback $X_{V_i} \to V_i$ of $X \to S$). Toward this end, we may take S_{red} and assume that S is reduced. Let $\cup Y_j = S$ be the decomposition into irreducible components and fix a component Y_0. Let $U_0 \subset Y_0$ be the complement of where Y_0 meets Y_i:

$$U_0 = Y_0 \setminus \{Y_i \cap Y_1\}_{i \neq 0}.$$

Now U_0 is an integral scheme. Let Spec $A \subset U_0$ be a dense open affine subscheme where A is an integral domain. Then we may apply the generic freeness theorem to the pullback $\mathcal{F}_A$ to $X_A \to$ Spec A. This gives us an open subscheme Spec $A_f \subset$ Spec $A \subset U_0$ such that $\mathcal{F}_{A_f}$ is flat over Spec A_f. Let $V_0 =$ Spec A_f and let $S_1 = S \setminus V_0$ its closed complement. Now we repeat in this way to produce an integral open subset $V_i \subset S_i$ where $\mathcal{F}_{V_i}$ is flat over V_i. By Noetherian induction, this process terminates so we get a stratification of S by locally closed subsets $\{V_0, \dots, V_m\}$ with the desired property.

Let us denote by $f_i : X_i \to V_i$ the pullback of $X \to S$ and by $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to $\mathcal{F}_i$. Now by construction, $\mathcal{F}_i$ is flat over V_i and by constancy of Hilbert polynomials, for each $s \in V_i$, the Hilbert polynomial $P_{\mathcal{F}_s}(d)$ is constant say equal to a polynomial P_m. Thus there are finitely many polynomials $\{P_1(d), \dots, P_m(d)\}$ such that for each $s \in S$, $P_{\mathcal{F}_s}(d) = P_m(d)$ for some m. Next, by Serre vanishing, there exists a d_i such that

$$R^j(f_i)_*\mathcal{F}_i(d) = 0$$

for all $d \geq d_i$. In this case, by cohomology and base change, $(f_i)_*\mathcal{F}_i(d)$ is locally free of rank $P_{\mathcal{F}_s}(d)$ and the basechange map

$$(f_i)_*\mathcal{F}_i(d) \otimes k(s) \to H^0(X_s, \mathcal{F}_s(d))$$

is an isomorphism. Letting $N = \max\{d_i\}$, we now have the following:

(1) There are finitely many polynomials $P_1, \dots, P_m$ such that for each $s \in S$, $P_{\mathcal{F}_s}(d) = P_i(d)$ for some i;

(2) $H^i(X_s, \mathcal{F}_s(d)) = 0$ for all $d \geq N$;

(3) $f_*\mathcal{F}(d) \otimes k(s) \cong H^0(X_s, \mathcal{F}_s(d))$ has dimension $P_i(d)$ for all $d \geq N$.

Now we will construct the flattening stratification for $\mathcal{F}$ using properties $(1), (2)$ and (3). Note that the preliminary stratification into reduced strata V_i above was just an auxillary tool to prove properties $(1), (2)$ and (3).

Fix n such that $\deg P_{\mathcal{F}_s}(d) \leq n$ for all $s \in S$ which exists by (1).[19] We have the following fact.

Fact 10.5. *Let Pol_n be the set of polynomials over $\mathbb{Q}$ of degree at most n. Then for any N,*

$$Pol_n \to \mathbb{Z}^{n+1} \tag{13}$$

$$P \mapsto (P(N), P(N+1), \ldots, P(N+n)) \tag{14}$$

is a bijection.

Now we can apply the flattening stratification in the special case of the coherent sheaves $\{\mathcal{E}_i := f_*\mathcal{F}(N+i)\}_{i=0}^n$ on S. Thus for each i and e, we have a stratum $W_{i,e}$ that is universal for the property that $\mathcal{E}_i$ is locally free of rank e. In particular, for any $s \in W_{i,e}$, by the base change properties (2) and (3), we have $e = \mathrm{rk}\mathcal{E}_i|_{W_{i,e}} = P_{\mathcal{F}_s}(N+i)$. Now for any sequence $(e_0, \ldots, e_n) \in \mathbb{Z}^{n+1}$, which by the basic fact corresponds to a polynomial P, we have the scheme theoretic intersection

$$W_P^0 := \bigcap_{i=0}^{n} W_{i,e_i}.$$

By definition, a map $\varphi : T \to S$ factors through W_P^0 if and only if $\varphi^* f_*\mathcal{F}(N+i)$ is locally free of rank $e_i = P(N+i)$ for $i = 0, \ldots, n$. In particular, $s \in W_P^0$ if and only if $P_{\mathcal{F}_s}(d) = P(d)$ and so by finiteness of the Hilbert polynomials, $\{W_P^0\}$ is a finite locally closed stratification of S which has the correct closed points. However, we need to determine the correct scheme structure.

By the vanishing condition (2), we know that the formation of $f_*\mathcal{F}(N+a)$ is compatible with arbitrary base change for all $a \geq 0$. Now for each $d \geq 0$, let we can apply the flattening stratification to the sheaf $f_*\mathcal{F}(N+n+d)|_{W_P^0}$ to obtain a locally closed subscheme W_P^d universal for this sheaf being locally free of rank $P(N+n+d)$. Note that at every closed point of W_P^0, the rank of $f_*\mathcal{F}(N+n+d)$ is equal to $P(N+n+d)$ and so W_P^d has the same underlying reduced subscheme. In particular, it is actually a closed subscheme of W_P^0 and so is cut out by some ideal I_P^d. Now consider the chain

$$I_P^1 \subset I_P^1 + I_P^2 \subset \ldots.$$

By the Noetherian condition, this sequence stabilizes to some ideal I cutting out a closed subscheme $S_P \subset W_P^0$ with the same underlying reduced scheme. Equivalently, S_P is the scheme theoretic intersection of the W_P^d for all d.

Since $S_P \subset W_P^0$ is a homeomorphism of underlying topological spaces, then $\{S_P\}$ is a finite locally closed stratification of S. By definition, $\varphi : T \to S$ factors through S_P if and only if for all $a \geq 0$,

$$\varphi^* f_*\mathcal{F}(N+a)$$

[19] Actually we already knew this because $X \subset \mathbb{P}_S^n$ for some n and we can take this n. However, this is only because we are using a stronger version of projectivity in this class.

is locally free of rank $P(N+a)$ but by the base change property,

$$\varphi^* f_* \mathcal{F}(N+a) = (f_T)_* \mathcal{F}_T(N+a).$$

Therefore, $\varphi : T \to S$ factors through S_P if and only if $(f_T)_* \mathcal{F}_T(N+a)$ is locally free of rank $P(N+a)$ for all $a \geq 0$ if and only if $\mathcal{F}_T$ is flat over T with Hilbert polynomial $P(d)$ by the corollary from lecture 5. Thus S_P has the required universal property.

□

10.2 Functoriality properties of Hilbert schemes

10.2.1 Closed embeddings

We proved the following in Step 1 of the construction of Hilbert schemes but its useful enough to make explicit.

Proposition 10.6. *Let $i : X \to Y$ be a closed embedding of projective S-schemes for S noetherian. Then there is a natural closed embedding $i_* : \mathrm{Hilb}^P_{X/S} \to \mathrm{Hilb}^P_{Y/S}$.*

10.2.2 Base-change

Let $f : X \to S$ be a projective morphism to a noetherian scheme and $S' \to S$ any morphism. Consider the pullback

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r] & S. \end{tikzcd}$$

Note that f' is projective. Suppose $\mathcal{E}$ is any coherent sheaf on X and let $\mathcal{E}'$ be the pullback of $\mathcal{E}$ to X'. The following is clear from the definition of the Quot functor.

Proposition 10.7. *The following is a pullback square of functors.*

$$\begin{tikzcd} \mathcal{Q}_{\mathcal{E}',X'/S'} \arrow[r] \arrow[d] & \mathcal{Q}_{\mathcal{E},X/S} \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

In particular, we conclude that

$$\mathrm{Quot}_{\mathcal{E}',X'/S'} \cong \mathrm{Quot}_{\mathcal{E},X/S} \times_S S'$$

10.2.3 Pullbacks

Let $f : X \to Y$ be a flat morphism of projective S-schemes. Since flatness and the Hilbert polynomial are stable under base-change, we have the following:

Proposition 10.8. *There is a pullback morphism*

$$f^* : \mathrm{Hilb}_{Y/S} \to \mathrm{Hilb}_{X/S}$$

induced by taking a closed subscheme $Z \subset T \times_S Y$ to the pullback $g^{-1}(Z) \subset T \times_S X$.

Proof. Since $f : X \to Y$ is flat, then so is $f_T : T \times_S X \to T \times_S Y$ and so $g^{-1}(Z) \to Z$ is also flat. Then the composition $g^{-1}(Z) \to Z \to T$ is also flat and so is an element of $\mathrm{Hilb}^P_{X/S}(T)$.

□

10.2.4 Pushforwards

Let $f : X \to Y$ be a morphism of projective S-schemes. We can ask if there is a "push-forward" map f_* in general. We saw above there is if f is a closed embedding.

Let us consider the diagram

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "p"'] & T \times_S X \arrow[r, "f_T"] \arrow[d] & T \times_S Y \arrow[dl] \\ & T & \end{tikzcd}$$

where $p : Z \to T$ is an element of $\mathrm{Hilb}_{X/S}(T)$. The question is whether the composition $f_T \circ i : Z \to T \times_S Y$ is a closed embedding and thus gives an element of $\mathrm{Hilb}_{Y/S}(T)$ which we can use to define $f_*(p : Z \to T)$. Of course this won't be true in general but it turns out it holds on an open subscheme of $\mathrm{Hilb}_{X/S}$.

Theorem 10.9. *There exists a universal open subscheme* $\mathrm{Hilb}_{X \to Y/S} \subset \mathrm{Hilb}_{X/S}$ *on which f_* defined by $Z \subset T \times_S X$ maps to $f_T \circ i : Z \to T \times_S Y$ gives a morphism*

$$f_* : \mathrm{Hilb}_{X \to Y/S} \to \mathrm{Hilb}_Y.$$

Proof. Equivalently, we want to show there exists an open subscheme $U \subset \mathrm{Hilb}_{X/S}$ such that a morphism $\varphi : T \to \mathrm{Hilb}_{X/S}$ coresponding to a closed subscheme $i : Z \subset T \times_S X$ factors through U if and only if the composition $f_T \circ i : Z \to T \times_S Y$ is a closed embedding. Indeed if such a subscheme exists its clearly universal and then $f_T \circ i$ is an element of $\mathrm{Hilb}_{Y/S}$ such $Z \to T$ is flat and proper regardless of embedding. Now the existence of this $U = \mathrm{Hilb}_{X \to Y/S}$ follows from applying the next lemma to the universal map $g := f_{\mathrm{Hilb}_{X/S} \circ i}$.

$$\begin{tikzcd} \mathrm{Hilb}_{X/S} \times_S X \arrow[dr] & \mathcal{Z} \arrow[l, hook', "i"'] \arrow[r, "g"] \arrow[d] & \mathrm{Hilb}_{X/S} \times_S Y \arrow[dl] \\ & \mathrm{Hilb}_{X/S} & \end{tikzcd}$$

□

Lemma 10.10. *Let $p : Z \to T$ and $q : Y \to T$ be projective T-schemes with p flat and let $g : Z \to T$ a morphism. Then there exists an open subscheme $U \subset T$ such that $\varphi : T' \to T$ factors through U if and only if $\varphi^* g : Z_{T'} \to Y_{T'}$ is a closed embedding.*

Proof. Since Z and Y are projective, the morphism g is proper and so we may replace Y by the image of g and assume without loss of generality that Y is the scheme theoretic image of g. Then g is a closed embedding if and only if it is an isomorphism and so the result reduces to the lemma proved during our study of Hom-schemes.

□

11 Weil restriction, quasi-projective schemes

11.1 Weil restriction of scalars

Let $S' \to S$ be a morphism of schemes and $X \to S'$ an S'-scheme. The *Weil restriction of scalars* $R_{S'/S}(X)$, if it exists, is the S-scheme whose functor of pointsis given by

$$\mathrm{Hom}_S(T, R_{S'/S}(X)) = \mathrm{Hom}_{S'}(T \times_S S', X).$$

Classically, the restriction of scalars was studied in the case that $S' \to S$ is a finite extension of fields $k \subset k'$. In this case, $R_{S'/S}(X)$ is roughly given by taking the equations of X/k' and viewing that as equations over the smaller field k.

Theorem 11.1. *Let $f : S' \to S$ be a flat projective morphism over S Noetherian and let $g : X \to S'$ be a projective S'-scheme. Then the restriction of scalars $R_{S'/S}(X)$ exists and is isomorphic to the open subscheme $\mathrm{Hilb}^P_{X\to S'/S} \subset \mathrm{Hilb}^P_{X/S}$ where P is the Hilbert polynomial of $f : S' \to S$.*

Proof. Note that $\mathrm{Hilb}^P_{S'/S} = S$ with universal family given by $f : S' \to S$. Then on $\mathrm{Hilb}^P_{X\to S/S'}$ we have a well defined pushforward

$$g_* : \mathrm{Hilb}^P_{X\to S'/S} \to S$$

given by composing a closed embedding $i : Z \subset T \times_S X$ with $g_T : T \times_S X \to T \times_S S'$. On the other hand, since the Hilbert polynomials agree, then the closed embedding $g_T \circ i : Z \to T \times_S S'$ must is a fiberwise isomorphism and thus an isomorphism. Therefore, $g_T \circ i : Z \to T \times_S X = (T \times_S S') \times_{S'} X$ defines the graph of an S' morphism

$$T \times_S S' \to X.$$

This gives us a natural transformation

$$\mathrm{Hilb}^P_{X\to S'/S} \to \mathrm{Hom}_{S'}(- \times_S S', X). \tag{15}$$

On the other hand, a T-point of the right hand side, $\varphi \in \mathrm{Hom}_{S'}(T \times_S S', X)$, gives us a graph

$$\Gamma_\varphi \subset T \times_S S' \times_{S'} X$$

which maps isomorphically to $T \times_S S'$. Since $S' \to S$ is flat, so is $\Gamma_\varphi \to T$ and thus defines an element of $\mathrm{Hilb}^P_{X\to S'/S}(T)$ giving an inverse to (15).

□

11.2 Hilbert and Quot functors for quasi-projective schemes

Next, we will generalize Hilbert and Quot functors to quasi-projective morphisms $f : X \to S$. Given a coherent sheaf $\mathcal{E}$ on X, we define the Quot functor just as before.

$$\mathcal{Q}_{\mathcal{E},X/S}(T) = \{q : \mathcal{E}_T \twoheadrightarrow \mathcal{F} \mid \mathcal{F} \text{ flat and proper over } T\}$$

Theorem 11.2. *Let $f : X \to S$ be a quasi-projective morphism over S noetherian and let $\mathcal{E}$ be a coherent sheaf on X. Then*

$$\mathcal{Q}_{\mathcal{E},X/S} = \bigsqcup \mathcal{Q}^P_{\mathcal{E},X/S}$$

over Hilbert polynomials P and each component $\mathcal{Q}^P_{\mathcal{E},X/S}$ is representable by a quasi-projective scheme over S.

Proof. Since $f : X \to S$ is quasi-projective, there is a projective $g : Y \to S$ and an open embedding $i : X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

commutes.

Lemma 11.3. *There exists a coherent sheaf $\mathcal{E}'$ on Y such that $\mathcal{E}'|_X = \mathcal{E}$.*

Proof. Exercise. $\square$

Given an element $(q, \mathcal{F}) \in \mathcal{Q}^P_{\mathcal{E},X/S}(T)$, we can consider the composition $q' : \mathcal{E}' \to \mathcal{E}|_X \to \mathcal{F}$, an object of $\mathcal{Q}^P_{\mathcal{E}',Y/S}(T)$. This gives us a natural transformation

$$\mathcal{Q}^P_{\mathcal{E},X/S} \to \mathcal{Q}^P_{\mathcal{E}',Y/S}.$$

We wish to show this is an open embedding. This follows from the following lemma.

Lemma 11.4. *Let $p : Y \to S$ be a proper morphism, $Z \subset Y$ a closed subscheme, and $\mathcal{F}$ a coherent sheaf on Y. Then there exists an open subscheme $U \subset S$ such that a morphism $\varphi : T \to S$ factors through U if and only if the support of the sheaf $\mathcal{F}_T$ on Y_T is disjoint from the closed subscheme Z_T.*

Proof. Exercise. $\square$

Now we apply the lemma to $\mathcal{F}^P_{\mathcal{E}',Y/S}$ the universal sheaf on $p : \mathrm{Quot}^P_{\mathcal{E}',Y/S} \times_S Y \to \mathrm{Quot}^P_{\mathcal{E}',Y/S}$ with the closed subscheme Z being the complement of the open subscheme $\mathrm{Quot}^P_{\mathcal{E}',Y/S} \times_S X$. Then we get an open

$$U \subset \mathrm{Quot}^P_{\mathcal{E}',Y/S}$$

such that $\varphi : T \to \mathrm{Quot}^P_{\mathcal{E}',Y/S}$ factors through U if and only if the support of $\mathcal{F}_T$ lies in X_T. This is precisely the subfunctor $\mathcal{Q}^P_{\mathcal{E},X/S}$ so we conclude this subfunctor is representable by the quasi-projective scheme U. $\square$

11.2.1 Hironaka's example

The Hilbert functor need not be representable by a scheme outside of the quasi-projective case. Indeed we have the following example due to Hironaka. For simplicity, we will work over the complex numbers.

Let X be a smooth projective 3-fold with two smooth curves C and D intersecting transversely in two points x and y. Consider the open subset $U_x = X \setminus x$ and let V_x be the variety obtained by first blowing up $C \setminus x$ inside U_x, then blowing up the strict transform of $D \setminus x$ inside the first blowup. Similarly, let $U_y = X \setminus y$ and let V_y be obtained by first blowing up $D \setminus y$ then blowing up the strict transform of $C \setminus y$.

Let π_x, π_y be the natural morphisms from the blowups to the open subsets of X. Then by construction $\pi_x^{-1}(U_x \cap U_y) \cong \pi_y^{-1}(U_x \cap U_y)$ so we can glue them together to obtain a variety Y with a morphism $\pi : Y \to X$.

Claim 11.5. *The variety Y is proper but not projective.*

Proof. The morphism π is proper by construction so Y is proper. Let l and m be the preimages of a general point on C and D respectively. These are algebraic equivalence classes of curves on Y. The preimage $\pi^{-1}(x)$ is a union of two curves l_x and m_x where $m \sim_{alg} m_x$ and $l \sim_{alg} l_x + m_x$. Similarly, $\pi^{-1}(y)$ is a union of l_y and m_y where $l \sim_{alg} l_y$ and $m \sim_{alg} l_y + m_y$. Putting this together, we get $l_x + m_y \sim_{alg} 0$. But l_x and m_y are irreducible curves so if Y is projective, it would have an ample line bundle which has positive degree on $l_x + m_y$, a contradiction.

$\square$

Now we pick X, C and D such that X has a a fixed point free involution τ which sends C to D and x to y. Then τ lifts to an involution on Y. We will study quotients of varieties in more detail later, but for now we can consider the quotient Y/τ as a complex manifold.

Claim 11.6. *The quotient Y/τ is not an algebraic variety.*

Proof. Let l', m', l_0, and m_0 be the images under $Y \to Y/\tau$ of l, m, l_x and m_x respectively, viewed as homology classes. Note that l_y and m_y map to the same classes. Then the algebraic equivalences show the following equalities of classes in homology:

$$[m_0] = [m'] = [l'] = [m_0] + [l_0]$$

which implies that the homology class of l_0 vanishes.

Suppose Y/τ is a variety let $t \in l_0$ be a point. Let U be an affine open neighborhood of t in Y/τ. Pick an irreducible surface $S_0 \subset U$ passing through t but not containing $l_0 \cap U$, and S be the closure of S_0 in Y/τ. Then on the one hand, the intersection number $S \cap l_0 > 0$ since its the intersection of two irreducible subvarieties meeting at a finite number of points, but on the other hand $S \cap l_0 = 0$ since $[l_0] = 0$, a contradiction. □

Claim 11.7. *The Hilbert functor $\mathcal{H}_{Y/\mathbb{C}}$ is not representable by a scheme.*

Proof. Let $R \subset Y \times Y$ be the closed subset defined as

$$R = Y \times_{Y/\tau} Y.$$

We can consider the action of $G = \langle \tau \rangle \cong \mathbb{Z}/2\mathbb{Z}$ on Y as a morphism $m : G \times Y \to Y$. There is also a projection $p_Y : G \times Y \to Y$. Then the product of these two maps gives a proper morphism $G \times Y \to Y \times Y$ that is an isomorphism onto R. In particular, the projection $R \to Y$ is flat and proper and so $R \subset Y \times Y$ defines a flat family of closed subschemes of Y parametrized by Y, i.e., a morphism

$$Y \to \mathcal{H}_{Y/\mathbb{C}}.$$

Suppose the latter is representable by a scheme $\mathrm{Hilb}_{Y/\mathbb{C}}$. Since Y is proper so is $\mathrm{Hilb}_{Y/\mathbb{C}}$[20] Then $Y \to \mathrm{Hilb}_{Y/\mathbb{C}}$ is a proper morphism and so its image Z is a closed subscheme of $\mathrm{Hilb}_{Y/\mathbb{C}}$. On the other hand, the underlying map complex spaces $Y \to Z$ is exactly the quotient map $Y \to Y/\tau$ since the fibers of $R \to Y$ are exactly the orbits of τ and so $Y \to \mathrm{Hilb}_{Y/\mathbb{C}}$ sends a point to its orbit. This contradicts the fact that Y/τ is not a scheme.

