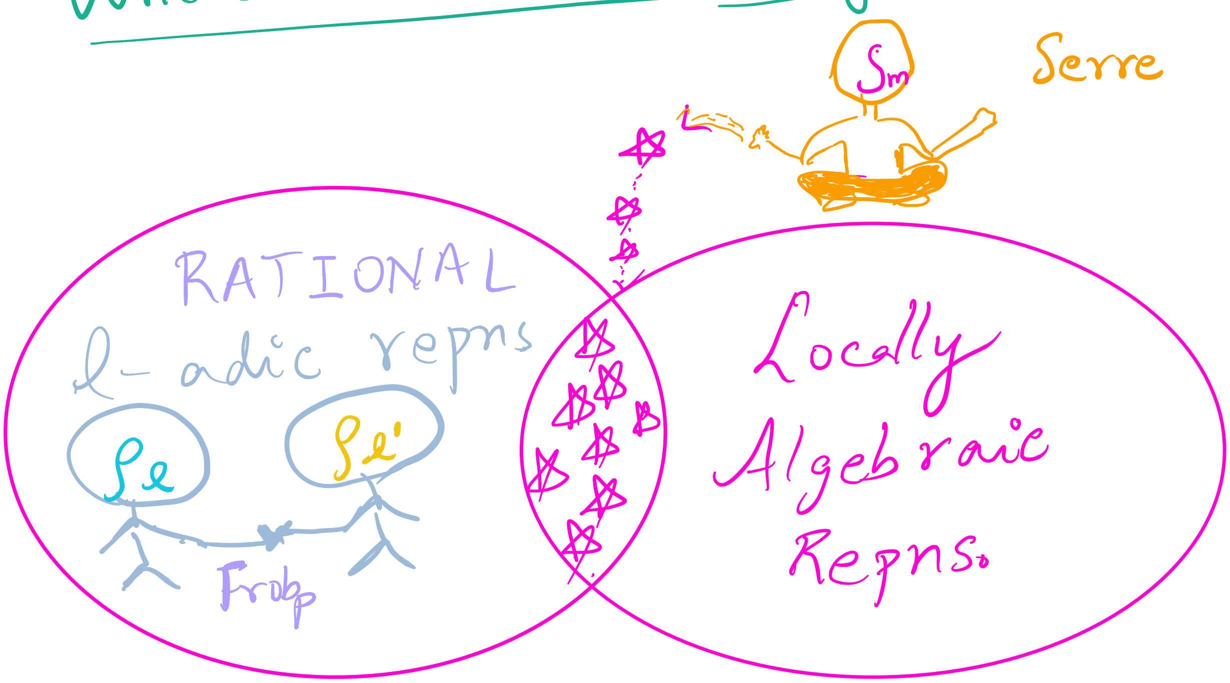
HODGE - TATE THEORY

Alternate Title:

Like linear algebra???
Try semi-linear algebra!
It's linear algebra with a
☆ fun twist! ☆

Where are we? Why Hodge-Tate Thy?



☆: Special repns. coming from S_m.

Last time:

If G_K-repn. is rational + locally algebraic,
it is special.

Future talks.

Shafarevich finiteness +
Properties of special G_K-repns.
$\Longrightarrow$ Open-Image Thm. for non-CM E

TODAY:

a) Tate's criterion for when
abelian G_K-repn. is locally algebraic

b) Verify criterion in our case (Tate) $\leftarrow$ FUTURE?

§ Tate's criterion.

Setup: $K/\mathbb{Q}_p$ local field

$$\rho : \mathrm{Gal}(\bar{K}/K)^{ab} \to GL_{\mathbb{Q}_p}(V)$$

LCFT:

$$\begin{tikzcd}
 & & & GL_{\mathbb{Q}_p}(V) & \\
1 \arrow[r] & T_K = G_K^1 \arrow[r] & \mathrm{Gal}(\bar{K}/K)^{ab} \arrow[r] \arrow[ur, "\rho"] & \hat{\mathbb{Z}} \arrow[r] & 1 \\
1 \arrow[r] & U_K \arrow[r] \arrow[u, "\cong"] & K^* \arrow[r, "v"] \arrow[u, "i"] & \mathbb{Z} \arrow[r] \arrow[u] & 1 \\
 & & T(\mathbb{Q}_p) \arrow[u, equal] \arrow[r, "i"] & T(\mathbb{Q}_p) \arrow[uuu, bend right] &
\end{tikzcd}$$

Defn: ρ is locally algebraic if $\exists$ alg. morphism $r : T \to GL_V$ s.t. $\rho \circ i(x) = r(x^{-1})$ $\forall$ $x \in K^*$ suff.-close to 1.

KEY EXAMPLE 1

$K = \mathbb{Q}_p$, $\dim V = 1$

$$\begin{tikzcd}
T(\mathbb{Q}_p) = \mathbb{Q}_p^* \arrow[r, "x \mapsto x^{\nu}"] \arrow[rr, bend left] \arrow[d, "i"'] & \mathbb{Q}_p^* = GL_1(V) = & \mathbb{G}_m(\mathbb{Q}_p) \\
\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{ab} \arrow[ur, "\rho"'] & &
\end{tikzcd}$$

Serre, III, 2, Example 1

Hint: $\mathrm{Hom}_{\text{cts-grp}}(\mathbb{Z}_p, \mathbb{Z}_p) = \underline{\overset{\substack{\nu \\ \cap}}{\mathbb{Z}_p}}$

Use p-adic logs to linearize!

Now, $T = \underline{\mathbb{G}_m} = GL_1(V)$

ρ locally algebraic $\iff$ ν in $\underset{\substack{\wr \\ \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)}}{\underline{\mathbb{Z}}} \subset \mathbb{Z}_p$

Arvind: Why x^{-1} & not x?

Further specialize:

$$V = V(\mu) = (\varprojlim \mu_{p^\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

$$\text{LCFT} \Rightarrow \rho \circ i(x) = x^{-1} \leftarrow \nu = -1$$

KEY EXAMPLE 2

$$V = \mathcal{F}_{LT}(\text{tors}) \otimes \mathbb{Q}_p$$

Lubin-Tate formal grp.

$$K = \mathbb{Q}_p, \quad \mathcal{F}_{LT} \equiv \widehat{\mathbb{G}}_m$$

$\exists$ abstract ρ s.t. ρ not semi-simple.

Section $\underline{\text{III}}$ → §, End of §1 Appendix

Q: What's special abt. I_K?

Propn: If ρ is locally-algebraic

Then $\rho|_{I_K}$ is semi-simple.

Pf: • $\rho|_{U_K^m}$ comes from an alg. repn. of a torus!

$\Rightarrow$ It is semi-simple.

• U_K^m finite index in I_K.

Averaging trick!

$\hookrightarrow$ lift $\rho|_{U_K^m}$ inv. $\pi: V \to W \supseteq$ I_K-inv. sub.

to a $\rho|_{I_K}$ inv. π.

Structure of $\rho|_{I_K}$:

- Diagonalize! $\rho|_{I_K} = \chi_1 \oplus \cdots \oplus \chi_n$ $/E/K$

- $\chi_i \in X(T_{E/\mathbb{Q}_p})$

$\Longrightarrow$ If $\Gamma_K = \mathrm{Hom}_{\mathbb{Q}_p}(K, E)$,

then $$\chi_i(u) = \prod_{\sigma \in \Gamma_K} \sigma(u)^{-n_\sigma(i)}$$

Tate's criterion: [Main Thm]

ρ is locally algebraic

$\Updownarrow$

$\rho|_{I_K}$ is semi-simple + ρ is Hodge-Tate type

§ Hodge-Tate repns:

$$C_K = \widehat{\overline{K}} \circlearrowleft G_K \text{ cts.}$$

W C_K-vector space

Defn: A semi-linear action of G_K on W is $G_K \times W \to W$

$$g, w \mapsto gw$$

$$g(w_1 + w_2) = gw_1 + gw_2$$

$$g(cw) = g(c)\, g(w).$$

GOAL: Decompose $V \otimes_K C$ into simpler sub-repns.

= "Hodge-Tate type" -- particularly nice decomposition.

Favourite Galois repn:

$$\begin{tikzcd} & U_p \arrow[d, hook] \\ \chi : \underbrace{G_K}_{\text{comp}} \arrow[ur, bend left] \arrow[r] & \mathbb{Q}_p^* \end{tikzcd}$$

$$= \mathrm{Aut}\left(\left(\varprojlim \mu_{p^\infty}\right) \otimes \mathbb{Q}\right)$$

p-adic cyclotomic character

$$g \longmapsto \left(z \longmapsto z^{\chi(g)}\right)$$

p.