□

12 The Picard functor

12.1 Picard groups

Our goal now is to study the representability properties of the Picard functor. Recall the definition of the Picard group.

Definition 12.1. *Let X be a scheme. The Picard group* $\mathrm{Pic}(X)$ *is the set of line bundles (or invertible sheaves) on X with group operation given by tensor product.*

Recall the following well known fact.

Lemma 12.2. *There is a canonical isomorphism* $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$.

[20] Note that the proof of properness was purely functorial and did not require representability or projectivity.

12.2 Picard functors

Let $f : X \to S$ be a scheme over S. We want to upgrade the Picard group to a functor on Sch_S. Since line bundles pull back to line bundles, we have a natural functor given by

$$T \mapsto \mathrm{Pic}(X_T).$$

Thus functor is the *absolute Picard functor*.

It is natural to ask if this functor is representable. It turns out this is not the case.

Claim 12.3. *The absolute Picard functor is not a sheaf.*

Proof. Let L be a line bundle on T such that f_T^*L is not trivial. Let $\{U_\alpha\}$ be an open cover of T that trivializes the bundle L. Then the pullback of f_T^*L to X_{U_α} is trivial and so L is in the kernel of the map

$$\mathrm{Pic}(X_T) \to \mathrm{Pic}\left(\bigsqcup_\alpha X_{U_\alpha}\right) = \prod_\alpha \mathrm{Pic}(X_{U_\alpha}).$$

□

Since the problem is the line bundles pulled back from the base scheme T, one proposed way to fix this is the following definition of the relative Picard functor.

Definition 12.4. *The relative Picard functor* $\mathrm{Pic}_{X/S} : Sch_S \to Set$ *is given by*

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T)/f_T^*\mathrm{Pic}(T)$$

where $f_T^* : \mathrm{Pic}(T) \to \mathrm{Pic}(X)$ *is the pullback map.*

The representability of $\mathrm{Pic}_{X/S}$ is still a subtle question and even the sheaf property is subtle and doesn't always hold. However, it does under some assumptions.

Definition 12.5. *We call a proper morphism $f : X \to S$ an algebraic fiber space if the natural map $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism. We say it is a universal algebraic fiber space if for all $T \to S$, the natural morphism $\mathcal{O}_T \to (f_T)_*\mathcal{O}_{X_T}$ is an isomorphism.* [21]

Proposition 12.6. *Suppose $f : X \to S$ is a universal algebraic fiber space and that there exists a section $\sigma : S \to X$.*[22] *Then* $\mathrm{Pic}_{X/S}$ *is a Zariski sheaf.*

Proof. Let us consider the Zariski sheafification $\mathrm{Pic}_{X/S,\, Zar}$ of $\mathrm{Pic}_{X/S}$. This is the sheafification of the functor

$$T \mapsto H^1(X_T, \mathcal{O}_{X_T}^*).$$

If we restrict this to the category of open subschemes of a fixed T, what we get is the sheaf

$$R^1(f_T)_*\mathcal{O}_{X_T}^*.\text{[23]}$$

[21] Some sources in the literature require algebraic fiber spaces to be projective not just proper morphisms.

[22] That is, σ is a morphism with $f \circ \sigma = id_S$.

[23] Some take this as the definition of the higher direct image functors and then you have to prove their properties, otherwise you can define the higher direct image functors as derived functors and check they agree with this sheafification.

Thus $\mathrm{Pic}_{X/S,\ Zar}(T)$ is the global sections of the sheaf $R^1(f_T)_*\mathcal{O}^*_{X_T}$ on T:

$$\mathrm{Pic}_{X/S,\ Zar}(T) = H^0(T, R^1(f_T)_*\mathcal{O}^*_{X_T}).$$

Now consider the Leray spectral sequence

$$H^j(T, R^i(f_T)_*\mathcal{O}^*_{X_T}) \implies H^{i+j}(X_T, \mathcal{O}^*_{X_T}).$$

There is a 5-term exact sequence associated to any spectral sequence which in this case is given by

$$\begin{aligned} 0 \to H^1(T, (f_T)_*\mathcal{O}^*_{X_T}) \to H^1(X, \mathcal{O}^*_{X_T}) \to H^0(T, R^1(f_T)_*\mathcal{O}^*_{X_T}) \\ \to H^2(T, (f_T)_*\mathcal{O}^*_{X_T}) \to H^2(X, \mathcal{O}^*_{X_T}). \end{aligned}$$

Since f is a universal algebraic fiber space, $(f_T)_*\mathcal{O}^*_{X_T} \cong \mathcal{O}^*_T$ and so the first map in the exact sequence can be identified with the pullback

$$f_T^* : \mathrm{Pic}(T) \to \mathrm{Pic}(X_T).$$

Thus we have an exact sequence

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X/S,\ Zar}(T).$$

We want to show that the last map above is surjective so that

$$\mathrm{Pic}_{X/S,\ Zar}(T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T) = \mathrm{Pic}_{X/S}(T).$$

By exactness, it suffices to show that

$$H^2(T, (f_T)_*\mathcal{O}^*_{X_T}) \to H^2(X, \mathcal{O}^*_{X_T})$$

is injective. This map is given by pulling back by f_T. Since f_T has a section given by σ_T, then we have

$$\sigma_T^* \circ f_T^* = id : H^2(T, (f_T)_*\mathcal{O}^*_{X_T}) \to H^2(T, (f_T)_*\mathcal{O}^*_{X_T}).$$

Therefore f_T^* is injective. □

Remark 12.7. *In the simplest case when $S = \mathrm{Spec}\ k$ is the spectrum of a field, the condition that $f : X \to S$ is a unviersal algebraic fiber space is equivalent to X being geometrically connected and geometrically reduced. The condition that there exist a section $\sigma : S \to X$ is exactly saying that X has a k-rational point. Note that this is always true after a separable field extension of k, that is, it holds étale locally. This suggests that to study the relative Picard functor in greater generality, one should consider the sheafification of $\mathrm{Pic}_{X/S}$ in the étale topology. Indeed it turns out that in the most general setting one should consider the fppf[24] sheafification of $\mathrm{Pic}_{X/S}$. In the setting above where there exists a section, $\mathrm{Pic}_{X/S}$ is already an étale and even fppf sheaf. To avoid getting into details of Grothendieck topologies and descent theory at this point in the class, we will stick with the case where a section σ exists.*

[24] faithfully flat and of finite presentation

12.3 Some remarks and examples

Note that the relative Picard functor has the same k-points as the absolute Picard functor:

$$\mathrm{Pic}_X(k) = \mathrm{Pic}(X_k) = \mathrm{Pic}(X_k)/\mathrm{Pic}(\mathrm{Spec}\, k) = \mathrm{Pic}_{X/S}(k).$$

Thus the points of the relative Picard scheme of $f : X \to S$, if it exists, can be identified with line bundles on the fibers X_s of f. The difference between Pic_X and $\mathrm{Pic}_{X/S}$ is only in how we glue together fiberwise line bundles into line bundles on the total space X.

Even when $\mathrm{Pic}_{X/S}$ is a Zariski sheaf, it may still exhibit some pathologies.

Example 12.8. *(The Picard functor is not separated.) Let*

$$X = \{tf(x,y,z) - xyz = 0\} \subset \mathbb{P}^3_{\mathbb{A}^1_t}$$

be a family of cubic curves in the plane over $k = \bar{k}$ an algebraically closed field, where $f(x,y,z)$ is a generic cubic polynomial so that the generic fiber of the projection $f : X \to \mathbb{A}^1_t = S$ is smooth and irreducible. Note that f is a universal algebraic fuber space and we can pick $f(x,y,z)$ appropriately so that a section σ exists. The special fiber at $t = 0$, given by $V(xyz)$, is the union of three lines l_1, l_2 and l_3. We will show that in this case, $\mathrm{Pic}_{X/S}$ fails the valuative criteria for the property of being separated.

Indeed suppose L^0 is a line bundle on $X \setminus X_0$ viewed as an element of $\mathrm{Pic}_{X/S}(\mathbb{A}^1 \setminus 0)$ and suppose there exists some line bundle L on X such that $L|_{X \setminus X_0}$ gives the same element of $\mathrm{Pic}_{X/S}(\mathbb{A}^1 \setminus 0)$ as L^0. Explicitly, this means that there exists some line bundle G on $\mathbb{A}^1 \setminus 0$ such that

$$L|_{X \setminus X_0} \otimes f^* G = L^0.$$

Then we claim that the twist $L(l_i) = L \otimes \mathcal{O}_X(l_i)$ by a component l_i of the central fiber gives another element of $\mathrm{Pic}_{X/S}(\mathbb{A}^1 \setminus 0)$ extending L^0 that is not equal to L. In particular, the map $\mathrm{Pic}_{X/S}(\mathbb{A}^1) \to \mathrm{Pic}_{X/S}(\mathbb{A}^1 \setminus 0)$ is not injective and so the valuative criterion fails.

To verify the claim, note that $\mathcal{O}_X(l_i)|_{X \setminus X_0} \cong \mathcal{O}_{X_0}$ and so $L(l_i)$ does indeed give an extension of L^0 in $\mathrm{Pic}_{X/S}$. On the other hand, L and $L(l_i)$ give the same element of $\mathrm{Pic}_{X/S}(\mathbb{A}^1)$ if and only of $\mathcal{O}_X(l_i)$ is pulled back from $\mathbb{A}^1$ which does not hold since the restriction to the scheme theoretic fiber $\mathcal{O}_X(l_i)|_{f^{-1}}(\{t = 0\})$ is a nontrivial line bundle.

This example shows that to get a well behaved relative Picard functor, we should restrict to the case that the fibers of $f : X \to S$ are integral. Indeed if the fibers are integral, then any fiber component $\mathcal{O}_X(F)$ that we can twist by is pulled back from the base and so this problem doesn't occur.

Example 12.9. *(The Picard functor need not be universally closed.) Let $X = \{y^2 z - x^2(x - z) = 0\}$ in $\mathbb{P}^3_k$ with $S = \mathrm{Spec}\, k$ for $k = \bar{k}$. Then $f : X \to \mathrm{Spec}\, k$ is a universal algebraic fiber space and it has a section given by the rational point $[0,1,0]$. Consider the subscheme $D \subset \mathbb{A}^1 \times_k X$ given by the graph of the morphism*

$$\varphi : \mathbb{A}^1 \to X, \quad t \mapsto [t^2 + 1, t(t^2 + 1), 1].$$

Let $t_\pm = \pm\sqrt{(-1)}$, $U = \mathbb{A}^1 \setminus \{t_+, t_-\}$, and $X_U = X \setminus \{X_{t_+} \cup X_{t_-}\}$. Then $D|_U \subset X_U$ is contained in the regular locus of $X_U = U \times X$ and so the ideal sheaf of $D|_U$ is a line bundle denoted

$\mathcal{O}_{X_U}(-D|_U)$. If an extension of $\mathcal{O}_{X_U}(-D|_U)$ to $\mathbb{A}^1$ exists as an element of the relative Picard functor, then it must be represented by a line bundle on X, and in particular, it must be flat over $\mathbb{A}^1$. On the other hand, we know that the ideal sheaf $\mathcal{I}_D$ is a flat extension of $\mathcal{O}_{X_U}(-D|_U)$. One can check that if the extension of $\mathcal{O}_{X_U}(-D|_U)$ exists as an element of the relative Picard functor, it must be equal to $\mathcal{I}_D$ up to twisting by a line bundle on the base. Since $D \to \mathbb{A}^1$ is flat, $\mathcal{I}_D|_{t_\pm} = \mathcal{I}_{D_{t_\pm}}$ but $D_{t_\pm}$ is the closed point $[0,0,1]$. The completed local ring of $X_{t_\pm}$ at this point is given by $k[\![x,y]\!]/(xy)$ and it has maximal ideal (x,y) corresponding to the point $[0,0,1]$. It is easy to see that (x,y) is not *a free $k[\![x,y]\!]/(xy)$-module and thus $\mathcal{I}_D$ is not a line bundle and so no extension of $\mathcal{O}_{X_U}(-D|_U)$ as an element of the relative Picard functor can exist.*

In the above example, what goes wrong is that the flat limit of the given family of line bundles is not a line bundle, but rather the rank 1 torsion free sheaf $\mathcal{I}_{D_{t_\pm}}$. Thus suggests that at least in the case of an integral curve, one can compactify the Picard functor by allowing such sheaves. We will study this *compactified Picard scheme* in the case of integral curves lying on a smooth surface[25] later in the class.

12.4 Outline of the proof of the representability theorem

Our goal will be to prove the following theorem.

Theorem 12.10. *Let $f : X \to S$ be a flat projective morphism with integral fibers such that f is a universal algebraic fiber space and suppose there exists a section $\sigma : S \to X$.[26] Then the relative Picard functor $\mathrm{Pic}_{X/S}$ is a representable by a locally of finite type, separated S-scheme with quasi-projective connected components.*

The proof roughly proceeds in the following steps:

(I) Given a Cartier divisor, that is, a codimension one closed subschemes $D \subset X$ with locally free ideal sheaf $I_D \subset \mathcal{O}_X$, we can dualize to obtain a line bundle $L = I_D^{-1}$ with section $s : \mathcal{O}_X \to L$. This gives a set theoretic bijection

$$\{(L,s) \mid s : \mathcal{O}_X \to L \text{ is injective}\} \leftrightarrow \{\text{Cartier divisors}\}.$$

(II) We define a relative notion of Cartier divisors and prove that the moduli functor $CDiv_{X/S}$ of relative Cartier divisors is representable by an open subscheme of the Hilbert scheme $\mathrm{Hilb}_{X/S}$. In particular, we have a disjoint union

$$CDiv_{X/S} = \bigsqcup_P CDiv^P_{X/S}$$

over Hilbert polynomials where each component is quasi-projective.

(III) Using the bijection in (I), we construct a morphism of functors

$$CDiv^P_{X/S} \to \mathrm{Pic}^{P_1}_{X/S}$$

which on k-points is given by $(D \subset X_k) \mapsto L = I_D^{-1}$. Here P_1 is the Hilbert polynomial of I_D^{-1} which depends only on the Hilbert polynomial P of D and that of $f : X \to S$.

[25] so-called locally planar curves

[26] In fact all we will need is that $\mathrm{Pic}_{X/S}$ is a sheaf.

Note that since f is flat, any line bundle on X_T is flat over T for any $T \to S$ and so there is a disjoint union

$$\mathrm{Pic}_{X/S} = \bigsqcup_P \mathrm{Pic}^P_{X/S}.$$

(IV) For a suitable choice of P and P_1, after twisting by a large enough multiple of $\mathcal{O}_X(1)$, the map

$$CDiv^P_{X/S} \to \mathrm{Pic}^{P_1}_{X/S}$$

is the quotient of $CDiv^P_{X/S}$ in the category of sheaves by a proper and smooth (and in particular flat) equivalence relation.

(V) We will study proper and flat equivalence relations and show that the quotient of a quasi-projective scheme by such equivalence relation exists as a quasi-projective scheme. This uses the existence of Hilbert schemes.

13 Relative effective Cartier divisors

13.1 The universal line bundle on $\mathrm{Pic}_{X/S}$

Recall last time we defined for an S-scheme $f : X \to S$ the relative Picard functor

$$\mathrm{Pic}_{X/S} : Sch_S \to Set \quad T \mapsto \mathrm{coker}(\mathrm{Pic}(T) \to \mathrm{Pic}(X_T)).$$

Under the assumption that f is a universal algebraic fiber space[27] and f admits a section $\sigma : S \to X$, we showed that $\mathrm{Pic}_{X/S}$ is a Zariski sheaf.

Our main goal will be to show the following:

Theorem 13.1. *Let $f : X \to S$ be a flat projecitve scheme over S Noetherian. Suppose S is a universal algebraic fiber space and admits a section $\sigma : S \to X$ and that the fibers of f are geometrically integral. Then $\mathrm{Pic}_{X/S}$ is representable by a locally of finite type scheme over S with quasi-projective connected components.*

Note that the elements of $\mathrm{Pic}_{X/S}(T)$ are not line bundles, but rather equivalence classes of line bundles under the equivalence given by tensoring by line bundles from the base T. In particular, even if the relative Picard functor is representable, it is not immediate that there exists an actual line bundle on $\mathrm{Pic}_{X/S} \times_S X$ that pulls back to the appropriate class in $\mathrm{Pic}_{X/S}(T)$ for all T. To show this, let us introduce the following variant of the relative Picard functor.

Definition 13.2. *Let $f : X \to S$ be a universal algebraic fiber space with section $\sigma : S \to X$. The σ-rigidified Picard functor is the functor*

$$\mathrm{Pic}_{X/S,\sigma} : Sch_S \to Set$$

such that

$$\mathrm{Pic}_{X/S,\sigma}(T) = \{(L, \alpha) \mid L \text{ is a line bundle on } X_T,\ \alpha : \mathcal{O}_T \to \sigma_T^* L \text{ is an isomorphism}\}/\sim$$

[27]For any $T \to S$, $(f_T)_* \mathcal{O}_{X_T} = \mathcal{O}_T$. Note this holds in particular if f is projective and the fibers of are geometrically integral.

where $(L,\alpha) \sim (L',\alpha')$ *if and only if there exists an isomorphism* $\epsilon : L \to L'$ *such that* $\sigma_T^*\epsilon \circ \alpha = \alpha'$. $\mathrm{Pic}_{X/S,\sigma}$ *is made into a functor by pullback.*

Remark 13.3. *Using the* σ*-rigidification and the assumptions on* f *one can check directly that* $\mathrm{Pic}_{X/S,\sigma}$ *is a sheaf in the Zariski topology. In fact under these assumptions it is even a sheaf in the fppf topology.*

Proposition 13.4. *Suppose* $f : X \to S$ *is a universal algebraic fiber space with section* σ*. Then* $\mathrm{Pic}_{X/S,\sigma} \cong \mathrm{Pic}_{X/S}$ *as functors.*

Proof. There is a natural transformation

$$\mathrm{Pic}_{X/S,\sigma} \to \mathrm{Pic}_{X/S}$$

given by forgetting the data of α and composing with the projection $\mathrm{Pic}_X \to \mathrm{Pic}_{X/S}$ from the absolute Picard functor. On the other hand, given an element $\mathrm{Pic}_{X/S}(T)$ represented by some line bundle L on X_T, the line bundle

$$L \otimes (f_T)^*\sigma^* L^{-1}$$

has a canonical rigidification given by the inverse of the isomorphism

$$\sigma^* L \otimes \sigma^* L^{-1} \to \mathcal{O}_T$$

and this gives an inverse

$$\mathrm{Pic}_{X/S}(T) \to \mathrm{Pic}_{X/S,\sigma}.$$

□

Corollary 13.5. *Suppose* $f : X \to S$ *is a universal algebraic fiber space with section* $\sigma : S \to X$*. Assume that the relative Picard functor is representable. Then there exists a* $\sigma_{\mathrm{Pic}_{X/S}}$*-rigidified line bundle* $\mathcal{P}$ *on* $X \times_S \mathrm{Pic}_{X/S}$ *that is universal in the following sense. For any* S*-scheme* T *and any line bundle* L *on* X_T*, let* $\varphi_L : T \to \mathrm{Pic}_{X/S}$ *be the corresponding morphism. Then* $\varphi_L^*\mathcal{P}$ *is* σ_T*-rigidified and*

$$L \cong \varphi_L^* \mathcal{P} \otimes f_T^* M$$

for some line bundle M *on* T*. In particular, if* $T = \mathrm{Spec}\, k$*, then for any* k*-point* $[L] \in \mathrm{Pic}_{X/S}(k)$*,* $\mathcal{P}|_{X_k} \cong L$*.*

13.2 Relative Cartier divisors

Recall that an effective Cartier divisor $D \subset X$ is a closed subscheme such that at each point $x \in D$, $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/f_x$ where $f_x \in \mathcal{O}_{X,x}$ is a regular element. That is, D is a pure codimension one locally principal subscheme. Then the ideal sheaf of D is a line bundle $\mathcal{O}_X(-D)$ and the inclusion $\mathcal{O}_X(-D) \cong \mathcal{I}_D \hookrightarrow \mathcal{O}_X$ induces a section

$$s_D : \mathcal{O}_X \to \mathcal{O}_X(D)$$

of the dual line bundle $\mathcal{O}_X(D)$ which is everywhere injective.

Definition 13.6. *Let L be a line bundle. A section $s \in H^0(X, L)$ is regular if $s : \mathcal{O}_X \to L$ is injective. Two pairs (s, L) and (s', L') of line bundles with regular sections are said to be equivalent if there exists an pair (α, t) where*

$$\alpha : L \to L'$$

is an isomorphism of line bundles and $t \in H^0(X, \mathcal{O}_X^*)$ is an invertible function such that $\alpha(a) = ts'$.

Given a line bundle and a regular section (s, L), the vanishing locus $V(s)$ is an effective Cartier divisor with ideal sheaf $s^\vee : L^{-1} \hookrightarrow \mathcal{O}_X$ and in this way we have a bijection

$$\{\text{effective Cartier divisors}\} \leftrightarrow \{(s, L) \mid s \text{ is a regular section}\} / \sim$$

where $\sim$ is the equivalence relation on pairs (s, L) given above. We wish to consider the relative notion.

Definition 13.7. *Let $f : X \to S$ be a morphism of schemes. A relative effective Cartier divisor is an effective Cartier divisor $D \subset X$ such that the projection $D \to X$ is flat.*

We will show that this notion is well behaved under base-change by any $S' \to S$.

Lemma 13.8. *Suppose $D \subset X$ is a relative effective Cartier divisor for $f : X \to S$. For any $S' \to S$, denote by $f' : X' \to S'$ the pullback. Then $D' = S' \times_S D \subset X'$ is a relative effective Cartier divisor for f'.*

Proof. Flatness of $D' \to S'$ is clear. We need to check that D' is cut out at each local ring $\mathcal{O}_{X',x'}$ by a regular element. Let x be the image of x' and consider the exact sequence

$$0 \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \to \mathcal{O}_{D,x} \to 0$$

where the first map is multiplication by the regular element f_x. Pulling back along $S' \to S$ gives us a sequence

$$0 \to \mathcal{O}_{X',x'} \to \mathcal{O}_{X',x'} \to \mathcal{O}_{D',x'} \to 0$$

which is exact since $\mathcal{O}_{D,x}$ is flat so the Tor term on the left vanishes. The first map is multiplication by f'_x, the pullback of f_x. Since it is injective, f'_x is a regular element. $\square$

Corollary 13.9. *Let $f : X \to S$ be a flat morphism and $D \subset X$ a subscheme flat over S. The following are equivalent:*

(a) D is a relative effective Cartier divisor;

(b) $D_s \subset X_s$ is an effective Cartier divisor for each $s \in S$.

Proof. (a) $\implies$ (b) by the previous lemma. Suppose (b) holds. We need to show that for all $x \in X$, $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/f_x$ where f_x is a regular element. By (b), we have that $\mathcal{O}_{D,x} \otimes k(s) = \mathcal{O}_{X,x} \otimes k(s)/\bar{f}_x$ where $\bar{f}_x$ is a regular element of $\mathcal{O}_{X,x} \otimes k(s) = \mathcal{O}_{X_s,x}$. Now by Nakayama's lemma we can lift this to an generator f_x of $\mathcal{I}_D$ so that $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/f_x$ and f_x a regular element. $\square$

Now we can define the functor

$$CDiv_{X/S} : Sch_S \to Set$$

given by

$$CDiv_{X/S}(T) = \{\text{relative effective Cartier divisors } D \subset X_T\}$$

Proposition 13.10. *Let $f : X \to S$ be a flat and projective morphism over a Noetherian scheme S. Then $CDiv_{X/S}$ is representable by an open subscheme of* $\mathrm{Hilb}_{X/S}$*. If moreover f is a smooth morphism, then $CDiv_{X/S}$ is proper over S.*

Proof. Since an element $CDiv_{X/S}(T)$ is a closed subscheme $D \subset X_T$ which is flat and proper over T, $CDiv_{X/S}$ is a subfunctor of $\mathrm{Hilb}_{X/S}$. We need to show that the inclusion $CDiv_{X/S} \to \mathrm{Hilb}_{X/S}$ is an open subfunctor.

That is, suppose $D \subset X_T$ is flat and proper over T. We need to show there exists an open subset $U \subset T$ such that $\varphi : T' \to T$ factors through U if and only if $D_{T'} \subset X_{T'}$ is an effective Cartier divisor which by the previous lemma is equivalent to the the requirement that $D_t \subset X_t$ is an effective Cartier divisor for each $t \in T'$.

Toward this end, let H be the union of irreducible components of $\mathrm{Hilb}_{X/S}$ which contain the image of $CDiv_{X/S}$ and let $D \subset X \times_S H = X_H$ be the universal proper flat cloesed subscheme over H. Note that the non-Cartier locus of $D \subset X_H$ is exactly the locus where I_D is not locally free of rank 1. Since X_H is locally Noetherian and I_D is coherent, the locus where I_D is locally free of rank 1 is locally closed by the locally free stratification (special case of flattening). On the other hand, for any point $x \in X_H \setminus D$, $\mathcal{I}_{D,x} = \mathcal{O}_{X,x}$ is free of rank 1 and thus the stratum contains a dense open subscheme of X_H.[28] Therefore this stratum is in fact open. Let $Z \subset X_H$ be its complement so that $x \in Z$ if and only if D is not Cartier at $x \in X$.

Now we let

$$U := H \setminus f_H(Z) \subset H.$$

Then U is open since f_H is proper and $t \in U$ if and only if for all $x \in X_t$, D is Cartier at x if and only if $D_t \subset X_t$ is an effective Cartier divisor (by the prevoius lemma). Then a T-point of H factors through U if and only if for all $t \in T$, $D_t \subset X_t$ is an effective Cartier divisor if and only if $D_T \subset X$ is an effective Cartier divisor so U represents the subfunctor $CDiv_{X/S}$.

Suppose now that f is smooth. We will use the valuative criterion. Let $T = \mathrm{Spec}\, R$ be the spectrum of a DVR with generic point $\eta = \mathrm{Spec}\, K$ and closed point $0 \in T$ and let D_η be an η point of $CDiv_{X/S}$. By properness of the Hilbert functor, we know there exists a unique $D \subset \mathrm{Hilb}_{X/S}(T)$ such that $D|_\eta = D_\eta$. We need to check that $D \subset X_T$ is in fact a relative effective Cartier divisor. This is equivalent to $D_0 \subset X_0$ being Cartier. By flatness over a DVR, the subscheme D has no embedded points and is pure of codimension 1 since D_η is pure of codimension 1. Thus $D_0 \subset X_0$ is a pure codimension 1 subscheme with no embedded points. Since X_0 is smooth, the local rings are UFDs and by a fact of commutative algebra, height 1 primes on UFDs are principal and thus $I_{D_0,x}$ is a principal ideal of $\mathcal{O}_{X_0,x}$ generated by a regular element for each $x \in X_0$.

[28] Why is $X_H \setminus D$ dense in X_H? This is clear if we add the assumption that the fibers of $f : X \to S$ are integral. In general I want to use the fact that H is the union of components with Hilbert polynomial equal to that of a Cartier divisor to show that $X_H \setminus D$ is dense inside each irreducible component of each fiber. For our purposes we can assume the fibers of $f : X \to S$ are integral since that is the only case we will consider when constructing Picard schemes.

□

Example 13.11. *(A non-proper space of effective Cartier divisors) Let $X \subset \mathbb{P}^3_{\mathbb{A}^1_t}$ be defined by the following equation.*

$$t(xw - yz) + x^2 - yz = 0$$

For $t \neq 0$, this is the smooth quadric surface which has a family of lines defined by the ideal (x, y). Since X_t is smooth this is a Cartier divisor. However, over $t = 0$, X_0 is a singular quadric cone $x^2 - yz$ and one can check that the deal (x, y) is not locally principal at the point $[0, 0, 0, 1]$. Therefore this family of lines gives an element in $CDiv_{X/\mathbb{A}^1}(\mathbb{A}^1 \setminus 0)$ which does not extend to $CDiv_{X/\mathbb{A}^1}(\mathbb{A}^1)$.

14 The Abel-Jacobi map

14.1 Representable morphisms

Throughout the course we have used the notion of a subfunctor being an open or closed subfunctor. This is a special case of the notion of representable morphism between functors.

Definition 14.1. *Let F, G be two functors $Sch_S \to Set$. A morphism $F \to G$ between functors is said to be representable by schemes if for any scheme T and any morphism $T \to G$, the pullback of functors $T \times_G F$ is representable by a scheme.*

Using Yoneda's lemma, the morphism $F \to G$ is representable by schemes if and only if the following condition holds. Given an element $\xi \in G(T)$, there exists a scheme T' and a morphism $T' \to T$ such that for any scheme T'' and any morphism $\varphi : T'' \to T$, φ factors through T' if and only of $\xi_{T''} \in G(T'')$ is the image of an element $\zeta \in F(T'')$. From this it is clear that open and closed subfunctors are representable morphisms. More generally, we can define the following properties for representable morphisms.

Definition 14.2. *For any of the following properties, we say that a morphism of functors $F \to G$ is representable by $\mathcal{P}$ if and only if $F \to G$ is representable by schemes and for all $T \to G$, the morphism $T \times_G F \to T$ has property $\mathcal{P}$:*

(a) $\mathcal{P}$ = "closed embeddings",

(b) $\mathcal{P}$ = "open embeddings",

(c) $\mathcal{P}$ = "affine morphisms",

(d) $\mathcal{P}$ = "projective bundles"[29],

(e) $\mathcal{P}$ = "proper morphisms",

(f) $\mathcal{P}$ = "flat morphisms",

(g) $\mathcal{P}$ = "smooth morphisms",

(h) $\mathcal{P}$ = "finite morphisms",

[29] Recall that a projective bundle on a scheme S is an S-scheme of the form $\mathbb{P}(\mathcal{E})$ for $\mathcal{E}$ a coherent sheaf on S.

(i) $\mathcal{P}$ = *"étale morphisms"*.

Remark 14.3. *In fact this definition makes sense for any property $\mathcal{P}$ of morphisms such that (1) $\mathcal{P}$ is stable under base change, and (2) the property can be checked for a morphism of schemes after taking a Zariski open cover of the target*[30]*. More generally, one can work with algebraic spaces or algebraic stacks and then one can ask for morphisms of (pseudo)functors*[31] *to be representable by schemes, or by algebraic spaces, or by algebraic stacks. Then it makes sense for such morphisms to be representable by $\mathcal{P}$ for any property that is fppf or fpqc local on the target.*

14.2 The Abel-Jacobi map

Recall last time we constructed the moduli space $CDiv_{X/S}$ of relative effective Cartier divisors for any flat and projective $f : X \to S$ over a Noetherian scheme S. The elements of $CDiv_{X/S}(T)$ are Cartier divisors $D \subset X_T$ which are flat over T. Then the ideal sheaf $\mathcal{O}_{X_T}(-D)$ of D is a line bundle on X_T with dual $\mathcal{O}_{X_T}(D)$. We wish to show that sending D to $\mathcal{O}_{X_T}(D)$ defines a natural transformation of functors $CDiv_{X/S} \to \mathrm{Pic}_{X/S}$.

Proposition 14.4. *The natural map*

$$CDiv_{X/S}(T) \to \mathrm{Pic}_{X/S}(T)$$

given by $(D \subset X_T) \mapsto [\mathcal{O}_{X_T}(D)]$ is a natural transformation of functors $CDiv_{X/S} \to \mathrm{Pic}_{X/S}$.

Proof. We need to check this map is functorial in T. Each side is made into a functor under pullback so concretely, we need to check that for any $\varphi : T' \to T$, $\mathcal{O}_{X_{T'}}(D_{T'}) = \varphi^*\mathcal{O}_{X_T}(D)$ up to twisting by a line bundle pulled back from T'. Now $\mathcal{O}_D$ is flat over T, so pulling back the ideal sequence

$$0 \to \mathcal{O}_{X_T}(-D) \to \mathcal{O}_{X_T} \to \mathcal{O}_D \to 0$$

gives

$$0 \to \varphi^*\mathcal{O}_{X_T}(D) \to \mathcal{O}_{X_{T'}} \to \mathcal{O}_{D_{T'}} \to 0$$

so in particular, $\varphi^*\mathcal{O}_{X_T}(-D) = \mathcal{O}_{X_{T'}}(-D_{T'})$. Now since $\mathcal{O}_{X_T}(-D)$ is locally free, we have

$$\varphi^*\mathcal{O}_{X_T}(D) = \varphi^*\mathcal{O}_{X_T}(-D)^{-1} = \mathcal{O}_{X_{T'}}(-D_{T'})^{-1} = \mathcal{O}_{X_{T'}}(D_{T'})$$

as required. □

This map is often called the Abel-Jacobi map and is denoted by

$$AJ_{X/S} : CDiv_{X/S} \to \mathrm{Pic}_{X/S}.$$

Let us study the fibers of $AJ_{X/S}$ over a k-point. A k-point $t \in \mathrm{Pic}_{X/S}(k)$ for $\mathrm{Spec}\, k \to S$ a point of S corresponds to a line bundle L on X_k. Then by the bijection between Cartier divisors and line bundles with regular sections up to isomorphism, the fiber $AJ_{X/S}^{-1}(t)$ is the set of pairs (s, L) where $s : \mathcal{O}_X \hookrightarrow L$ is a regular section up to scaling:

$$AJ_{X/S}^{-1}(t) = H^0(X_k, L)^{reg}/H^0(X_k, \mathcal{O}_{X_k}^*).$$

If X_k is geometrically integral, then every nonzero section is regular and we have $AJ_{X/S}^{-1}(t) = \mathbb{P}(H^0(X_k, L))$ is a projective space. This observation is generalized by the following theorem.

[30] That is, the property is local on the target.

[31] Functors to the category of groupoids rather than to sets. We will discuss this in more detail later.

Theorem 14.5. *Suppose $f : X \to S$ is a flat projective universal algebraic fiber space with section $\sigma : S \to X$. Suppose further that the fibers of f are geometrically integral. Then the Abel-Jacobi map $AJ_{X/S} : CDiv_{X/S} \to \mathrm{Pic}_{X/S}$ is representable by a projective bundle. More precisely, for any scheme T and T-point $\varphi_L : T \to \mathrm{Pic}_{X/S}$ corresponding to a line bundle L on X_T, there exists a coherent sheaf $\mathcal{E}$ on T such that the pullback $AJ^{-1}_{X/S}(T) \to T$ is isomorphic to the projective bundle $\mathbb{P}(\mathcal{E})$ over T. Moreover, if $R^1(f_T)_*L = 0$, then (1) $(f_T)_*L$ commutes with base change, (2) $\mathcal{E}$ and $(f_T)_*L$ are locally free and dual to each other, and (3) the formation of $\mathcal{E}$ commutes with base change. In particular, if $R^1(f_T)_*L = 0$ for all T-points, then $AJ_{X/S}$ is representable by smooth morphisms.*

To prove this theorem, we will use the following proposition, which is on problem set 2, and follows from the existence of the Grothendieck complex.

Proposition 14.6. *Let $f : X \to S$ be a proper morphism over a Noetherian scheme S and let $\mathcal{F}$ be a coherent sheaf on X which is flat over S. Then there exists a coherent sheaf $\mathcal{Q}$ on S with a functorial isomorphism*

$$\theta_{\mathcal{G}} : f_*(\mathcal{F} \otimes f^*\mathcal{G}) \to \mathcal{H}om_S(\mathcal{Q}, \mathcal{G}).$$

Corollary 14.7. *Suppose $R^1 f_*\mathcal{F} = 0$ so that $f_*\mathcal{F}$ is locally free and commutes with base change by cohomology and base change. Then $\mathcal{Q}$ is locally free and is dual to $f_*\mathcal{F}$. In particular, the formation of $\mathcal{Q}$ commutes with base change.*

Proof. Apply the proposition to the special case $\mathcal{G} = \mathcal{O}_S$ and use that dualizing commutes with tensor products for locally free modules. □

Remark 14.8. *Note that by definition, $\mathrm{Pic}_{X/S}$ is compatible with base change in the following sense. For any $S' \to S$ such that $f' : X' \to S'$ is the pullback of $f : X \to S$,*

$$\mathrm{Pic}_{X'/S'} = \mathrm{Pic}_{X/S} \times_S S'$$

as functors. Moreover, if f satisfies any of the above assumptions then so does f'. Note also that $CDiv_{X/S} \times_S S' = CDiv_{X'/S'}$. Indeed we already discussed the Hilbert schemes have this property and since the condition of being a relative effective Cartier divisor is compatible with base change, the claim follows. Then it is clear to see from the definition of $AJ_{X/S} \times_S S' = AJ_{X'/S'}$.

Proof. (Proof of theorem) By the remark, we may suppose that $T = S$. Let $\varphi_L : S \to \mathrm{Pic}_{X/S}$ be an S-point corresponding to the class of a line bundle L on X[32] Then the pullback $AJ^{-1}(\varphi_L)$ by definition is the functor which we will denote $D_{[L]}$ that takes an S-scheme T to the set of relative effective Cartier divisors $D \subset X_T$ such that $\mathcal{O}_{X_T}(D) = L_T \otimes f_T^*M$ where M is some line bundle on T.

Since f is a universal algebraic fiber space, $f_T^* : \mathrm{Pic}(T) \to \mathrm{Pic}(X_T)$ is injective thus if M' is some other line bundle such that $\mathcal{O}_{X_T}(D) = L_T \otimes f_T^*M'$, then $f_T^*M \cong f_T^*M'$ so $M \cong M'$. Thus M is unique up to isomorphism and D corresponds to a regular section s of $L_T \otimes f_T^*M$. Equivalently, s is a regular section of $(f_T)_*(L_T \otimes f_T^*M)$. By the proposition, since L is flat there exists a coherent sheaf $\mathcal{Q}$ along with a functorial isomorphism

$$f_*(L \otimes f^*\mathcal{G}) = \mathcal{H}om_S(\mathcal{Q}, \mathcal{G})$$

for all quasi-coherent $\mathcal{G}$ on S. We will show next time that $\mathcal{E} = \mathcal{Q}$ is our required sheaf. □

[32]Here is where we are using the assumption that σ has a section so that T-points correspond to actual line bundles on X_T. Otherwise, we would need to sheafify and thus T-points would correspond to line bundles on some cover $T' \to T$.

15 The Abel-Jacobi map (cont.), boundedness, quotients by equivalence relations

15.1 $AJ_{X/S}$ is representable by projective bundles

Recall last time we defined the Abel-Jacobi map

$$AJ_{X/S} : CDiv_{X/S} \to \mathrm{Pic}_{X/S}$$

by $(D \subset X_T) \mapsto \mathcal{O}_{X_T}(D)$. We are proving the following.

Theorem 15.1. *Suppose $f : X \to S$ is a flat projective universal algebraic fiber space with section $\sigma : S \to X$. Suppose further that the fibers of f are geometrically integral. Then the Abel-Jacobi map $AJ_{X/S} : CDiv_{X/S} \to \mathrm{Pic}_{X/S}$ is representable by a projective bundle. More precisely, for any scheme T and T-point $\varphi_L : T \to \mathrm{Pic}_{X/S}$ corresponding to a line bundle L on X_T, there exists a coherent sheaf $\mathcal{E}$ on T such that the pullback $AJ^{-1}_{X/S}(T) \to T$ is isomorphic to the projective bundle $\mathbb{P}(\mathcal{E})$ over T. Moreover, if $R^1(f_T)_*L = 0$, then (1) $(f_T)_*L$ commutes with base change, (2) $\mathcal{E}$ and $(f_T)_*L$ are locally free and dual to each other, and (3) the formation of $\mathcal{E}$ commutes with base change. In particular, if $R^1(f_T)_*L = 0$ for all T-points, then $AJ_{X/S}$ is representable by smooth morphisms.*

Proof. We have reduced to the case that $T = S$ and are considering an S-point $\varphi_L : S \to \mathrm{Pic}_{X/S}$ corresponding to a line bundle L on X. Let $D_{[L]}$ denote the fiber product $AJ^{-1}_{X/S}(\varphi_L)$. We saw that T-point of $D_{[L]}$ corresponds to a line bundle M on T as well as a regular section of $L_T \otimes f_T^*M$. Sections of this sheaf are the same as sections of $(f_T)_*(L_T \otimes f_T^*M)$ so we are led to consider the universal coherent sheaf $\mathcal{Q}$ on S such that

$$f_*(K \otimes f^*\mathcal{G}) = \mathcal{H}om_S(\mathcal{Q}, \mathcal{G})$$

for all quasi-coherent $\mathcal{G}$ on S.

We want to take $\mathcal{G}$ to be g_*M for $g : T \to S$ the structure morphism[33]. Since L and M are locally free, we have the projection formula:

$$g_*(L_T \otimes_{\mathcal{O}_{X_T}} f_T^*M) = g_*(g^*L \otimes_{\mathcal{X}_T} f_T^*M) = L \otimes_{\mathcal{O}_X} g_*f_T^*M.$$

Now f is flat so by flat base change, we have $g_*f_T^*M \cong f^*g_*M$. Putting this together, we get

$$\begin{aligned} H^0(L_T \otimes_{\mathcal{O}_{X_T}} f_T^*M) &= H^0(L \otimes_{\mathcal{O}_X} f^*g_*M) = H^0(f_*(L \otimes_{\mathcal{O}_X} f^*g_*M)) \\ &= H^0(\mathcal{H}om_S(\mathcal{Q}, g_*M)) = \mathrm{Hom}_S(\mathcal{Q}, g_*M) = \mathrm{Hom}_T(\mathcal{Q}_T, M). \end{aligned}$$

In fact if one is more careful about the construction of $\mathcal{Q}$, one can show that it commutes with arbitrary base change so that $(f_T)_*(L_T \otimes f_T^*M) = \mathcal{H}om_T(\mathcal{Q}_T, M)$ as sheaves rather than just global sections, that is, the universal sheaf from the proposition for L_T over T is the pullback of the one for L over S.