Defn:

$$W^i := \left\{ w \in W \;\middle|\; gw = \chi(g)^i w \ \ \forall g \in G_K \right\}$$

$$\left(= W^{\chi^i}\right)$$

W^i is a K-linear subspace

<u>Example</u> $W = C$, $i = 0$

$$C^0 = \underset{\text{Elements of } C \text{ fixed by } G_K\text{-action}}{C^{G_K}}$$

?

<u>Thm</u>: $C^0 = K$

<u>(Tate)</u> $C^i = 0 \quad i \neq 0$

$$\bigoplus_i W^i \otimes_K C \xrightarrow{\alpha} W$$

G_K G_K

is <u>injective</u>.

Defn: The module W is Hodge-Tate type if α is an bijection.

☆☆☆ **Thm:** F p-div grp of finite height.

Ref: Tate's p-divisible groups paper.

$V = T(F) \otimes \mathbb{Q}_p$

(For F coming from A, good redn. $T(F) = T_p A$)

$H^1(A_c, \mathbb{Q}_p)$

$W = V \otimes \mathbb{C}_p = W(0) \oplus W(1)$

$W(0) = H^1(A_c, \Omega^0) \oplus H^0(A_c, \Omega^1)(1)$

p-adic Hodge Thy:

$A \rightsquigarrow X$

$H^i \rightsquigarrow H^i(X, \mathbb{Q}_p)$

$\mathbb{C}_p \rightsquigarrow$ other period rings to recover extra structure in $H^1_{dR}(A/K)$

§ Back to Tate's criterion:

Recall, we want to show

ρ is locally algebraic

$\Updownarrow$

$\rho|_{I_K}$ is semi-simple $+$ ρ is Hodge-Tate type

Pf: WLOG Assume $\rho|_{I_K}$ is semi-simple.

(§1, §2 of Appendix)

Useful Fact: Both sides can be checked after passing to a finite extension.

(Twist + reduce to $i=0$ $W_K^0 \otimes K' = W_{K'}^0$
Non-commutative Hilbert-Thm 90)

- Some explicit LCFT!

Split the sequence!

$$\begin{tikzcd} 1 \arrow[r] & U_K \arrow[r] & G_K \arrow[r] \arrow[l, bend left, "pr_\pi"] & \hat{\mathbb{Z}} \arrow[r] & 1 \end{tikzcd}$$

$\boxed{\text{Lubin-Tate totally ramified tower}}$

$$\overline{K}^{\ker(pr_\pi)} = \bigcup \text{ ab. extn.}$$

where π is a norm

Eg: $K = \mathbb{Q}_p$, $\pi = p$, $\mathbb{Q}_p(\mu_{p^\infty})$

$$\begin{tikzcd} U_K \arrow[r, shift left] \arrow[dr, "\rho"'] & G_K \arrow[l, shift left, "pr_\pi"] \arrow[d, "\rho'"] \\ & GL_V \end{tikzcd}$$

$$\rho' = \rho \circ pr_\pi$$

$$\rho'|_{U_K} = \rho|_{U_K}$$

ρ' is semi simply by our assumption.

Enough to prove propn. for simple sub-repn of ρ'.

WLOG ρ' is simple.

$$\rho' : G_K \longrightarrow GL_V$$

$E \subset End(V)$

= All endo. that commute with ρ'

ρ' simple + Schur's Lemma

$\Rightarrow E$ is a division algebra.

+ $\dim_E V = 1$

ρ' abelian + simple $\Rightarrow$ E is a field

$$\rightsquigarrow \rho' : G_K \to GL_E(V) = \underline{E^{*}}.$$

By Fact, can

Replace K by a large extn. that contains all conjugates of K.

§: ~~Decomposing~~ Hodge-Tate 1-d repns of G_K

$$\rho : G \longrightarrow U_E, \underset{\cong E}{\cancel{X}} \quad \dim 1 / E$$

$$W = C \otimes_{\mathbb{Q}_p} E$$

$$= C \otimes_K (K \otimes_{\mathbb{Q}_p} E)$$

$$\downarrow 1 \otimes \prod_{\sigma \in \mathrm{Hom}(E,K) = \Gamma_E} \sigma$$

$$\underset{G_K\text{-equiv}}{\cong} \quad C \otimes_K K^{d} = C^d$$

$$\Big\downarrow \sigma\text{-proj}, \ \sigma \in \Gamma_E$$

$$W^\sigma \subset W$$

$\underline{1}$ dim'l. / C $\in$ twisted C

G_K-action is twisted $C(\sigma \circ \rho)$

$$C(\sigma\cdot\rho) : G \times \mathbb{C} \to \mathbb{C}$$

$$(g, c) \mapsto \sigma\cdot\rho(g)\cdot g(c)$$

$$\bigoplus_{\sigma\in\Gamma_E} C(\sigma\cdot\rho) = \bigoplus_{\sigma\in\Gamma_E} W_\sigma = W$$