Now the condition that a section s of $L_T \otimes_{\mathcal{O}_{X_T}} f_T^*M$ is a regular section is equivalent to $s_t \in H^0(X_t, L_t)$ being nonzero for each $t \in T$ and thus the corresponding morphism $u_s : \mathcal{Q}_T \to M$ must be nonzero at each fiber over $t \in T$. Since M is a line bundle, $M \otimes k(t)$ is

[33] Here we have to assume that g is qcqs so that this pushforward is quasi-coherent. You can convince yourself that it is enough to prove representability in the category of qcqs S-schemes.

a rank 1 vector space and so $u_s \otimes k(t)$ is nonzero if and only if it is surjective. By Nakayama's lemma, this implies u_s is surjective as a map of sheaves for all $t \in T$[34]. Thus $u_s : \mathcal{Q}_T \to M$ is a rank 1 locally free quotient of $\mathcal{Q}_T$. By definition, this is a T point of the projective bundle $\mathbb{P}(\mathcal{Q})$ over S.

On the other hand, given a T-point of $\mathbb{P}(\mathcal{Q})$, we can reverse the equalities above to obtain a locally free quotient $u : \mathcal{Q}_T \to M$ corresponding to a section $s : \mathcal{O}_{X_T} \to L_T \otimes f_T^* M$ which is nonzero on every fiber and thus regular. Therefore the vanishing subscheme $D \subset X_T$ of s satisfies that for all $t \in T$, $D_t \subset X_t$ is a Cartier divisor. We need to check that $D \to T$ is flat so that it is a relative effective Cartier divisor. Then by construction $\mathcal{O}_{X_T}(D) = L_T \otimes f_T^* M$ and so $D \subset X_T$ gives a T-point of $D_{[L]}$. For flatness, we have the following lemma.

Lemma 15.2. *Let $f : X \to S$ be a flat morphism of finite type over a Noetherian scheme S and let $D \subset X$ be a closed subscheme such that for each $s \in S$, $D_s \subset X_s$ is an effective Cartier divisor. Then $D \to S$ is flat.*

Proof. Let $x \in D \subset X$ with $s = f(x)$. We need to show that $\mathcal{O}_{D,x}$ is a flat $\mathcal{O}_{S,s}$-module. By the local criterion for flatness, this is equivalent to the vanishing of $\mathrm{Tor}_1^{\mathcal{O}_{S,s}}(k(s), \mathcal{O}_{D,x})$. Consider the long exact sequence associated to the ideal sequence

$$0 \to \mathcal{I}_{D,x} \to \mathcal{O}_{X,x} \to \mathcal{O}_{D,x} \to 0.$$

We have

$$\mathrm{Tor}_1^{\mathcal{O}_{S,s}}(k(s), \mathcal{O}_{X,x}) \to \mathrm{Tor}_1^{\mathcal{O}_{S,s}}(k(s), \mathcal{O}_{D,x}) \to I_{D,x} \otimes k(s) \to \mathcal{O}_{X,x} \otimes k(s) = \mathcal{O}_{X_s,x}.$$

Since the first term is zero by flatness of $X \to S$, the required vanishing would follow from injectivity of the last map. To see this injectivity, let $\bar{f}_x \in \mathcal{O}_{X_s,x}$ be a regular element cutting out D_s at $x \in X_s$ and let $f_x \in \mathcal{O}_{X,x}$ be a lift. Now multiplication by f_x induces a map $\mathcal{O}_{X,x} \otimes k(s) \to \mathcal{O}_{X,x} \otimes k(s)$ which is injective with image $I_{D_s,x}$. Thus we have an injective map which factors as

$$\mathcal{O}_{X,x} \otimes k(s) \to \mathcal{I}_{D,x} \otimes k(s) \to \mathcal{O}_{X,x} \otimes k(s)$$

where the first map is a surjection and so the required map is an injection. □

This shows that $\mathcal{E} = \mathcal{Q}$ is our required sheaf so that $AJ_{X/S}^{-1}(\varphi_{[L]}) \to S$ is representable by the projective bundle $\mathbb{P}(\mathcal{E})$. If $R^1 f_* L = 0$, then by the previous corollary, $\mathcal{Q}$ and $f_* L$ are locally free, dual to eachother, and commute with basechange. In particular, if this holds for all T-points, then $AJ_{X/S}$ is representable by smooth morphisms since a projective bundle is smooth over the base when $\mathcal{E}$ is locally free. □

15.2 Boundedness

Definition 15.3. *We say the a moduli functor $F : Sch_S \to Set$ is bounded, or that the objects parametrized by F form a bounded family, if there exists a finite type scheme T over S as well as a T-point $\xi \in F(T)$ such that for any field $t : \mathrm{Spec}\, k \to S$ and k-point $\xi_t \in F(k)$, there exists a field extension k'/k and a k'-point $t' \in T(k')$ such that $\xi|_{t'} = \xi_t \otimes_k k'$.*

[34]Here we have to use a finite presentation trick to reduce to T Noetherian as usual.

Intuitively, a bounded moduli problem, or a bounded family of geometric objects, is one where there exists a family $f : U \to T$ over a finite type base scheme such that every isomorphism class of objects in our moduli problem, appears as a fiber of f. In particular, if F is representable by some fine moduli space $\mathcal{M}$, then this induces a *surjective* morphism $T \to \mathcal{M}$ which exhibits the fine moduli space as being finite type over $\mathcal{M}$. Essentially, boundedness is a way of showing our moduli spaces are finite type.

Example 15.4. *Let $F = H^P_{X/S}$ be the Hilbert functor for P a fixed Hilbert polynomial and $f : X \to S$ a projective morphism over a Noetherian scheme. Then the boundedness of F was a result of uniform CM regularity which allowed us to embed F into a fixed Grassmannian, which is of finite type.*

Now let us consider our situation for the Picard functor: $f : X \to S$ is a flat projective universal algebraic fiber space with section $\sigma : S \to X$. Then for any $T \to S$ and any line bundle L on X_T, L is flat over T. Therefore the Hilbert polynomial $P_{L_t}(d)$ is constant for $t \in T$ so the relative Picard functor can be written as a union

$$\bigsqcup_P \mathrm{Pic}^P_{X/S}.$$

Our goal is for each of these components $\mathrm{Pic}^P_{X/S}$ to be bounded. As with the case of the Hilbert functor, this boils down to a uniform CM regularity result.

Theorem 15.5. *(SGA 6, Exp XIII) Let $f : X \to S$ be a projective morphism over a Noetherian scheme S. Suppose the fibers of f are geometrically integral and of equal dimension r and fix a Hilbert polynomial P. Then there exists an integer m such that for any field k and k-point $\xi \in \mathrm{Pic}^P_{X/S}(k)$ corresponding to a line bundle L on X_k, L is m-regular.*

Proposition 15.6. *For each Hilbert polynomial P, there exists an m such that the Abel-Jacobi map $AJ^{P(d+m)}_{X/S} : CDiv^{P(d+m)}_{X/S} \to \mathrm{Pic}^{P(d+m)}_{X/S}$ is the projectivization of a locally free sheaf. In particular it is a smooth, proper surjection.*

Proof. Pick m so that L on X_k is m-regular for each k-point of $\mathrm{Pic}^P_{X/S}$ and consider the Abel-Jacobi map for $P(d+m)$. For any T-point of $\mathrm{Pic}^{P(d+m)}_{X/S}$ corresponding to L on X_T, $L(-m)$ has Hilbert polynomial P, and in particular is m-regular. Therefore

$$H^i(X_t, L|_{X_t}) = 0 \;\; i \geq 1$$

for all $t \in T$. By cohomology and base change, $R^i(f_T)_*L = 0$ for all $i \geq 1$ and so $(f_T)_*L$ is locally free of rank $P(m)$ and $AJ^{P(d+m)}_{X/S} \times_{\mathrm{Pic}^{P(d+m)}_{X/S}} T \cong \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is the locally free sheaf $((f_T)_*L)^\vee$ on T. $\square$

Fact 15.7. *For each $m \in \mathbb{Z}$, twisting by $\mathcal{O}_X(m)$ induces an isomorphism*

$$\mathrm{Pic}^{P(d)}_{X/S} \cong \mathrm{Pic}^{P(d+m)}_{X/S}.$$

Corollary 15.8. *The functor $\mathrm{Pic}^P_{X/S}$ is bounded.*

Proof. By the Proposition, for each P, there exists an m such that the Abel-Jacobi map for $P(d+m)$ is surjective. Since $CDiv^{P(d+m)}_{X/S}$ is of finite type, so $\mathrm{Pic}^{P(d+m)}_{X/S}$ is bounded. By the previous fact, this functor is isomorphic to $\mathrm{Pic}^{P(d)}_{X/S}$ and so we are done. $\square$

Now let us say a few words about the proof of the uniform CM-regularity theorem for the Picard functor. The proof is very similar to the prevoius uniform CM-regularity theorem for the Quot funcotr. Recall that the idea there was to induct on the dimension of the ambient projective space and restrict to a hyperplane section. More precisely, the proof there showed the following.

Proposition 15.9. *Let $\mathcal{F}$ be a coherent sheaf on a projective variety X over a field* Spec k*. Suppose we have an exact sequence*

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0$$

given by restricting to a hyperplane section H such that $\mathcal{F}_H$ is m-regular. Then:

(a) $H^i(X, \mathcal{F}(n)) = 0$ for $n \geq m - i$, and

(b) the sequence $\{\dim H^1(X, \mathcal{F}(n)\}$ is monotonically decreasing to zero for $n \geq m - 1$.

In particular, $H^1(X, \mathcal{F}(n)) = 0$ for $n \geq (m-1) + \dim H^1(X, \mathcal{F}(m-1))$ so that $\mathcal{F}$ is $[m + \dim H^1(X, \mathcal{F}(m-1))]$-regular.

In the previous incarnation of uniform CM-regularity the only place where we used that $\mathcal{F}$ was a subsheaf of $\mathcal{O}_X^{\oplus r}$ was to bound $H^0(X, \mathcal{F}(m))$ in terms of the Hilbert polynomial P. By the above proposition, we get that

$$\dim H^1(X, \mathcal{F}(m-1)) = H^0(X, \mathcal{F}(m-1)) - P(m-1)$$

and so a uniform bound for $H^0(X, \mathcal{F}(m-1))$ gives us a uniform bound for the regularity of $\mathcal{F}$. Thus, the main technical part of the proof of the theorem is to bound the dimension of the space of global sections of a line bundle with fixed Hilbert polynomial.

Definition 15.10. *The degree of a projective variety $X \subset \mathbb{P}^n_k$ of pure dimension r is given by the intersection number H^r where H is a section of $\mathcal{O}_X(1)$.*

The idea then is to relate the Hilbert polynomial, the degree, and the space of global sections and their restrictions to hyperplane sections (for the inductive step!) using Grothendieck-Riemann-Roch and Serre duality. We won't say more about the general proof here but let us consider the easier case of X a smooth projective curve.

15.2.1 Boundedness of Picard for smooth projective curves

For C an integral projective curve over a field k, we can define the arithmetic genus

$$p_a := \dim H^1(C, \mathcal{O}_C).$$

If $C^\nu \to C$ is the normalization of C, we define the geometric genus by

$$p_g(C) := p_a(C^\nu).$$

When C is already normal, and thus regular, we have $p_g = p_a$ and we simply call this the genus $g = g(C)$. On such a C we have the canonical bundle

$$\omega_C := \Omega^1_{C/k}.$$

Recall the statement of Serre duality.

Theorem 15.11. *(Serre duality) Let C be a projective, integral and regular curve over a field. Then for any locally free coherent sheaf $\mathcal{E}$ on C, there is a natural isomorphism*

$$H^1(C,\mathcal{E})^\vee \cong H^0(C,\omega_C \otimes \mathcal{E}^\vee).$$

We also have the Riemann-Roch theorem which allows us to compute the Hilbert polynomial of a line bundle.

Theorem 15.12. *(Riemann-Roch) Let L be a line bundle on C a projective, integral and regular curve over a field. Then*

$$\chi(L) := \dim H^0(C,L) - \dim H^1(C,L) = \deg(L) - g + 1.$$

Note that for C an integral, regular, projective curve, the degree of C in the sense of Definition 2 above is the same as the degree of $\mathcal{O}_C(1)$. Then by Riemann-Roch, for L any line bundle, we have

$$P_L(m) = Dm + \deg(L) - g + 1$$

where $D = \deg(_{\mathcal{C}}(\infty)) = \deg(\mathcal{C})$. Therefore, the Hilbert polynomial of a line bundle depends only on the degree $\deg(L)$.

On the other hand, a Cartier divisor $D \subset C$ is simply a zero dimensional subscheme and it has constant Hilbert polynomial $d = \dim \mathcal{O}_D$ and the degree of $\mathcal{O}_C(D)$ is simply d. Thus, we can label the components of the Picard functor by the degree $d = \deg L$ and the Abel-Jacobi map takes the form

$$AJ^d_{X/S} : CDiv^d_{C/k} = \mathrm{Hilb}^d_{C/k} \to \mathrm{Pic}^d_{C/k}.$$

Applying Riemann-Roch and Serre duality to $L = \omega_C$, we get that $\dim H^0(C,\omega_C) = g$ and $\deg(\omega_C) = 2g-2$. Then if L is a line bundle with $\deg L > 2g-2$,

$$\dim H^1(C,L) = \dim H^0(C,\omega_C \otimes L^{-1}) = 0$$

since $\deg(\omega_C \otimes L^{-1}) < 0$. Therefore the Abel-Jacobi map is a smooth projective bundle for $d > 2g-2$, in fact equal to the projectivization $\mathbb{P}((\pi_*\mathcal{L})^\vee)$ where $\mathcal{L}$ is the universal line bundle on $C \times \mathrm{Pic}^d_{C/k}$ and π is the second projection. In particular, this gives boundedness.

15.3 Quotients by flat and proper equivalence relations

The last technical ingredient we need before we can prove representability of the Picard functor is the existence of quotients by finite equivalence relations for quasi-projective schemes. We begin with some generalities on categorical quotients.

Let $\mathcal{C}$ be a category with fiber products and a terminal object $*$. An equivalence relation on an object X of $\mathcal{C}$ is an object R along with a morphism $R \to X \times_* X$ such for each object T, the map of sets $R(T) \subset X(T) \times X(T)$ is the inclusion of an equivalence relation on the set $X(T)$. The two projections give us two morphisms $p_i : R \to X$ from an equivalence relation to X.

Definition 15.13. *A categorical quotient of X by the equivalence relation R is an object Z as well as a morphism $u : X \to Z$ such that $u \circ p_1 = u \circ p_2$ such that (Z,u) is initial with respect to this property. That is, for any $f : X \to Y$ such that $f \circ p_1 = f \circ p_2$, there exists a unique morphism $g : Z \to Y$ such that f factors through u, $f = g \circ u$.*

If such a pair (Z, u) exists, then it is unique up to unique isomorphism and is denoted X/R. If X/R exists, we say that it is an effective quotient if the natural map

$$R \to X \times_{X/R} X$$

is an isomorphism.

More generally, we can consider maps $R \to X \times_* X$ that are not necessarily monomorphisms but such that the image of $R(T) \to X(T) \times X(T)$ is an equivalence relation. In this can we can replace R with its image in $X \times_* X$ if it exists to reduce to the previous situation.[35]

Example 15.14. *(The case of a group quotient) Let G be a S-group scheme acting on an S-scheme X. Then the action is given by a morphism $m : G \times_S X \to X$ and the product $m \times pr_X : G \times_S X \to X \times_S X$ is an equivalence relation on X in the category of S-schemes. If an effective quotient exists, we will denote it X/G. Note that in this case, the fibers of the natural map $u : X \to X/G$ are exactly the orbits of G.*

Example 15.15. *(A non-effective quotient) Consider $\mathbb{A}^1$ over an algebraically closed field $k = \bar{k}$. The group $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by scaling and there are two orbits, $U = \mathbb{A}^1 \setminus \{0\}$ and $\{0\}$. Now $\mathbb{A}^1 \to \operatorname{Spec} k$ is a categorical quotient but the fibers of this map are not orbits so the quotient isn't effective.*

Given an equivalence relation $R \to X \times_S X$ on an S-scheme X, we say that R has property $\mathcal{P}$ for any property of morphisms if the morphisms $p_i : R \to X$ have this property.

Theorem 15.16. *Let $X \to S$ be a quasi-projective scheme over a Noetherian scheme S and suppose that $R \to X \times_S X$ is a flat and proper equivalence relation. Then an effective quotient X/R exists and moreover it is a quasi-projective S-scheme.*

16 Sheaves, quotients, representability of the Picard functor

16.1 fpqc Descent

Given a morphism $p : S' \to S$, we can consider the pullback functor p^*.

$$p^* : QCoh(S) \to QCoh(S')$$
$$\mathcal{F} \mapsto p^*\mathcal{F}$$

Denoting by $q_i : S'' := S' \times_S S' \to S'$ the two projections, then the sheaf $p^*\mathcal{F}$ carries a natural isomorphism

$$\varphi : q_1^* p^* \mathcal{F} \to q_2^* p^* \mathcal{F}.$$

given by the isomorphism of functors

$$q_1^* \circ p^* \cong (p \circ q_1)^* = (p \circ q_2) \cong q_2^* \circ p^*.$$

Now we can consider the various projections from the triple product,

$$q_{ij} : S''' := S' \times_S S' \times_S S' \to S''.$$

[35]This doesn't make a difference for us now but when we consider the more general category of algebraic stacks, taking quotients by a finite map $R \to X \times_S X$ versus its image $R' \subset X \times_S X$ is exactly the difference between a stack quotient and its coarse moduli space.

Then for any sheaf $\mathcal{F}$ on S, we have the following commutative diagram.

$$\begin{tikzcd} q_{12}^*q_1^*p^*\mathcal{F} \arrow[r, "q_{12}^*\varphi"] \arrow[d, equal] & q_{12}^*q_2^*p^*\mathcal{F} = q_{23}^*q_1^*p^*\mathcal{F} \arrow[r, "q_{23}^*\varphi"] & q_{23}^*q_2^*p^*\mathcal{F} \arrow[d, equal] \\ q_{13}^*q_1^*p^*\mathcal{F} \arrow[rr, "q_{13}^*\varphi"] & & q_{13}^*q_2^*p^*\mathcal{F} \end{tikzcd}$$

Said succinctly, we have the cocycle condition

$$q_{13}^*\varphi = q_{23}^*\varphi \circ q_{12}^*\varphi. \tag{16}$$

Let $QCoh(p : S' \to S)$ denote the category of pairs $(\mathcal{F}', \varphi)$ where $\mathcal{F}'$ is a quasi-coherent sheaf on S' and $\varphi : q_1^*\mathcal{F}' \to q_2\mathcal{F}'$ is an isomorphism satisfying the cocycle condition (16). Then p^* gives a functor

$$p^* : QCoh(S) \to QCoh(p : S' \to S).$$

The question of descent is the question of when p^* is an equivalence of categories. We say that quasi-coherent sheaves satisfy descent along p or that descent holds for p when this functor is an equivalence.[36]

Example 16.1. *Suppose $\{U_i\}_{i\in I}$ is a Zariski open cover of S and let $p : S' = \bigsqcup_{i\in I} U_i \to S$. Then S'' is the disjoint union of intersections $U_i \cap U_j$, S''' is the disjoint union of triple intersections, and the cocycle condition is the usual cocycle condition for gluing sheaves so quasi-coherent sheaves satisfy descent along p.*

Definition 16.2. *A morphism $p : S' \to S$ is fpqc[37] if it is faithfully flat and each point $s' \in S'$ has a quasi-compact open neighborhood $U \subset S'$ with $f(U)$ an open affine subset of S. A morphism $p : S' \to S$ if fppf if it is faithfully flat and of finite presentation.*

fpqc and fppf morphisms satisfy many nice properties.

Fact 16.3. *(i) the property of being fpqc or fppf is compatible under base-change and composition;*

(ii) if $p : S' \to S$ is fpqc, then S has the quotient topology of S' by p. That is, $U \subset S$ is open if and only if $f^{-1}(U) \subset S'$ is open;

(iii) an open faithfully flat morphism is fpqc;

(iv) an fppf morphism is open, and in particular, fpqc.

Many properties of schemes (resp. morphisms) are fpqc local (resp. fpqc local on the target), meaning they can be checked after pulling back by an fpqc morphism. This includes the propeties we defined for representability of morphisms in a previous lecture. The following is Grothendieck's main theorem of descent.

Theorem 16.4. *Let $p : S' \to S$ be an fpqc morphism. Then quasi-coherent sheaves satisfy descent by p:*

$$p^* : QCoh(S) \cong QCoh(p : S' \to S).$$

[36] The cateogry $QCoh(p : S' \to S)$ is sometimes called the category of descent data and the descent data in the image of p^* is called effective.

[37] "faithfully flat and quasi-compact"

We can also talk about the question of descent for other objects on S. For example we can consider the case of schemes $X \to S$. Given such an X we can pull it back along p to obtain $X' = p^*X \to S'$ an S'-scheme with an isomorphism $\varphi : q_1^*X' \to q_2^*X'$ of S''-schemes satisfying the cocycle condition on the triple fiber product S'''. We have a category of S'-schemes with descent data $Sch_{S'\to S}$ consisting of (X', φ) where $\varphi : q_1^*X \to q_2^*X$ is an isomorphism. Then p^* gives a functor

$$p^* : Sch_S \to Sch_{S'\to S}$$

and we can ask when p^* is an equivalence.

Corollary 16.5. *Let $p : S' \to S$ be an fpqc morphism. Let Aff_S be the category of affine S-schemes and $Aff_{S'\to S}$ the category of affine S'-schemes with descent data. Then*

$$p^* : Aff_S \to Aff_{S'\to S}$$

is an equivalence. That is, affine S-schemes satisfy fpqc descent. In particular, closed subschemes of S satisfy fpqc descent.

Proof. The relative spec functor Spec_S gives an equivalence of categories between affine S-schemes and quasi-coherent $\mathcal{O}_S$-algebras over S. Moreover, $p^* : QCoh(S) \to QCoh(S')$ is compatible with tensor products. Thus, it sends quasi-coherent $\mathcal{O}_S$-algebras to $\mathcal{O}_{S'}$-algebras and the canonical isomorphism $\varphi : q_1^*p^*\mathcal{A} \to q_2^*p^*\mathcal{A}$ is an algebra homomorphism. Thus, the equivalence

$$p^* : QCoh(S) \to QCoh(p : S' \to S)$$

restricts to an equivalence on the subcategories of algebra objects which by the Spec_S equivalence gives us the first claim. For the second statement, closed subschemes of S correspond to affine morphisms $f : X \to S$ such that $\mathcal{O}_S \to f_*\mathcal{O}_X$ is a surjection, or equivalently, algebras such that the canonical map $\mathcal{O}_S \to \mathcal{A}$ is a surjection. As before, since p^* is an equivalence onto the category of quasi-coherent sheaves with descent data, $\mathcal{O}_S \to \mathcal{A}$ is a surjection if and only if $\mathcal{O}_{S'} \to \mathcal{A}'$ is a surjection so closed subschemes descend to closed subschemes. □

More generally, for any fpqc morphism, it is a fact that

$$p^* : Sch_S \to Sch_{S'\to S}$$

is fully faithful. Essential surjectivity (ie effectivity of descent data) is more subtle but the affine case above suggests that one should restrict to desent data (X, φ) for which there exists an open affine cover of X by U such that φ restricts to an isomorphism $q_1^*U \to q_2^*U$ for each U.

16.2 Grothendieck topologies and sheaves

Given a category $\mathcal{C}$ which has pullbacks, a collection of morphisms $\mathcal{T}$ generates[38] a *Grothendieck topology* if

[38] Technically, the collection $\mathcal{T}$ is not the Grothendieck topology, but rather a Grothendieck pre-topology, and different pre-topologies may generate the same topology. The notion of sheaves which we will define shortly depends only on the topology not the pre-topology but we won't need this distinction here. One should think of $\mathcal{T}$ is a sub-base for the topology.

(1) any isomorphism $X \to Y$ is contained in $\mathcal{T}$;

(2) for any $U \to X$ in $\mathcal{T}$ and any $X' \to X$, the pullback $U' \to X'$ is in $\mathcal{T}$;

(3) For any $U \to X$ and $V \to U$ in $\mathcal{T}$, the composition $V \to X$ is in $\mathcal{T}$.

Example 16.6. *If Top is the category of topological spaces, then the collection of morphisms of the form $U = \bigsqcup_{i \in I} U_i \to X$ where $\{U_i\}$ is an open cover X generate a grothendieck topology. Similarly, replacing Top by Sch_S and open cover by Zariski open cover, we obtain the Zariski topology.* [39]

Definition 16.7. *Let $\mathcal{C}$ be a category with Grothendieck topology generated by $\mathcal{T}$. A presheaf $F : \mathcal{C} \to Set$ is a sheaf for $\mathcal{T}$ if*

1. *for any collection of objects $\{T_i\}$, $F(\bigsqcup T_i) = \prod F(T_i)$, and*

2. *for any object X and morphism $U \to X$ in $\mathcal{T}$, the sequence*

$$F(X) \to F(U) \rightrightarrows F(U \times_X U)$$

is an equalizer where the maps are induced by the two projections.

The topologies we will consider now, beyond the Zariski topology, are the fpqc and fppf topologies, where $\mathcal{T}$ is the collection of fpqc, respectively fppf morphisms. The main theorem is the following.

Theorem 16.8. *Let F be a representable functor $Sch_S \to Set$. Then F is a sheaf for the fpqc topology.*

We will leave this as an exercise, with the hint that this follows from fpqc descent. More precisely, one uses the fact that for an fpqc morphism $p : S' \to S$, the functor $p^* : Sch_S \to Sch_{S' \to S}$ is fully faithful.

Finally, we recall the notion of sheafification. Let F be any presheaf on a category $\mathcal{C}$ with Grothendieck topology generated by a collection $\mathcal{T}$. For any $p : U \to X$ in $\mathcal{T}$, we define

$$H^0(F, p) = Eq(F(U) \rightrightarrows F(U \times_X U)).$$

Now we define the presheaf F^+ by

$$F^+(X) = \mathrm{colim}_{(p:U \to X) \in \mathcal{T}} H^0(F, p).$$

There is a natural morphism of presheaves $F \to F^+$. We have the following theorem.

Theorem 16.9. *The construction $F \to (F \to F^+)$ is functorial in F. Moreover, for any F, the presheaf F^{++} is a sheaf. Moreover, it is universal for sheaves receiving a map from F.*

We call F^{++} the sheafification of F for the topology generated by $\mathcal{T}$.

[39] Note that here we are using a convenient notational trick of replacing an open covering $\{U_i\}$ with their disjoint union $\bigsqcup U_i$ mapping to X.

16.3 Quotients by flat and proper equivalence relations

We now return to the existence of quotients by flat and proper equivalence relations. Given an equivalence relation $R \to X \times_S X$ on an S-scheme X, one can consider the categorical quotient in the category of fppf sheaves on Sch_S. In this category, all quotients exist. Indeed, we can define the quotient $(X/R)_{fppf}$ as the fppf-sheafification of the presheaf

$$T \mapsto X(T)/R(T)$$

where the latter denotes the quotient of the set $X(T)$ by the set theoretic equivalence relation $R(T)$.

Lemma 16.10. *Let $f : X \to Z$ be an fppf morphism of S-schemes and let $R = X \times_Z X \subset X \times_S X$. Then Z is an effective quotient of X by R in the category of schemes, and moreover, Z represents the fppf-sheafification $(X/R)_{fppf}$.*

Proof. Let $g : X \to Y$ be any morphism such that $g \circ p_1 = g \circ p_2$ where $p_i : R \to X$ are the two projections. Then as a morphism of sheaves for the fppf topology, g factors uniquely through $(X/R)_{fppf}$ as $X \to (X/R)_{fppf} \to Y$. We conclude that if $(X/R)_{fppf}$ is representable by a scheme, then it must be the categorical quotient of X by R. On the other hand, by fppf descent, in particular, fully faithfulness of the functor

$$f^* : Sch_Z \to Sch_{X \to Z},$$

Z represents the functor $(X/R)_{fppf}$ and so Z is a categorical quotient of X by R. By assumption, $R = X \times_Z X$ so the quotient is effective. $\square$

Remark 16.11. *This same analysis could have been carried out with the fppf topology replaced by the fpqc topology.*

Theorem 16.12. *Let $f : X \to S$ be a quasi-projective scheme over a Noetherian scheme S and let $R \subset X \times_S X$ be a flat and proper equivalence relation on X. Then an effective quotient X/R exists and it is a quasi-projective S-scheme. Moreover, the map $q : X \to X/R$ is fppf.*

Proof. Since $R \to X$ is flat and proper over a Notherian scheme, and so in particular, of finite presentation, there are a finite number of Hilbert polynomials $\{P_1, \ldots, P_n\}$ such that the fibers of $R \to X$ have Hilbert polynomial $P = P_i$ for some i. Let $H = \bigsqcup \mathrm{Hilb}^{P_i}_{X/S}$ be the quasi-projective S-scheme obtained as the union of these components of the Hilbert scheme of X/S and let $\mathcal{Z} \subset X \times_S H$ be the universal family of subschemes over H. Then $R \subset X \times_S X$ gives an X-point of H, that is, a morphism

$$g : X \to H$$

such that $g^* \mathcal{Z} = R$.

Let $\Gamma_g \subset X \times_S H$ be the graph of g. Since $H \to S$ is separated, $\Gamma_g \subset X \times_S H$ is a closed embedding. Now for any T, let $x_1, x_2 \in X(T)$ be two T-points which by the isomorphism $\Gamma_g \to X$ can be identified with T-points of the graph. Now we have

$$(x_1, x_2) \in R(T) \iff (x_1, gx_2) \in \mathcal{Z}(T) \iff gx_1 = gx_2.$$

The first equality follows by the fact that $g^*\mathcal{Z} = R$ and the second from the fact that g is the second projection from the graph and the properties of an equivalence relation. In particular, since $(x_1, x_1) \in R(T)$, then $(x_1, gx_1) \in \mathcal{Z}(T)$ so $\Gamma_g \subset \mathcal{Z}$ is a closed subscheme of $\mathcal{Z}$.

Now $\mathcal{Z} \to H$ is an fppf morphism. We claim that as subschemes of the fiber product $\mathcal{Z} \times_H \mathcal{Z}$, $\Gamma_g \times_H \mathcal{Z} = \mathcal{Z} \times_H \Gamma_g$. A T-point of either, by the string of equalities above and the definition of a graph, corresponds to a pair $(x_1, x_2) \in R(T)$ and so we conclude the required equality. Then by fpqc descent of closed subschemes, $\Gamma_g \subset \mathcal{Z}$ descends to a closed subscheme $Y \subset H$ with an fppf morphism $\Gamma_g \to Y$. By definition of the graph this can be identified with the morphism $g : X \to H$ and so g factors through an fppf morphism $g : X \to Y$. By the lemma, this fppf morphism is an effective categorical quotient of X by $X \times_Y X$ but again by the above string of equalities, this fiber product is just R. Therefore $g : X \to Y$ is an effective categorical quotient of X by R. Finally, Y is a closed subscheme of the quasi-projective S-scheme H so Y is quasi-projective.

□

16.4 Representability of the Picard functor

We are now ready to prove the main representability result, assuming the above theorem on quotients by flat and proper equivalence relations. We need the following preliminary result.

Proposition 16.13. *Let $f : X \to S$ be a flat projective universal algebraic fiber space over a Noetherian scheme S. Suppose S has a section $\sigma : S \to X$ and the fibers of f are geometrically integral. Then $\mathrm{Pic}_{X/S}$ is an fppf sheaf.*

Proof. Under these assumptions[40], the functor $\mathrm{Pic}_{X/S}$ is isomorphic to the functor of σ-rigidified line bundles $\mathrm{Pic}_{X/S,\sigma}$. Now using fppf descent one can check that this latter functor is an fppf sheaf. □

Theorem 16.14. *Let $f : X \to S$ be a flat projective universal algebraic fiber space over a Noetherian scheme S. Suppose f has a section $\sigma : S \to X$ and that the fibers of f are geometrically integral. Then for each Hilbert polynomial P, the functor $\mathrm{Pic}^P_{X/S}$ is representable by a quasi-projective S-scheme.*

Proof. By uniform CM regularity, there exists an m such that for any k-point of $\mathrm{Pic}^P_{X/S}$ corresponding to a line bundle L on X_k, then $H^i(X_k, L(m)) = 0$ for all $i > 0$. In particular, the Abel-Jacobi map for $P_1(d) := P(d+m)$ is representable by smooth and proper surjections. Let P_2 be the Hilbert polynomial of the component of $CDiv_{X/S}$ such that for any $D \subset X_T$ with Hilbert polynomial P_2, $\mathcal{O}_{X_T}(D)$ has Hilbert polynomial P_1 and let us denote

$$\mathcal{D}(P_2) := CDiv^{P_2}_{X/S}.$$

Let R denote the fiber product $\mathcal{D}(P_2) \times_{CDiv^{P_1}_{X/S}} \mathcal{D}(P_2)$.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \mathcal{D}(P_2) \arrow[d] \\ \mathcal{D}(P_2) \arrow[r] & \mathrm{Pic}^{P_1}_{X/S} \end{tikzcd}$$

[40]Check which assumptions we actually need.

Then $R \to \mathcal{D}(P_2) \times_S \mathcal{D}(P_2)$ is a flat and proper equivalence relation. By the fppf sheaf condition,

$$\mathrm{Pic}^{P_1}_{X/S}$$

is the quotient in the category of fppf sheaves of $\mathcal{D}(P_2)$ by R and by the previous theorem, an effective quotient $\mathcal{D}(P_2)/R$ exists in the category of quasi-projective S-schemes so this quotient represents $\mathrm{Pic}^{P_1}_{X/S}$. Then tensoring by $\mathcal{O}_X(m)$ induces an isomorphism

$$\mathrm{Pic}^{P_1}_{X/S} \cong \mathrm{Pic}^{P}_{X/S}$$

so we conclude representability of $\mathrm{Pic}^{P}_{X/S}$ by a quasi-projective S-scheme. □

17 Deformation theory of line bundles, compactified Jacobians of integral curves

17.1 Deformation theory of line bundles

Our goal now is to compute the local structure of the Picard scheme. In particular, we can ask is it regular or smooth? The first step is to compute the tangent space. Recall the following basic proposition from scheme theory.

Proposition 17.1. *Let Y be a scheme and $\xi : \operatorname{Spec} k \to Y$ a k-point. The tangent space $T_\xi Y$ is the the set of maps $\operatorname{Spec} k[\epsilon] \to Y$ where $\epsilon^2 = 0$ such that the following diagram commutes*

$$\begin{tikzcd} \operatorname{Spec} k[\epsilon] \arrow[r] & Y \\ \operatorname{Spec} k \arrow[u] \arrow[ur, "\xi"'] & \end{tikzcd}$$

In the case Y is the Picard scheme $\mathrm{Pic}_{X/S}$, ξ corresponds to a line bundle L on X_k and the tangent space is the fiber over $[L]$ of the map of groups

$$\mathrm{Pic}_{X/S}(k[\epsilon]) \to \mathrm{Pic}_{X/S}(k).$$

Using the group action, we can tensor by L^{-1} so that we get a new point $\xi' : \operatorname{Spec} k \to \mathrm{Pic}_{X/S}$ corresponding to the line bundle $L \otimes L^{-1} = \mathcal{O}_{X_k}$. Since tensoring by a line bundle is an isomorphism of functors, it suffices to compute the tangent space for $\mathcal{O}_{X_k}$. This is the identity of the group $\mathrm{Pic}_{X/S}(k)$ so weve deduced that the tangent space to the Picard scheme is isomorphic to the kernel of the map above. That is, we have an exact sequence

$$0 \to T_\xi \mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X/S}(k[\epsilon]) \to \mathrm{Pic}_{X/S}(k).$$

Proposition 17.2. *The tangent space to $\xi : \operatorname{Spec} k \to \mathrm{Pic}_{X/S}$ corresponding to the line bundle $\mathcal{O}_{X_k}$ is isomorphic to*

$$H^1(X_k, \mathcal{O}_{X_k}).$$

Proof. The scheme $X_{k[\epsilon]}$ has the same underlying topological space as X_k with structure sheaf $\mathcal{O}_{X_{k[\epsilon]}} = \mathcal{O}_X[\epsilon] := \mathcal{O}_X \otimes_k k[\epsilon]$. The map

$$\mathrm{Pic}_{X/S}(k[\epsilon]) \to \mathrm{Pic}_{X/S}(k)$$

can be identified with the map

$$H^1(X_k, \mathcal{O}_{X_k}[\epsilon]^*) \to H^1(X_k, \mathcal{O}_{X_k}^*).$$

Here we are taking cohomology of sheaves of abelian groups on the underlying topological space X_k. To compute the kernel, consider the short exact sequence of sheaves of abelian groups, written multiplicatively.

$$1 \to 1 + \epsilon\mathcal{O}_{X_k} \to \mathcal{O}_{X_k}[\epsilon]^* \to \mathcal{O}_{X_k}^* \to 1.$$

The multiplicative sheaf $1 + \epsilon\mathcal{O}_{X_k}$ is isomorphic to the sheaf of additive abelian groups $\mathcal{O}_{X_k}$ since $\epsilon^2 = 0$. Taking the long exact sequence of cohomology we get

$$H^0(X_k, \mathcal{O}_{X_k}[\epsilon]^*) \to H^0(X_k, \mathcal{O}_{X_k}^*) \to H^1(X_k, \mathcal{O}_{X_k}) \to \mathrm{Pic}_{X/S}(k[\epsilon]) \to \mathrm{Pic}_{X/S}(k).$$

The first map is surjective, so by exactness, the kernel of interest is $H^1(X_k, \mathcal{O}_{X_k})$ as claimed. □

Having computed the tangent space to $\mathrm{Pic}_{X/S}$, we can ask more generally if it is smooth over S. Recall the following definition of formally smooth.

Definition 17.3. *A map of schemes $X \to S$ is formally smooth if for any closed embedding of affine S-schemes $i : T \to T'$ defined by a square zero ideal, and any solid diagram as below, there exists a dotted arrow making the diagram commute.*

$$\begin{tikzcd} T \arrow[r] \arrow[d, "i"'] \arrow[dr, dashed] & X \arrow[d, "f"] \\ T' \arrow[r] & S \end{tikzcd}$$

The advantage of formal smoothness is that it is a condition on the functor of points of $X \to S$ on Sch_S. On the other hand, we have the following lifting criterion for smoothness.

Proposition 17.4. *A morphism $X \to S$ is smooth if and only if it is formally smooth and locally of finite presentation.*

The conditions under which we proved representability of the Picard functor also gurantee that $\mathrm{Pic}_{X/S} \to S$ is locally of finite presentation. Thus we can check smoothness using formal smoothness. It suffices to consider the case $S = \mathrm{Spec}\ R$ is affine. Then $i : T \to T'$ corresponds to a surjection $A' \to A$ of R-algebras with kernel I satisfying $I^2 = 0$. The lifting criterion to smoothness then asks the question of when

$$\mathrm{Pic}_{X/S}(A') \to \mathrm{Pic}_{X/S}(A)$$

is surjective. Repeating the argument from the computation of the tangent space, we get the following.

Proposition 17.5. *Suppose $f : X \to S$ is an S-scheme which is A-flat. Then we have an exact sequence*

$$0 \to H^1(X_A, f^*I) \to \mathrm{Pic}(X_{A'}) \to \mathrm{Pic}(X_A) \to H^2(X_A, f^*I).$$

Proof. By A-flatness of X, we have that the sequence

$$0 \to f^*I \to \mathcal{O}_{X_{A'}} \to \mathcal{O}_{X_A} \to 0,$$

obtained by pulling back $0 \to I \to A' \to A \to 0$ to X, is exact. Since I is square zero, we have the following exact sequence of multiplicative groups.

$$1 \to 1 + f^*I \to \mathcal{O}^*_{X_{A'}} \to \mathcal{O}^*_{X_A} \to 1$$

Taking the long exact sequence of cohomology and using the fact that the multiplicative sheaf $1 + f^*I$ is isomorphic to the additive sheaf f^*I and that the map $H^0(X_A, \mathcal{O}^*_{X_{A'}}) \to H^0(X_A, \mathcal{O}^*_{X_A})$ is surjective concludes the proof. □

Corollary 17.6. *Suppose $T \to T'$ is a square zero thickening of affine schemes corresponding to $A' \to A$. Then $\mathrm{Pic}(T') \to \mathrm{Pic}(T)$ is an isomorphism.*

Proof. By the proposition, we have an exact sequence

$$H^1(T, I) \to \mathrm{Pic}(T') \to \mathrm{Pic}(T) \to H^2(T, I).$$

Now I is quasi-coherent and T is affine so the first and last groups vanish. □

Putting this all together, we get the following result.

Theorem 17.7. *Let $S = \mathrm{Spec}\ R$ be an affine Noetherian scheme and suppose that $f : X \to S$ is as in the existence theorem for the Picard scheme.*[41] *Then for any $A' \to A$ a morphism of R-algebras with square-zero kernel I, and any diagram*

$$\begin{tikzcd} \mathrm{Spec}\ A \arrow[r, "\xi"] \arrow[d] & \mathrm{Pic}_{X/S} \arrow[d, "f"] \\ \mathrm{Spec}\ A' \arrow[r] & S \end{tikzcd}$$

there exists an element $\mathrm{obs}(\xi) \in H^2(X_A, f^*I)$ such that a lift $\xi' : \mathrm{Spec}\ A' \to \mathrm{Pic}_{X/S}$ exists if and only of $\mathrm{obs}(\xi) = 0$. Moreover, in this case, the set of such lifts is a torsor[42] *for the group $H^1(X_A, f^*I)$.*

Proof. Combining the above proposition and corollary, we see that for $T \to T'$ being a square-zero thickening of affine schemes, the exact sequence of Proposition 5 becomes an exact sequence

$$0 \to H^1(X_A, f^*I) \to \mathrm{Pic}_{X/S}(A') \to \mathrm{Pic}_{X/S}(A) \to H^2(X_A, f^*I).$$

[41] These assumptions can be relaxed for deformation theory but we keep them here for simplicity.