$\boxed{\text{Thm 2:}}$

$$\forall \sigma \in \Gamma_E, \quad W_\sigma \underset{G_k}{\cong} C(\chi^{n_\sigma})$$

$$\Longleftrightarrow \rho \underset{\uparrow}{\equiv} \prod_{\sigma\in\Gamma_E} \sigma^{-1}\circ \chi_{\sigma E}^{n_\sigma} \;\star$$

equal on an open subgrp of I_K.

where $\chi_{\sigma E} : G \twoheadrightarrow G(\bar{E}/E)^{ab}$

$$\begin{tikzcd} G(\bar{E}/E)^{ab} \arrow[d, "pr"] \\ U_E \arrow[d, hook, "i"'] \\ U_E \end{tikzcd}$$

$$\begin{tikzcd} & & K \\ E \arrow[r, hook] \arrow[rru, bend left, hook] & \sigma E \arrow[ru, hook] & \\ & \mathbb{Q}_p \arrow[lu, dash] \arrow[u, dash] \arrow[ruu, dash] & \end{tikzcd}$$

On $U_K \longrightarrow U_E$

(LCFT) $\quad x \longmapsto N_{K/\sigma E}(x^{-1})$.

Let's conclude Tate's criterion.

from Thm 2. (§A6)

(Fun with characters of tori!)

$$\begin{tikzcd} & & K \\ E \arrow[r, hook, "\sigma"] \arrow[dr, dash] & \sigma E \arrow[ur, hook] & \\ & \mathbb{Q}_p \arrow[uur, dash] & \end{tikzcd}$$

$T = \mathrm{Res}_{K/\mathbb{Q}_p} \mathbb{G}_m$

$T_E = \ldots$

$T_{\sigma E} = \ldots$

$$\begin{tikzcd} T \arrow[r, "N_{K/\sigma E}"] \arrow[rr, bend right, "\gamma_\sigma"'] & T_{\sigma E} \arrow[r, "\sigma^{-1}"] & T_E \subset GL(V) \end{tikzcd}$$

Take $\gamma := \prod_{\sigma \in \Gamma_E} \gamma_\sigma^{n(\sigma)}$

<u>CALCULATION</u>: $\gamma_\sigma(u^{-1}) = \sigma^{-1} \chi_{\sigma E}(u)$

$\forall u \in U_K.$

Thm 2 Sketch:

(Heart of the argument).

$$
\begin{tikzcd}
\chi_E : G_K \arrow[r] & U_E \arrow[d, hook] \\
& \mathrm{Aut}(\underbrace{T_\pi \otimes \mathbb{Q}_p}_{V_\pi})
\end{tikzcd}
$$

Tate module of the Lubin-Tate formal grp. F giving
$$
\begin{tikzcd}
U_E \arrow[r, shift left] & G(\bar{E}/E)^{ab} \arrow[l, shift left, hook', "\mathrm{pr}_\pi"]
\end{tikzcd}
$$

$$W_\pi = V_\pi \otimes_{\mathbb{Q}_p} \mathbb{C}$$

$$= W_\pi(0) \oplus W_\pi(1) \circlearrowleft \chi_E$$

(Tate)

$\nearrow$ dual of F $\cong$ E-action.

... $d-1$ dim'l $\chi \circlearrowright C(X) \otimes_K t_F$

1-d tgt. space of F

$E \curvearrowright t$ by σ_1

$$\Rightarrow (W_{\sigma_1}) = W_\pi(1)$$

$$\Rightarrow C(\chi_E) \simeq C(X)$$

$$\forall \sigma \neq \sigma_1 \Rightarrow C(\sigma \cdot \chi_E) \simeq C$$

Now, consider

$$\tilde{\rho} : G_K \longrightarrow E^*$$

$$\tilde{\rho} := \prod_{\sigma \in \Gamma_E} \sigma^{-1} \circ \chi_{\sigma E}^{n_\sigma}$$

Claim: $\tilde{\rho} = \rho$ on I_K

$$\Updownarrow$$

$$\tau \circ \tilde{\rho} = \tau \circ \rho \quad \forall\, \tau \in \mathrm{Hom}_{\mathbb{Q}_p}(E, K)$$

$(A2 + A3) \Longrightarrow$ Fun with log maps + cocycles!