[42] a set with a free and transitive action of

Here we have used the assumptions on $f : X \to S$ only to gurantee that $\mathrm{Pic}_{X/S}(A) = \mathrm{Pic}(X_A)/\mathrm{Pic}(A)$ and similarly for A'. Then the statement of the theorem is just a reinterpretation of exactness. Indeed ξ gives an element of $\mathrm{Pic}_{X/S}(A)$ and its image in $H^2(X_A, f^*I)$ is $\mathrm{obs}(\xi)$. Then ξ is in the image of the middle map if and only if it is in the kernel of the last map if and only of $\mathrm{obs}(\xi) = 0$. When this happens, the set of preimages of ξ under the middle map has a free and transitive action by the kernel of the middle map, which is exactly the image of $H^1(X_A, f^*I) \to \mathrm{Pic}_{X/S}(A')$. $\square$

This is an exampe of a deformation-obstruction theory, in this case for the Picard functor. The connecting map $\mathrm{obs} : \mathrm{Pic}_{X/S}(A) \to H^2(X, f^*I)$ and the association that takes a square-zero thickening of affine schemes $A' \to A$ to the groups $H^*(X, f^*I)$ is functorial in I. The group $H^2(X, f^*I)$ is the *obstruction group* and $H^1(X, f^*I)$ is the *group of first order deformations*. The special case where $A' = k[\epsilon] \to A = k$ gives us the tangent space to $\mathrm{Pic}_{X/S}$ and, analagously, the deformation-obstruction theory can be thought of as encoding functorially the local structure of $\mathrm{Pic}_{X/S}$.

In general, one can ask whether a moduli functor admits a deformation-obstruction theory which has the features above (an obstruction group which receives an obstruction map whose image vanishes if and only if a lift exists and a deformation group under which the set of lifts is a torsor if its nonempty which are functorial in the square-zero extension $A' \to A$). This forms the basis of Artin's axiomatic approach to representability of moduli problems by algebraic spaces and stacks.

From the previous result, we have the following corollary.

Corollary 17.8. *Let $f : X \to S$ as in the existence theorem for the Picard scheme and suppose further that the fibers are curves. Then $\mathrm{Pic}_{X/S}$ is smooth over S of relative dimension g, the arithmetic genus of the family of curves f.*

Proof. We can suppose without loss of generality that $S = \mathrm{Spec}\ R$ is affine. Then the statement follows from the lifting criterion for smoothness (since $\mathrm{Pic}_{X/S} \to S$ is locally of finite type and S is Noetherian). To check that the lifting holds, it suffices to check that obstruction group vanishes. By the theorem this is a second coherent cohomology group which vanishes since $X \to S$ is a curve. Finally, the tangent space to a fiber over $\mathrm{Spec}\ k \to S$ is computed by $H^1(X_k, \mathcal{O}_{X_k})$ which is $g = p_a$ dimensional. $\square$

17.2 Jacobians of integral curves

We saw previously that when X is a smooth projective curve over a field, the Hilbert polynomials of line bundles are just indexed by the degree d and the Abel-Jacobi map is given as

$$AJ^d_{X/k} : \mathrm{Hilb}^d_{X/k} \to \mathrm{Pic}^d_{X/k}$$

from the Hilbert scheme of zero dimensional subschemes with Hilbert polynomial constant d, that is, subschemes of length d, to the component of the Picard scheme of degree d line bundles. Moreover, we saw using Riemann-Roch and Serre duality that for $d > 2g - 2$, AJ^d_X is a smooth projective bundle with fiber dimension $d - g$. By the results of the previous section on deformation theory, we also know that $\mathrm{Pic}^d_{X/k}$ is smooth. In particular, since the genus g is constant in flat families, this holds in the relative setting so that

$$AJ^d_{X/S} : \mathrm{Hilb}^d_{X/S} \to \mathrm{Pic}^d_{X/S}$$

is a smooth projective bundle of rank $d - g$ for $d > 2g - 2$ whenever $f : X \to S$ is a smooth integral projective one dimensional universal algebraic fiber space with a section over a Noetherian base. Moreover, we saw in the smooth case $\mathrm{Pic}^d_{X/S}$ is in fact proper.

Definition 17.9. *The Jacobian* $\mathrm{Jac}_{X/S} = \mathrm{Pic}^0_{X/S}$ *is the degree zero component of the Picard scheme.*

In particular, when $f : X \to S$ is a smooth curve with the assumptions above $\mathrm{Jac}_{X/S} \to S$ is a smooth and proper group scheme over S so it is an abelian scheme. Moreover, in this case, if $f : X \to S$ has a section $\sigma : S \to X$, then twisting by $\mathcal{O}_X(\sigma(S))$ gives an isomorphism $\mathrm{Pic}^n_{X/S} \cong \mathrm{Pic}^{n+1}_{X/S}$ and so each component is isomorphic to the Jacobian.[43]

In the special case when $S = \mathrm{Spec}\ \mathbb{C}$ it is the abelian variety corresponding to the g-dimensional complex analytic torus

$$H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}).$$

Indeed using the exponential sheaf sequence

$$0 \longrightarrow 2\pi i \mathbb{Z}_X \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

one can identify the degree map $\deg : \mathrm{Pic}_{X/S} \to \mathbb{Z}$ with the connecting map $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ and exactness implies $\ker(\deg)$ is the claimed quotient.

17.2.1 Singular curves

We are interested more generally in the case that $f : X \to S$ is a family of integral but not necessarily smooth curves. Under the usual assumptions, we have proved the existence of the Picard scheme and have constructed an Abel-Jacobi map from an open subscheme of $\mathrm{Hilb}^n_{X/S}$. In the case where the fibers of f are also assumed to be Gorenstein, then the picture is almost identical.

Remark 17.10. *Recall that a quasi-projective scheme X/k is Gorenstein if it is Cohen-Macaulay and the dualizing sheaf $\omega_{X/k}$ is a line bundle. The most important case for us to note is that local complete intersection varieties, and in particular hypersurfaces, are Gorenstein. In this case, the theory of adjunction tells us that if X is a hypersurface in a smooth variety $\mathbb{P}$, then $\omega_{X/k} = \omega_{\mathbb{P}/k} \otimes \mathcal{O}_{\mathbb{P}}(X)|_X$ where for a smooth variety $\omega_{\mathbb{P}/k} = \Omega^{\dim X}_{X/k}$.*

The main input we need is that Riemann-Roch and Serre duality work as expected for X/k an integral Gorenstein curve.

Theorem 17.11. *Let X/k be an integral Gorenstein curve with arithmetic genus $g = p_g := \dim H^1(X, \mathcal{O}_X)$. Then for any line bundle L,*

$$H^i(X, L)^\vee \cong H^{1-i}(X, \omega_X \otimes L^\vee)$$

and there exists a number $d = \deg L$ such that

$$\chi(X, L) = \deg L + \chi(\mathcal{O}_X) = \deg L + 1 - g.$$

[43] More generally, the components $\mathrm{Pic}^n_{X/S}$ are torsors over $\mathrm{Jac}_{X/S}$.

By the previous remark, the theorem holds in particular whenever the integral curve X is embedded in a smooth surface S. In fact, we ony need to assume such an embedding locally since both the Cohen-Macualay condition and the condition of being a line bundle are local. In particular, we have that curves with singularities that can be embedded in the affine plane[44] are Gorenstein. We call such curves *locally planar*.

The upshot, is that for $f : X \to S$ a flat family of projective Gorenstein curves satisfying the usual assumtpions, the structure of $\mathrm{Pic}_{X/S}$ is almost identical to the smooth case. The components are indexed by degree d, the Abel-Jacobi map is a smooth projective bundle above degree $2g-2$ with fibers of dimension $d-g$, and the degree 0 component $\mathrm{Jac}_{X/S} \to S$ is a smooth group scheme of relative dimension g where $g = p_a$ is the arithmetic genus of the family. The one thing that fails is properness, as we have seen.

Let X/k be projective Gorenstein curve with arithmetic genus $g = p_a$ and let $\nu : X^\nu \to X$ be the normalization so that X^ν is a smooth projective curve of genus $p_g \leq g$, the geometric genus of X/k. Pulling back gives us a homomorphism

$$\nu^* : \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X^\nu/k}$$

of group schemes which preserves the degree.

On the other hand, consider the short exact sequence of sheaves on X

$$1 \to \mathcal{O}_X^* \to \nu_* \mathcal{O}_{X^\nu}^* \to \mathcal{F} \to 1$$

where $\mathcal{F}$ is the cokernel of the pullback map on invertible functions. Then $\mathcal{F}$ is a direct sum of skyscraper sheaves of abelian groups supported at the singular points of X. Now we take the long exact sequence of cohomology, noting that $H^1(X, \mathcal{F}) = 0$ since $\mathcal{F}$ is supported on points, that the pullback map on global functions is an isomorphism, and that $H^1(X, \nu_* \mathcal{O}_{X^\nu}^*) = H^1(X^\nu, \mathcal{O}_{X^\nu}^*)$ since ν is finite, we get the short exact

$$1 \to H^0(X, \mathcal{F}) \to H^1(X, \mathcal{O}_X^*) \to H^1(X^\nu, \mathcal{O}_{X^\nu}^*) \to 1$$

of abelian groups. The latter map is the pullback map ν^* on Picard groups and since ν^* preserves degrees, we get a short exact sequence

$$1 \to H^0(X, \mathcal{F}) \to \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X^\nu) \to 0.$$

The same analysis can be performed for X_T for any $T \to \operatorname{Spec} k$ and so we get an exact sequence of group schemes

$$1 \to F \to \mathrm{Jac}_{X/k} \to \mathrm{Jac}_{X^\nu/k} \to 0$$

where F is the commutative group scheme over k representing the sheafification of the functor

$$T \mapsto H^0(X_T, \mathcal{F}_T)^{45}$$

The main thing to note is that the group scheme F is a direct sum over each singular point $x \in X$ of a local factor F_x depending only on the stalk $\mathcal{F}_x$ of the skyscraper $\mathcal{F}$. On the other hand, $\mathcal{F}_x$ depends only on the completed local ring $\widehat{\mathcal{O}}_{X,x}$ which allows us to compute F in examples.

[44]equivalently, have tangent space dimension 2

[45]Note since $\mathcal{F}$ is a union of skyscrapers, this is just the locally constant sheaf associated to the group $H^0(X, \mathcal{F})$.

Example 17.12. *(The node) Suppose $x \in X$ has a split nodal singularity. That is, the completed local ring is isomorphic to $R = k[\![x,y]\!]/xy$. The normalization has completed local ring $\tilde{R} = k[\![x]\!] \times k[\![y]\!]$. Then the stalk of $\mathcal{F}_x$ can be computed from the sequence*

$$1 \to R^* \to \tilde{R}^* \to \mathcal{F}_x \to 1.$$

Then the map $\tilde{R}^* \to k^*$ given by $(f,g) \mapsto f(0)/g(0)$ identifies $\mathcal{F}_x$ with k^* so $\mathcal{F}$ is the skyscraper sheaf k^*_x and the group scheme F is simply $\mathbb{G}_m$. More generally, suppose X has exactly δ split nodal singular points and is smooth elsewhere. Then we have an exact sequence

$$1 \to \mathbb{G}_m^{\oplus\delta} \to \mathrm{Jac}_{X/k} \to \mathrm{Jac}_{X^\nu/k} \to 0$$

where $\mathrm{Jac}_{X^\nu/k}$ is a $g-\delta$ dimensional abelian variety.

Example 17.13. *(The cusp) Suppose $x \in X$ has a cuspidal singularity with completed local ring isomorphic to $k[\![x,y]\!]/\{y^2 = x^3\}$. Then $\tilde{R} = k[\![t]\!]$ with the map $R \to \tilde{R}$ given by $(x,y) \mapsto (t^3, t^2)$. The cokernel of $R^* \to \tilde{R}^*$ can be identified with the map $\tilde{R} \to k$ given by*

$$g(t) \mapsto \frac{g(t)-g(0)}{t}\Big|_{t=0}.$$

Therefore, $F = \mathbb{G}_a$ is the additive group and we have an exact sequence

$$0 \to \mathbb{G}_a \to \mathrm{Jac}_{X/k} \to \mathrm{Jac}_{X^\nu/k} \to 0$$

17.3 Compactified Jacobians

Our goal now is to compactify the Jacobian, or more generally Pic^d for $f : X \to S$ a family of locally planar, or more generally Gorenstein, integral curves.

The idea is to again leverage the Abel-Jacobi map as in the construction of $\mathrm{Pic}_{X/S}$. In the case of curves we have the space of degree d Cartier divisors $CDiv^d_{X/S}$ sitting inside of the Hilbert scheme $\mathrm{Hilb}^d_{X/S}$. While $CDiv^d_{X/S}$ is not proper, $\mathrm{Hilb}^d_{X/S}$ is and so the idea is to extend functor of $\mathrm{Pic}_{X/S}$ to something that admits an Abel-Jacobi map from the proper S-scheme $\mathrm{Hilb}^d_{X/S}$ and then construct a representing object as a quotient of $\mathrm{Hilb}^d_{X/S}$ by a flat and proper equivalence relation.

If $D \subset X$ is a length d subscheme that is not necessarily a Cartier divisor, then the ideal sheaf I_D is not a line bundle, but it is a rank 1 torsion free sheaf.

Definition 17.14. *Let X/k be an integral variety over a field. A torsion free sheaf on X is a coherent sheaf $\mathcal{E}$ such that the support* $\mathrm{Supp}(\mathcal{E})$ *has no embedded points. Equivalently, the annihilator of $\mathcal{E}$ is the* 0 *ideal. The rank of a torsion free sheaf is the rank of the generic fiber $\mathcal{E}_\eta$.*

Now to see that I_D for $D \subset X$ a closed subscheme of an integral curve X/k is a rank 1 torsion free sheaf, note that $I_D \subset \mathcal{O}_X$ and $\mathcal{O}_X$ is torsion free. Moreover, the inclusion is an isomorphism away from D so the rank of I_D is 1. In this setting, a point $D \subset X$ of the Hilbert scheme is called a generalized divisor.

Definition 17.15. *The degree of a rank* 1 *torsion free sheaf I on an integral curve X/k is defined as*

$$\chi(I) - \chi(\mathcal{O}_X).$$

This definition of degree generalizes the degree of a line bundle on a smooth projective curve as computed by Riemann-Roch. Now we can define the compactified Picard functor.

Definition 17.16. *Let $f : X \to S$ be a projective family of integral curves. A family of rank 1 degree d torsion free sheaves on X is an S-flat coherent sheaf $\mathcal{I}$ on X such that $\mathcal{I}|_{X_s}$ is a rank 1 degree d torsion free sheaf on X_s for each $s \in S$.*

Definition 17.17. *Let $f : X \to S$ be a flat projective family of integral curves. For each integer d, the compactified Picard functor $\overline{\mathrm{Pic}}^d_{X/S} : Sch_S \to Set$ given by*

$$T \mapsto \{\text{families of rank } 1 \text{ degree } d \text{ coherent sheaves on } X_T \to T\}/\mathrm{Pic}(T).$$

Note that a line bundle L on X_T is in particular a family of rank 1 degree d coherent sheaves on X_T so that $\mathrm{Pic}^d_{X/S}$ is a subfunctor of $\overline{\mathrm{Pic}}^d_{X/S}$. The special case $d = 0$, the compactified Jacobian, will be denoted by $\overline{\mathrm{Jac}}_{X/S}$.

Remark 17.18. *Note that as in the case of the usual Picard functor, if our family of $f : X \to S$ has a section σ that is contained in the regular locus, then $\sigma(S)$ is a relative Cartier divisor of degree 1 and twisting by $\mathcal{O}_X(-d\sigma(S))$ gives an isomorphism of functors*

$$\overline{\mathrm{Pic}}^d_{X/S} \to \overline{\mathrm{Jac}}_{X/S}.$$

This happens in particular if $S = \mathrm{Spec}\, k$ and $X \setminus X^{sing}$ has a rational point.

The idea now is to extend the Abel-Jacobi map $AJ^d_{X/S} : CDiv^d_{X/S} \to \mathrm{Pic}^d_{X/S}$ to the compactified Jacobian.

$$\begin{array}{ccc} \mathrm{Hilb}^d_{X/S} & \xrightarrow{AJ^d_{X/S}} & \overline{\mathrm{Pic}}^d_{X/S} \\ \uparrow & & \uparrow \\ CDiv^d_{X/S} & \xrightarrow[AJ^d_{X/S}]{} & \mathrm{Pic}^d_{X/S} \end{array}$$

The extension must be defined on points by sending a flat closed subscheme of degree d $D \subset X_T$ to rank 1 torsion free sheaf $I_D^\vee := \mathcal{H}om_{X_T}(I_D, \mathcal{O}_{X_T})$. The problem is that when I_D is not a line bundle, taking duals isn't well behaved in general so its not clear this is a well defined natural transformation of functors. However, we have the following results due to Hartshorne. Let us denote $I_D^\vee$ by $\mathcal{O}_X(D)$ in analogy with the Cartier case.

Proposition 17.19. *(Properties of generalized divisors on Gorenstein curves) Suppose X/k is a Gorenstein integral curve and let I be a rank 1 torsion free sheaf on X. We have the following:*

(a) the natural map $I \to (I^\vee)^\vee$ is an isomorphism[46],

(b) $\deg \mathcal{O}_X(D) = \deg D$,

(c) Riemann-Roch and Serre duality hold for $\mathcal{O}_X(D)$.

[46] that is, I is a reflexive sheaf

Moreover, we have the usual correspondence between the set of $D \subset X$ such that $\mathcal{O}_X(D) \cong L$ and sections $H^0(X, L)$. Note that these two are also in bijection with $\mathrm{Hom}_X(I_D, \mathcal{O}_X)$.

The above facts tell us that the Abel-Jacobi map for $\overline{\mathrm{Pic}}^d_{X/S}$ works as expected on the level of points when $f : X \to S$ is a flat projective family of integral Gorenstein curves. The missing piece is that it behaves well under base-change. This comes from a certain generalization of the cohomology and base change theorem for $\mathcal{E}xt^i$ groups rather than cohomology groups due to Altman and Kleiman. In this particular case we get the following.

Theorem 17.20. *(Altman-Kleiman) Suppose $f : X \to S$ is a flat projective family of integral Gorenstein curves and I a family of torsion free sheaves on $f : X \to S$. Then $\mathcal{H}om_X(I, \mathcal{O}_X)$ is flat and its formation commutes with arbitrary base change.*

The key point here is that the vanishing of H^1 that implies that pushforwards commute with basechange is replaced in this case with a vanishing $\mathrm{Ext}^1_{X_k}(I_k, \mathcal{O}_{X_k}) = 0$ which holds since X_k is Gorenstein. This implies that the Abel-Jacobi map is a well defined natural transformation of functors, and by repeating the argument for the Picard group we obtain the following theorem of Altman and Kleiman.

Theorem 17.21. *Let $f : X \to S$ be a flat projective family of integral Gorenstein curves over a Noetherian scheme satisfying conditions $(**)$. Then $\overline{\mathrm{Pic}}^d_{X/S}$ is representable by a projective S-scheme. Moreover, the Abel-Jacobi map*

$$AJ^d_{X/S} : \mathrm{Hilb}^d_{X/S} \to \overline{\mathrm{Pic}}^d_{X/S}$$

is identified with the projectivization of a coherent sheaf. When $d > 2g - 2$ where g is the arithmetic genus of $f : X \to S$, the Abel-Jacobi map is a smooth projective bundle of rank $d - g$.

Example 17.22. *Let X/k be a projective geometrically integral Gorenstein curve of genus* 1*. Then the the $d = 1$ Abel-Jacobi map $\mathrm{Hilb}^1_{X/k} = X \to \overline{\mathrm{Pic}}^1_{X/k}$ is a smooth projective bundle of rank $1 - 1 = 0$, that is, its an isomorphism. In this case, $\overline{\mathrm{Pic}}^1_{X/k}$ is the curve itself and the points in the boundary*

$$\overline{\mathrm{Pic}}^1_{X/k} \setminus \mathrm{Pic}^1_{X/k}$$

correspond to the maximal ideal I of the singular point, or more precisely, its $\mathcal{O}_X$ dual $\mathcal{H}om_{\mathcal{O}_X}(I, \mathcal{O}_X)$.

17.4 The topology of compactified Jacobians

For this section let us work over $k = \mathbb{C}$ the complex numbers. Let X/k be a projective integral Gorenstein curve. To study the topology of $\overline{\mathrm{Jac}}_X$, we will leverage the action of Jac_X by tensoring with a degree 0 line bundle.

Toward that end, let $\mathcal{I}$ a rank 1 degree 0 torsion free sheaf and L a line bundle. Consider the endormorphism algebra $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{I})$. This is a finite extension of $\mathcal{O}_X$ with generic fiber equal to the function field $k(X)$. Thus $X' := \mathrm{Spec}\ {}_X\mathcal{A}$ is an integral curve mapping finitely and birationally to X. That is, $f : X' \to X$ is a partial normalization of X. Moreover, $\mathcal{I}$ is an $\mathcal{A}$-algebra and by construction, $f_*\mathcal{O}_{X'} = \mathcal{A}$, therefore $\mathcal{I}$ is an $f_*\mathcal{O}_{X'}$-module and by pulling back sections we get an $\mathcal{O}_{X'}$-module $\mathcal{I}'$ such that $f_*\mathcal{I}' = \mathcal{I}$. In this way, every rank 1 torsion free sheaf on X is pushed forward from some partial normalization.

Lemma 17.23. *The sheaf $\mathcal{I} \otimes L$ is isomorphic to $\mathcal{I}$ if and only if $f^*L \cong \mathcal{O}_{X'}$ where $f : X' \to X$ is the partial normalization associated to $\mathcal{I}$.*

Proof. If f^*L is trivial, then $\mathcal{I} \otimes L = f_*(\mathcal{I}' \otimes f^*L) = f_*\mathcal{I}' = \mathcal{I}$ by the projection formula. Similarly, consider

$$\mathcal{H}om(\mathcal{I}, \mathcal{I} \otimes L) = \mathcal{E}nd(\mathcal{I}) \otimes L = f_*\mathcal{O}_{X'} \otimes L = f_*(\mathcal{O}_{X'} \otimes f^*L) = f_*f^*L.$$

Then if $\mathcal{I} \cong \mathcal{I} \otimes L$, such an isomorphism would give a nonzero section of f^*L. On the other hand, f^*L is a degree 0 line bundle so if it has a section it is trivial. $\square$

We will consider the topological Euler characteristic e_{top} of $\overline{\mathrm{Pic}}^d_X$. This is a topological invariant valued in the integers. We will need the following properties of the Euler characteristic.

Fact 17.24.
1. *If $Z \subset X$ is a closed subvariety and $X \setminus Z = U$ the open complement, then $e_{top}(X) = e_{top}(U) + e_{top}(Z)$.*
2. *If $f : X \to Y$ smooth and proper morphism then $e_{top}(X) = e_{top}(Y)e_{top}(F)$ where F is any fiber of f.*[47] *More generally, suppose f is a proper fibration, the same is true.*
3. *$e_{top}(point) = 1$ and $e_{top}(S^1) = 0$. In particular $e_{top}(torus) = 0$.*

When X is smooth, then the Jacobian is a $g(X)$ dimensional abelian variety. In particular, by the third fact, $e_{top}(\mathrm{Jac}_X) = 1$ if $g = 0$ since Jac_X is a point, and $e_{top}(\mathrm{Jac}_X) = 0$ for $g > 0$ since it is topologically a torus. That is, one can distinguish smooth rational curves by $e_{top}(\mathrm{Jac}_X)$. The following proposition generalizes this.

Proposition 17.25. *Suppose the normalization X^ν of X has genus $g(X^\nu) \geq 1$. Then $e_{top}(\overline{\mathrm{Jac}}_X) = 0$.*

Proof. Consider the exact sequence of group schemes

$$0 \to F \to \mathrm{Jac}_X \to \mathrm{Jac}_{X^\nu} \to 0.$$

We saw above that F is an extension of multiplicative and additive groups $\mathbb{G}_m$ and $\mathbb{G}_a$. In particular, F is divisible as an abstract abelian groups and so this sequence splits as a sequence of abelian groups[48]. Since $g(X^\nu) \geq 1$, then Jac_{X^ν} is an abelian variety of dimension at least 1. Thus for each n, there exists an element of order n. Using this noncanonical splitting, we can lift this to an element of order in Jac_X, that is, a line bundle L on X with $L^{\otimes n} \cong \mathcal{O}_X$. Then the pullback of L to X^ν is nontrivial by construction since it pulls back to an element of order n. In particular, for any partial normalization $f : X' \to X$, f^*L is nontrivial. Thus by the previous lemma, for any rank 1 torsion free sheaf $\mathcal{I}$ on X, $\mathcal{I} \otimes L \ncong \mathcal{I}$. That is, the action of L has no fixed points on $\overline{\mathrm{Jac}}_X$. In fact tensoring by L induces a free action of $\mathbb{Z}/n\mathbb{Z}$ on $\overline{\mathrm{Jac}}_X$. Therefore $e_{top}(\overline{\mathrm{Jac}}_X)$ is divisible by n, but n was arbitrary so $e_{top}(\overline{\mathrm{Jac}}_X) = 0$. $\square$

Now let $f : \mathcal{X} \to S$ be some flat and propre family of integral Gorenstein curves over $\mathbb{C}$ and suppose $S_{rat} := \{s \in S \mid \mathcal{X}_s \text{ is rational}\}$[49] is a finite set. Then we have the relative compactified Jacobian

$$\overline{\mathrm{Jac}}_{\mathcal{X}/S} \to S$$

[47] Note that the fibers of a smooth and proper morphism have diffeomorphic underlying complex manifolds.
[48] not necessarily as group schemes
[49] Recall a curve is rational of the genus of X^ν is 0, that is, $X^\nu \cong \mathbb{P}^1$.

which is proper over S. Now for any proper map $Y \to S$ of complex varieties, there is a locally closed decomposition $S = \sqcup S_\alpha$ such that $Y_\alpha \to S_\alpha$ is a proper fibration with fiber F_α. Using additivity and multiplicativity propertes of e_{top} above, we see that

$$e_{top}(Y) = \sum_\alpha e_{top}(S_\alpha) e_{top}(F_\alpha).$$

In our case at hand where $Y = \overline{\mathrm{Jac}}_{\mathcal{X}/S} \to S$, by the proposition, we see that $e_{top}(F_\alpha) = 0$ over any stratum where where the curve $\mathcal{X}_s$ has geometric genus ≥ 1. Therefore the whole sum collapses to the points S_{rat} which are assumed to be finite. Therefore we get the following computation.

Proposition 17.26.

$$e_{top}(\overline{\mathrm{Jac}}_{\mathcal{X}/S}) = \sum_{s \in S_{rat}} e_{top}(\overline{\mathrm{Jac}}_{\mathcal{X}_s}).$$

In particular, $e_{top}(\overline{\mathrm{Jac}}_{\mathcal{X}/S})$ counts the number of rational curves in the fibers $f : \mathcal{X} \to S$, weighted with multiplicity given by the topological Euler characteristic of their compactified Jacobian. Beauville used this to give a proof of a remarkable formula of Yau-Zaslow counting the number of rational curves on a K3 surface which we will now sketch.

17.5 The Yau-Zaslow formula

We continue working over $\mathbb{C}$. Recall that a *K3 surface* is a smooth projective surface X with trivial canonical sheaf

$$\omega_X := \Lambda^2 \Omega_X \cong \mathcal{O}_X$$

and $H^1(X, \mathcal{O}_X) = 0$. A *polarized* K3 surface is a pair (X, H) where X is a K3 surface and H is an ample line bundle. The degree of (X, H) is $d = c_1(H)^2$.

Consider the linear series $|H| = \mathbb{P}(H^0(X, H))$. It is a g-dimensional space where $d = 2g - 2$ and the curves in $|H|$ have arithmetic genus g by the adjunction formula. Inside $X \times |H|$ we have a universal family of curves $\mathcal{C} \to |H|$ with the fiber over a point being the curve in the linear series parametrized by that point. Indeed one can identify $|H|$ with a component of the Hilbert scheme corresponding to effective Cartier divisors D with $\mathcal{O}_X(D) \cong H$. In general, this is an Abel-Jacobi fiber but in this case we see that $H^1(X, \mathcal{O}_X) = 0$ by definition of a K3 so Pic_X is zero dimensional and so the fibers are the components. Then $\mathcal{C} \to |H|$ is simply the universal family of the Hilbert scheme over this component.

Lemma 17.27. *There are finitely many rational curves parametrized by $|H|$.*

Proof. Suppose that the locus in $|H| \cong \mathbb{P}^g$ parametrizing rational curves is higher dimensional. Then there exists an irreducible curve $B_0 \subset |H|$ contained in the rational locus and over B_0 there is a family of rational curves $R_0 \to B_0$. Taking the normalization of both sides, we obtain a family $R \to B$ where B is integral and the generic fibers are smooth rational curves. Thus $R \to B$ contains a generically ruled surface $R' \to B$ as a component. On the other hand, we have a map $R' \to X$ which is dominant. This is a contradiction as X is a K3. $\square$

Let $n(g)$ denote the number of rational curves in $|H|$ for a generic polarized complex K3 surface (X, H) Then we have the following formula of Yau-Zaslow. We will sketch the proof of Beauville which is based on the topology of compactified Jacobians.

Theorem 17.28 (Yau-Zaslow)**.**

$$1 + \sum_{g \geq 1} n(g) q^g = \prod_{n \geq 1} \frac{1}{(1 - q^n)^{24}}$$

In particular, the numbers $n(g)$ are constant for general (X, H).

Proof. (Sketch) Let (X, H) be a generic genus g K3 surface. It is hard theorem which we won't cover here that in this case, every curve in $|H|$ is integral and in fact has at worst nodal singularities.[50] Let $\mathcal{C} \to |H|$ be the universal family of curves in this linear series. By the integrality we have a relative compactified Picard. consider the degree g piece

$$\overline{\mathrm{Pic}}^g_{\mathcal{C}/|H|} \to |H|.$$

This family is fiberwise isomorphic to $\overline{\mathrm{Jac}}_{\mathcal{C}/|H|} \to |H|$ (though it could be globally different if there is no global section of $\mathcal{C} \to |H|$) so the argument in the previous section tells us that

$$e_{top}(\overline{\mathrm{Pic}}^g_{\mathcal{C}/|H|}) = \sum_{\mathcal{C}_s \text{ rational curves in } |H|} e_{top}(\overline{\mathrm{Jac}}_{\mathcal{C}_s}).$$

We saw in the above examples that C is a nodal cubic, $\overline{\mathrm{Jac}}_C \cong C$ and in particular has topological euler characteristic 1. This generalizes as follows.

Lemma 17.29. *If C is an integral rational curve with at worst nodal singularities, then $e_{top}(\overline{\mathrm{Jac}}_C) = 1$.*

We won't give the details of the proof but the idea is that topologically, $\overline{\mathrm{Jac}}_C$ is a product over local contributions that are each homeomorphic to the above example and so the euler characteristic is still 1. Thus we get that

$$n_g = e_{top}(\overline{\mathrm{Pic}}^g_{\mathcal{C}/|H|}).$$

Now given a point of $\overline{\mathrm{Pic}}^g_{\mathcal{C}/|H|}$ corresponding to a pair (C, L) where C is smooth and L is a line bundle of degree g, then

$$\chi(C, L) = 1$$

by Riemann-Roch. On the other hand, by semi-continuity of coherent cohomology, there exists an open subset $U \subset \overline{\mathrm{Pic}}^g_{\mathcal{C}/|H|}$ parametrizing such pairs where $H^1(C, L) = 0$. On this open subset, we in fact that that $H^0(C, L) = 1$ and so L has a unique section. The zero locus of this section is a zero dimensional degree g subscheme of X which gives a point of Hilb^g_X, the Hilbert scheme of g points. This gives a rational and generically injective map

$$\overline{\mathrm{Pic}}^g_{\mathcal{C}/|H|} \dashrightarrow \mathrm{Hilb}^g_X.$$

[50] Of course this isn't true for any (X, H) and here is where the genericity assumption comes in. In fact this statement was only conjectured at the time of the proof of the Yau-Zaslow formula and it was only proved a few years later.

The source and the target of this map are in fact smooth holomorphic symplectic varieties[51] of the same dimension. In particular, this map is birational and then it follows from a result of Batyrev and Kontsevich or a result of Huybrechts that the source and the target then have the same euler characteristics.[52]

Thus we have that

$$n_g = e_{top}(\mathrm{Hilb}_X^g).$$

Finally, in the next few classes we will study the geometry of the Hilbert scheme of points on surfaces and prove both the above smoothness and irreducibility claim, as well as the formula that in this particular case of X being a K3,

$$\sum_{g \geq 0} e_{top}(\mathrm{Hilb}_X^g) q^g = \prod_{n \geq 1} \frac{1}{(1-q^n)^{24}},$$

completing the proof. $\square$

18 The Hilbert scheme of points on surfaces

18.1 The topology of Hilbert schemes of points on surfaces

Let $S = \mathbb{C}$ and X/S a smooth quasi-projective surface. Our goal now is to study the topology of the Hilbert schemes of points Hilb_X^n parametrizing subschemes $Z \subset X$ with Hilbert polynomial constant n. That is, Z is a zero dimensional subscheme with

$$\dim_{\mathbb{C}} \mathcal{O}_Z = n.$$

Our goal is to sketch the proof of the following theorem, which is a combination of results due Fogarty, Briançon, and Göttsche.

Theorem 18.1. *Let $X/\mathbb{C}$ be a smooth quasi-projective surface. Then Hilb_X^n is a smooth and irreducible quasi-projective $2n$-fold. Moreover, the topological Euler characteristic of the Hilbert schemes of points on X are given by the following formula.*

$$\sum_{n \geq 0} e_{top}(\mathrm{Hilb}_X^n) q^n = \prod_{m \geq 0} \frac{1}{(1-q^n)^{e_{top}(X)}}$$

This completes the sketch of the proof of the Yau-Zaslow formula from last class, and in fact also implies the following about compactified Jacobians.

Theorem 18.2. *Let C be an integral locally planar[53] curve over $\mathbb{C}$. Then $\overline{\mathrm{Jac}}_C$ is an irreducible variety of dimension $g = g(C)$. In particular, $\mathrm{Jac}_C \subset \overline{\mathrm{Jac}}_C$ is dense.*

[51] A holomorphic symplectic variety V is one with a holomorphic 2-form $\omega H^0(V, \Omega_V^2)$ which is antisymmetric, closed, and nondegenerate. In this case the existence of such a form on these two moduli spaces follows from more general work of Mukai on moduli of sheaves on a K3 surface, which both of these spaces are examples of.

[52] In fact they are even diffeomorphic.

[53] That is, the tangent space dimension is at most 2 at each $p \in C$.

18.2 The case of $X = \mathbb{A}^2$

The Hilbert scheme $\mathrm{Hilb}^n_{\mathbb{A}^2}$ admits a particularly concrete combinatorial description due to Haiman. Let us denote $\mathrm{Hilb}^n_{\mathbb{A}^2}$ by $\mathbf{H}^n$. Since $\mathbb{A}^2$ is affine, we can identify $\mathbf{H}^n$ with the set

$$\{I \subset \mathbb{C}[x,y] \mid \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n\}.$$

18.2.1 The torus action

There is an action of the algebraic torus $T = \mathbb{G}^2_{m,t_1,t_2}$ on $\mathbb{A}^2_{x,y}$ which on polynomial functions is given

$$f(x,y) \mapsto f(t_1x, t_2y).$$

This extends to an action on $\mathbf{H}^n$ by

$$t \cdot I = \{f(t_1x, t_2y) \mid f \in I\}.$$

The torus fixed points, denoted by $(\mathbf{H}^n)^T$, correspond to those ideals generated by f such that $f(t_1x, t_2y) = t_1^a t_2^b f(x,y)$, that is, by monomials $x^a y^b$.

Recall that a *partition of* n, $\lambda \vdash n$, is a decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ such that

$$\sum \lambda_i = n.$$

The number $k = l(\lambda)$ is the *length* of λ, the size n is denoted by $|\lambda|$, and the λ_i are the *parts* of λ. We can represent a partition by its *Young diagram*, a left aligned arrangement of boxes with λ_j boxes in the j^{th} row. We identify λ with its Young diagram and label λ as a subset of $\mathbb{N}^2$ with the box in the i^{th} column and j^{th} row labeled by (i,j).

Lemma 18.3. *There is a bijection between monomial ideals in $\boldsymbol{H}^n$ and partitions of n given by*

$$I \mapsto \lambda(I) = \{(i,j) \mid x^i y^j \notin I\}$$

and

$$\lambda \mapsto I_\lambda = (\{x^r y^s \mid (r,s) \notin \lambda\}.$$

This is enough to compute the Euler characteristic of $\mathbf{H}^n$ by the following fact. Let Y be a finite type $\mathbb{C}$-scheme with an action of an algebraic torus $T = \mathbb{G}^r_m$ and let Y^T be the T-fixed locus. Then

$$e_{top}(Y) = e_{top}(Y^T).$$

Indeed $e_{top}(T) = 0$ for any $r > 0$ and non-zero dimensional orbit of T is homeomorphic to an algebraic torus and so only the set of zero dimensional orbits, i.e., Y^T, contributes to the Euler characteristic. As a corollary, we obtain

Corollary 18.4. *The Euler characteristic $e_{top}(\boldsymbol{H}^n) = p(n)$ the number of partitions of n. Moreover,*

$$\sum_{n \geq 0} e_{top}(\boldsymbol{H}^n) q^n = \prod_{m \geq 1} \frac{1}{1 - q^m}.$$

Proof. By the previous fact,
$$e_{top}(\mathbf{H}^n) = e_{top}((\mathbf{H}^n)^T)$$
by $(\mathbf{H}^n)^T$ is the finite set of monomial ideals so its Euler characteristic is just the cardinality. Thus,
$$e_{top}(\mathbf{H}^n) = \#\{\text{monomial partitions}\} = \#\{\lambda \vdash n\}.$$
Thus, it suffices to compute the generating series
$$\sum_{n\geq 0} p(n)q^n.$$
Consider the infinite product
$$\prod_{m\geq 1} \frac{1}{1-q^m}$$
and expand using $1/(1-x) = \sum x^i$. Given a partition λ, we can write it as
$$\sum mk_m = n$$
where there are are k_m parts of size m. Then the infinite product exactly counts expressions of this form. □

18.2.2 Local structure

Let $B_\lambda = \{x^i y^j \mid (i,j) \in \lambda\}$. Note that B_λ forms a basis for $\mathbb{C}[x,y]/I_\lambda$. We will define open subfunctors $U_\lambda \subset \mathbf{H}^n$ as follows. Given any test scheme S and a map $S \to \mathbf{H}^n$ corresponding to a closed subscheme $Z \subset S \times \mathbb{A}^2$ flat over S, the pushforward $\pi_* \mathcal{O}_Z$ along $\pi : Z \to S$ carries a canonical section $s_{ij} : \mathcal{O}_S \to \pi_*\mathcal{O}_Z$ for each monomial $x^i y^j$. We can take the direct sum
$$s_\lambda = \sum_{(i,j)\in\lambda} s_{ij} : \mathcal{O}_S^{\oplus n} \to \pi_*\mathcal{O}_Z.$$
Then U_λ is the subfunctor representing those S-points such that s_λ is an isomorphism. The points of U_λ are exactly those ideals I such that B_λ is a basis for $\mathbb{C}[x,y]/I$.

Proposition 18.5. *U_λ is a T-invariant open affine neighborhood of I_λ.*

Proof. It is clear that $[I_\lambda] \in U_\lambda$ and that U_λ is T-invariant. We will write down explicit coordinates for U_λ. Given a monomial $x^r y^s$ we have a unique expansion
$$x^r y^s = \sum_{(i,j)\in\lambda} c_{ij}^{rs}(I) x^i y^j \mod I$$
for any $I \in U_\lambda$ where $c_{ij}^{rs}(I)$ are coefficients depending on I. The c_{ij}^{rs} are in fact global sections of $\mathcal{O}_{U_\lambda}$. To see this, using the notation above, note that for any S-point $(Z \subset S \times \mathbb{A}^2)$ of U_λ and any monomial $x^r y^s$, we have a section $s_{rs} : \mathcal{O}_S \to \pi_*\mathcal{O}_Z$. Pulling back by the isomorphism s_λ, we obtain a section $s_\lambda^* s_{rs} : \mathcal{O}_S \to \mathcal{O}_S^{\oplus n}$ where the components of the target are indexed by $(i,j) \in \lambda$. Then the functions c_{ij}^{rs} on S are exactly the components of $s_\lambda^* s_{rs}$.

Since I is an ideal, it is closed under multiplication by x and y. Multiplying the above equation x and y respectively, re-expanding both sides in the basis B_λ, and equating coefficients gives us the following.

$$c_{ij}^{r+1,s} = \sum_{(h,k)\in\lambda} c_{hk}^{rs} c_{i,j}^{h+1,k} \tag{17}$$

$$c_{ij}^{r,s+1} = \sum_{(h,k)\in\lambda} c_{hk}^{rs} c_{ij}^{h,k+1} \tag{18}$$

Now we leave it to the reader to check that U_λ is represented by Spec of the ring

$$\mathcal{O}_\lambda := \mathbb{C}[c_{ij}^{rs} \mid (i,j) \in \lambda]/(\text{relations (1) \& (2)}).$$

□

Remark 18.6. *In fact the U_λ are the pullbacks of the natural open affine subfunctors that cover a Grassmannian under the embedding of the Hilbert scheme into a Grassmannian used to construct $\boldsymbol{H}^n$. In particular, U_λ over all λ cover $\boldsymbol{H}^n$.*

Now we will compute the cotangent space to $\mathbf{H}^n$ at a monomial ideal I_λ. For this ideal, we have

$$c_{ij}^{rs}(I_\lambda) = \begin{cases} 1 & (i,j) = (r,s) \in \lambda \\ 0 & \text{else} \end{cases}$$

Thus the maximal ideal $\mathfrak{m}_\lambda \subset \mathcal{O}_\lambda$ corresponding to the point $[I_\lambda] \in U_\lambda \subset \mathbf{H}^n$ is generated by c_{ij}^{rs} for $(r,s) \notin \lambda$. The cotangent space to an affine scheme is given by $\mathfrak{m}_\lambda/\mathfrak{m}_\lambda^2$. Examining the relations above, we see that all the terms on the right are in $\mathfrak{m}_\lambda^2$ except for the term

$$c_{i-1,j}^{rs} c_{ij}^{ij} = c_{i-1,j}^{rs}.$$

Here we are using that $c_{ij}^{ij} = 1$. Thus we have that

$$c_{ij}^{r+1,s} = c_{i-1,j}^{rs} \mod \mathfrak{m}_\lambda^2.$$

Similarly, $c_{ij}^{r,s+1} = c_{i,j-1}^{rs} \mod \mathfrak{m}_\lambda^2$. For each box $(i,j) \in \lambda$ we define two special functions u_{ij} and d_{ij} as in the following diagram.

put in picture and discussion of arrows

Now a simple combinatorial argument shows that each function is either zero or equivalent to one of the d_{ij} or $u_{i,j}$ in $\mathfrak{m}_\lambda^2$. Since there are $2n$ such functions, we conclude the following.

Proposition 18.7. *The cotangent space to $\boldsymbol{H}^n$ at I_λ has dimension at most $2n$.*

18.2.3 Initial degenerations

Let $\rho : \mathbb{G}_m \to T$ be a character of the torus so that $\rho(t) = (t^a, t^b)$. Then we define the initial ideal, if it exists, to be the flat limit

$$in_\rho I := \lim_{t\to 0} \rho(t) \cdot I.$$

More precisely, the action of T on $\mathbf{H}^n$ composed with ρ induces an action of $\mathbb{G}_m$. Then the orbit of I can be viewed as a morphism

$$\varphi_I : \mathbb{G}_m \to \mathbf{H}^n$$

corresponding to the family of ideals $I_\rho = (\{f(t^a x, t^b y) \mid f \in I\}) \subset \mathcal{O}_{\mathbb{G}_m}[x,y]$. If this morphism extends to an equivariant morphism

$$\bar{\varphi}_I : \mathbb{A}^1 \to \mathbf{H}^n,$$

the initial ideal is exactly the ideal corresponding to the point $\bar{\varphi}_I(0)$.

Fact 18.8. *There exists a generic enough ρ such that the fixed points of the $\mathbb{G}_m$ action under ρ are the same as those for T, $(\mathbf{H}^n)^\rho = (\mathbf{H}^n)^T$, and such that $\bar{\varphi}_I$ exists for each I.*

Exercise 18.1. *Check that the co-character $\rho(t) = (t^{-p}, t^{-q})$ for $p \gg q > 0$ works.*

Since $\bar{\varphi}_I$ has $\mathbb{G}_m$-equivariant it sends fixed points to fixed points so $\bar{\varphi}_I(0) \in (\mathbf{H}^n)^T$ is a monomial ideal.

Proposition 18.9. *$\mathbf{H}^n$ is connected.*

Proof. By the above fact, every point of $\mathbf{H}^n$ is connected to a monomial ideal by an initial degeneration over $\mathbb{A}^1$. Thus, it suffices to show that the monomial ideals lie in the same connected component. Let λ_i be partitions corresponding to ideals I_i for $i = 1,2$. Suppose the partitions differ in exactly one box.

$$\lambda_2 = (\lambda_1 \setminus (i,j)) \cup (r,s)$$

Let $J = I_1 \cap I_2$ corresponding to the partition $\mu = \lambda_1 \cup \lambda_2 \subset \mathbb{N}^2$. Then the family of ideals

$$I_{\alpha,\beta} = J + (\alpha x^i y^j - \beta x^r y^s)$$

gives map $\varphi : \mathbb{P}^1 \to \mathbf{H}^n$ with $\varphi(0,1) = I_1$ and $\varphi(1,0) = I_2$. Now any partition can be obtained from the row partition (n) by moving one box at a time. This shows each monomial ideal is in the same connected component as the row so $\mathbf{H}^n$ is connected.

□

Example 18.10. *Let $\lambda_1 = (1,1,1,1)$ and $\lambda_2 = (2,1,1)$ so that the ideals λ_i differ by one box. These correspond to the ideals $I_1 = (x, y^4)$ and $I_2 = (x^2, xy, y^3)$ respectively. Consider the ideal $J = I_1 \cap I_2 = (x^2, xy, y^4)$ corresponding to the partition $\mu = \lambda_1 \cup \lambda_2 \subset \mathbb{N}^2$. Then the one parameter family $(x^2, xy, y^4, \alpha y^3 - \alpha x)$ of ideals connects these two points of $\mathbf{H}^4$.*

Remark 18.11. *Note that the curves in $\mathbf{H}^n$ constructed above to connect two monomial ideals parametrize subschemes Z supported on the origin. That is, such that $Z_{red} = \{(0,0)\}$.*

18.2.4 The Hilbert-Chow morphism for $\mathbb{A}^2$

19 The moduli of curves

19.1 The functor of genus g curves

First try: ($g \geq 2$ morphisms are automatically projective)

Definition 19.1. *A smooth curve over S is a flat and proper morphism $f : X \to S$ with smooth geometrically connected 1-dimensional fibers. The genus of $X \to S$ is the genus of a geometric fiber.*[54]

$$\begin{aligned} \pi_0\mathcal{M}_g : Sch_{\mathbb{Z}} &\to Set \\ S &\to \{f : X \to S \text{ a smooth curve of genus } g\}/\sim \end{aligned}$$

$\pi_0\mathcal{M}_g$ is *not representable.*

Example 19.2. $C \times \mathbb{P}^1 \to$ *node.*

Fix, upgrade the functor to a *pseudofunctor*

$$\mathcal{M}_g : Sch_S \to Gpd$$

Define groupoids + Stacks = pseudofunctors to groupoid + sheaf.
Diagram relating all notions. Explain notation $\pi_0\mathcal{M}_g$.

19.2 Stacks

Definition 19.3. *Category fibered in groupoids (CFG) $p : \mathscr{X} \to \mathcal{C}$ such that blah. If $f : T' \to T$ in $\mathscr{C}$, and E an object over T, then there exists a E' unique up to unique isom and a map $E' \to E$ lying over f.*

Denote $E' = f^*E$. $p^{-1}(T) :=$ objects over T + morphisms over id_T. Makes precise the idea of a "pseudofunctor" to groupoids. $T \mapsto p^{-1}(T)$ which is a groupoid. We will denote $p^{-1}(T)$ by $\mathscr{X}_T$. Presheaves are CFG by viewing a set as a category with only identities. Objects S may be indentified with the category $\mathcal{C}/S$ (equivalent to the data of the functor of points of S) and maps $S \to \mathscr{X}$ identified with objects of $\mathscr{X}_S$ by where $id : S \to S$ maps. There is a 2-categorical Yoneda lemma.

Example 19.4. *$B\mathbb{G}_m$, BGL_n, quotient stack, Picard stack, $\mathcal{M}_g$ as a CFG.*

Fact 19.5. *Fiber products of CFGs exist. I'll let you work out the details of the definition.*

Consider Sch_S with a Grothendieck topology $\mathcal{T} =$ (Zariski, étale, fppf, fpqc, etc).

Definition 19.6. *A $\mathcal{T}$-stack is a category $p : \mathscr{X} \to Sch_S$ over Sch_S such that*

(1) p is a CFG,

(2) for each scheme $T \to S$ and each pair of objects $\xi, \psi \in \mathscr{X}_T$, the functor $Sch_T \to Set$ given by $f : V \to T$ maps to
$$\mathrm{Hom}_{\mathscr{X}_V}(f^*\xi, f^*\psi)$$
is a $\mathcal{T}$-sheaf, and

(3) objects of $\mathcal{X}$ satisfy effective $\mathcal{T}$-descent.

Example 19.7. *All examples above are fppf stacks (and thus also Zariski and étale) (need $g \neq 1$).*

A morphism of stacks is representable by schemes if the usual thing. Can define all properties $\mathcal{P}$ for representable morphisms.[55]

[54]Note this is constant over connected components of S by flatness.
[55]something about representable = representable by spaces

19.3 Algebraic stacks

From now on work with étale or fppf topology, won't make a difference which.

Lemma 19.8. *Let $\mathscr{X}$ be a stack over Sch_S. Then the diagonal map*

$$\Delta_{\mathscr{X}} : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$$

is representable by schemes if and only if for all schemes $T_1, T_2 \to \mathscr{X}$, the fiber product $T_1 \times_{\mathscr{X}} T_2$ is a scheme.

That is, the diagonal is representable by schemes if and only if for any morphism $T \to \mathscr{X}$ from a scheme is representable. For a stack $\mathscr{X}$ with representable diagonal, we can define all the usual separation axioms.

Remark 19.9. *How do we check if Δ is representable? We need to show that for any $T \to \mathscr{X} \times_S \mathscr{X}$, the pullback $T \times_{\mathscr{X} \times_S \mathscr{X}, \Delta} \mathscr{X}$ is a scheme. $T \to \mathscr{X} \times_S \mathscr{X}$ corresponds to a pair of objects $\xi, \psi \in \mathscr{X}(T)$ over T as well as an isomorphism $\xi \to \psi$, that is, an element of $\mathcal{H}om_{\mathscr{X}(T)}(\xi, \psi) = Isom_T(\xi, \psi)$. By definition of a stack, the functor sending a $T' \to T$ to $\mathrm{Isom}_{T'}(\xi_{T'}, \psi_{T'})$ is a sheaf which is isomorphic to the pullback*

$$T \times_{\mathscr{X} \times_S \mathscr{X}} \mathscr{X}.$$

Thus the condition that the diagonal is representable is the condition that for any T and any objects ξ, ψ over T, the isom sheaf is representable by a scheme.

Definition 19.10. *A stack $\mathscr{X}$ is an algebraic stack (resp. Deligne-Mumford stack) if*

*(1) the diagonal $\Delta_{\mathscr{X}}$ is representable**, and*

(2) there exists a scheme U and a smooth (resp. étale) surjection $U \to \mathscr{X}$.

Remark 19.11. *(A remark on algebraic spaces) If $\mathscr{X}$ is a stack where the groupoids are sets, that is, $\mathscr{X}$ is simply a sheaf, that satisfies the conditions of the above theorem, then we say that the sheaf $\mathscr{X}$ is an algebraic space. In this case, $\mathscr{X}$ is the quotient in the category of sheaves of the equivalence relation*

$$U \times_{\mathscr{X}} U \rightrightarrows U$$

of schemes. Once one develops the theory of algebraic spaces, then the right notion of a representable map of stacks is one that is representable by algebraic spaces, rather than representable by schemes.

Remark 19.12. *Let us unravel the definition, we need $\mathscr{X}$ to be a sheaf so that we can do geometry locally, we need representability of the diagonal to make sense of the having a smooth or étale cover by a scheme, and then we can use descent by this cover to "do geometry" on $\mathscr{X}$.*

Algebraic stacks have a Zariski topology generated by morphisms that are representable by open immersions, and an underlying topological space $|\mathscr{X}|$ given by equivalence classes of K-points for fields K and the Zariski topology. Universally closed makes sense with the usual definition that for any $\mathscr{Z}$, the map $|\mathscr{X} \times_S \mathscr{Z}| \to |\mathscr{Y} \times_S \mathscr{Z}|$ is closed.

Theorem 19.13. *(Valuative criterion for properness) Proper = universally closed + separated by defition. Diagram... Existence + uniqueness separate.*

Theorem 19.14. *Suppose $\mathscr{X}$ is a quasi-separated algebraic stack such that $\Delta_{\mathscr{X}}$ is unramified. Then $\mathscr{X}$ is a Deligne-Mumford stack.*

Definition of coarse moduli space... unique up to unique iso. Example...

Theorem 19.15. *(Keel-Mori) Suppose $\mathscr{X}$ is a proper Deligne-Mumford stack over S. Then there exists a proper coarse moduli space $X \to S$.*

The CFG of stable curves $\overline{\mathcal{M}}_g$

Theorem 19.16. *(Deligne-Mumford) $\overline{\mathcal{M}}_g$ for $g \geq 2$ is a smooth and proper Deligne-Mumford stack of dimension $3g-3$ with projective coarse moduli space $\overline{M}_g$.*

20 Homework

20.1 Problem Set 1

1. If you haven't done so before, prove the Yoneda lemma. That is, if $\mathcal{C}$ is a category, X is an object of $\mathcal{C}$ and $F : \mathcal{C}^{op} \to Set$ is a presheaf, show that there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(h_X, F) \cong F(X).$$

   Conclude that the functor $\mathcal{C} \to Fun(\mathcal{C}^{op}, Set)$ given by $X \mapsto h_X$ is fully faithful.

2. Are the following functors representable? If so find the representing object and universal family and if not show why not.
   - The functor $GL_n : Sch_{\mathbb{Z}}^{op} \to Set$ taking a scheme T to $GL_n(\mathcal{O}_T(T))$;
   - The functor $Nil_n : Sch_{\mathbb{Z}}^{op} \to Set$ taking a scheme T to the set of elements $f \in \mathcal{O}_T(T)$ with $f^n = 0$.
   - The functor $Nil : Sch_{\mathbb{Z}}^{op} \to Set$ taking a scheme T to the set of all nilpotent elements $f \in \mathcal{O}_T(T)$.
   - The functor $F_n : Sch_{\mathbb{Z}}^{op} \to Set$ taking T to the set $(f_1, \ldots, f_n) \in \mathcal{O}_T(T)^{\oplus n}$ such that for each $t \in T$, there exists an i with $f_i(t) \neq 0$.
   - The functor $F_n/\mathbb{G}_m : Sch_{\mathbb{Z}}^{op} \to Set$ taking T to the set $F_n(T)/\sim$ where

$$(f_1, \ldots, f_n) \sim (f_1', \ldots, f_n')$$

     if and only if there exists a unit $u \in \mathcal{O}_T(T)^\times$ with $f_i' = uf_i$ for all i.

3. In the last example above, what happens if we sheafify the functor $F_n/\mathbb{G}_m$?

4. Prove the following statement we used in class. Let $f : X \to Y$ be a morphism locally of finite type between locally Noetherian schemes. Show that f is a closed embedding if and only if it is a proper monomorphism. Recall that a morphism is a monomomorphism if for all T, the induced map $\mathrm{Hom}_{\mathcal{C}}(T, X) \to \mathrm{Hom}_{\mathcal{C}}(T, Y)$ is injective. (Hint: you might need to use the famous result that a morphism is finite if and only if it is proper and has finite fibers.)

5. In this exercise, we will prove the generic freeness theorem.

Theorem 20.1. *Let A be a Noetherian integral domain and B a finitely generated A-algebra. For any finite B-module M, there exists an $f \in A$ such that M_f is a free A_f-module.*

To make the argument easier, let us say that an A-algebra B satisfies generic freeness if for any finite B-module M, there exists $f \in A$ such that M_f is a free A_f module. Then we want to show that any finitely generated A-algebra satisfies generic freeness.

(a) Show that it suffices to prove that the polynomial algebra $A[t_1, \ldots, t_n]$ satisfies generic freeness.

(b) Show that A itself satisfies generic freeness.

(c) Let B be an A-algebra and let M be a finitely generated $B[t]$-module. Let $m_1, \ldots, m_k$ be a finite set of $B[t]$-module generators. We will define a filtration of M by finite B-modules as follows. Let M_0 be the B-module generated by $m_1, \ldots, m_k$ and let $M_{n+1} = M_n + tM_n$. Note that each M_n is a finite B-module. Show that for each $n \gg 0$, multiplication by t is an isomorphism $M_n/M_{n-1} \to M_{n+1}/M_n$. Hint: consider the graded $B[t]$ module

$$\bigoplus_{n \geq 0} M_n/M_{n-1}.$$

(d) Let M be an A-module with a filtration $M_0 \subset M_1 \subset \ldots M_n \subset \ldots M$. Suppose that M_{n+1}/M_n is a free A-module for each n. Show that M is a free A-module.

(e) Suppose that B is an A-algebra which satisfies generic freeness. Use parts (c) and (d) above to conclude that $B[t]$ satisfies generic freeness.

(f) Conclude that any finitely generated A-algebra B satisfies generic freeness.

20.2 Problem Set 2

1. Let R be a Noetherian ring, $I \subset R$ a proper ideal, M and R-module such that M/IM is flat over R/I. Suppose $\mathrm{Tor}_1^R(R/I, M) = 0$. Show the following holds for any $n \geq 1$:

(a) $M/I^n M$ is a flat R/I^n module, and

(b) for any R-module N that is annihilated by I^n, we have

$$\mathrm{Tor}_1^R(N, M) = 0.$$

Hint: for (a) use the fact that an R-module M is flat if and only for every ideal $J \subset R$,

$$J \otimes_R M \to M$$

is injective.

2. Let S be a Noetherian scheme, $U \subset S$ an open subscheme, and $\mathcal{E}$ a coherent sheaf on U. Show that there exists a coherent sheaf $\mathcal{E}'$ on S such that $\mathcal{E}'|_U = \mathcal{E}$.

3. Let $p : Y \to S$ be a proper morphis, $Z \subset Y$ a closed subscheme, and $\mathcal{F}$ a coherent sheaf on Y. Show that there exists an open subscheme $U \subset S$ such that a morphism $\varphi : T \to S$ factors through U if and only if the support of $\mathcal{F}_T$ on Y_T is disjoint from Z_T.

4. Recall the existence theorem on the Grothendieck complex.

   Theorem 20.2. *Let $f : X \to S$ be a proper morphism over a Noetherian affine scheme $S = \operatorname{Spec} A$ and $\mathcal{F}$ a coherent sheaf on X flat over S. Then there exists a finite complex*

   $$K^\bullet = (0 \to K^0 \to K^1 \ldots \to K^m \to 0)$$

   of finitely generated projective A-modules such that for all A-modules M, there are functorial isomorphism

   $$H^p(X, \mathcal{F} \otimes_A M) \cong H^p(K^\bullet \otimes_A M).$$

   Let $f : X \to S$ be a proper morphism over a Noetherian scheme S and let $\mathcal{E}, \mathcal{F}$ be coherent sheaves on X with $\mathcal{F}$ flat over S. Suppose that $\mathcal{E}$ admits a locally free resolution. The goal of this exercise is to use the Grothendieck complex to show that the functor $Sch_S \to Set$ given by

   $$T \mapsto \operatorname{Hom}_{X_T}(\mathcal{E}_T, \mathcal{F}_T)$$

   is representable by a scheme over S.

   In fact we will show more. Given coherent sheaf $\mathcal{Q}$ on S, we may form the scheme

   $$\mathbb{V}(\mathcal{Q}) := \operatorname{Spec}_S \operatorname{Sym}^* \mathcal{Q}$$

   where $\operatorname{Sym}^* \mathcal{Q}$ is the symmetric algebra of $\mathcal{Q}$. When $\mathcal{Q}$ is locally free, $\mathbb{V}(\mathcal{Q})$ is what one might call the total space or geometric vector bundle associated to $\mathcal{Q}$. By the definition of relative Spec, the universal property of the symmetric algebra, and the push-pull adjunction, $\mathbb{V}(\mathcal{Q})$ is characterized by the universal property that

   $$\operatorname{Hom}_S(T, \mathbb{V}(\mathcal{Q})) = \operatorname{Hom}_{\mathcal{O}_T}(\varphi^* \mathcal{Q}, \mathcal{O}_T) = \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{Q}, \varphi_* \mathcal{O}_T)$$

   for any S-scheme $\varphi : T \to S$.

   (a) Suppose $S = \operatorname{Spec} A$ is affine. Show that for any coherent $\mathcal{F}$ flat over S, there exists an S-module Q as well as a functorial isomorphism

   $$\theta_M : f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* M) \to \mathcal{H}om_A(Q, M)$$

   for any quasi-coherent A-module M.

   (b) Suppose S is an arbitrary Noetherian scheme and $\mathcal{F}$ as above. Show that there exists a coherent sheaf $\mathcal{Q}$ on S and a funtorial isomorphism

   $$\theta_{\mathcal{G}} : f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G}) \to \mathcal{H}om_S(\mathcal{Q}, \mathcal{G})$$

   for all quasi-coherent sheaves $\mathcal{G}$ on S.

(c) Conclude from the previous proposition that for any coherent $\mathcal{F}$ which is flat over S, there exists a coherent $\mathcal{Q}$ on S such that

$$\mathrm{Hom}_S(T, \mathbb{V}(\mathcal{Q})) = H^0(X_T, \mathcal{G}_T)$$

for each S-scheme T. Hint: you might need to use the projection formula for coherent sheaves.

(d) Suppose $\mathcal{F}, \mathcal{E}$ are coherent sheaves on X with $\mathcal{E}$ locally free and $\mathcal{F}$ flat over S. Use part (c) to show that there exists a coherent $\mathcal{Q}$ on S such that

$$\mathrm{Hom}_S(T, \mathbb{V}(\mathcal{Q})) = \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{E}_T, \mathcal{F}_T).$$

(e) Extend (d) to the case where $\mathcal{E}$ has a locally free resolution.