1 Chapter I Solutions

1.1 Section 1

(TODO)

2 Chapter II Solutions

2.1 Section 1

1.16b. Given an exact sequence of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ over a topological space X with $\mathcal{F}'$ flasque show that for every open $U \subset X$ that the sequence $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is exact.

Proof. Since the section functor $\Gamma(U, -)$ is left exact, we only need to show the map $\beta\colon \mathcal{F}(U) \to \mathcal{F}''(U)$ is surjective. Fix a section $s \in \mathcal{F}''(U)$. Given any point $P \in U$, the sequence $0 \to \mathcal{F}'_P \to \mathcal{F}_P \to \mathcal{F}''_P \to 0$ is exact since the stalk functor is exact. So there is a germ $t_P \in \mathcal{F}_P$ that is mapped by β_P to s_P. Since these are germs of functions, there exists an open neighborhood $U_i \subset U$ and a section $t \in \mathcal{F}(U)$ with $\beta(t)|_{U_i} = s|_{U_i}$. Now suppose we have two sections $t_i, t_j \in \mathcal{F}(U)$ whose images under β agree with s on open neighborhoods U_i and $U_j \subset U$ of P respectively. Then $\beta(t_i - t_j)|_{U_i \cap U_j} = 0$, so since $0 \to \mathcal{F}'(U_i \cap U_j) \to \mathcal{F}(U_i \cap U_j) \to \mathcal{F}''(U_i \cap U_j)$, we have a section $w' \in \mathcal{F}'(U_i \cap U_j)$ that maps to $t_i - t_j$ on $U_i \cap U_j$. $\mathcal{F}'$ flasque gives a section $w \in \mathcal{F}'(U)$ that maps to w' under restriction, and maps to $t_i - t_j$ under $\alpha\colon \mathcal{F}'(U) \to \mathcal{F}(U)$. Thus the sections t_i and $t_j + \alpha(w)$ agree on $U_i \cap U_j$, so we can glue them on $U_i \cap U_j$ to get a section $t \in \mathcal{F}(U)$ such that $t|_{U_i} = t_i|_{U_i}$ and $t|_{U_j} = (t_j + \alpha(w))|_{U_j}$. Since $\beta(\alpha(w)) = 0$, we see $\beta(t)|_{U_i \cup U_j} = s|_{U_i \cup U_j}$, thus extending the sections on U_i and U_j to a section mapping to s over all $U_i \cup U_j$. Then Zorn's Lemma allows us to extend to a section t' over all U such that $\beta(t) = s$. Thus $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is exact. $\square$

3 Chapter III Solutions

3.1 Section 1

3.2 Section 2

2.1a. Let $X = \mathbb{A}^1_k$ be the affine line over an infinite field k. Let P, Q be distinct closed points of X, and let $U = X - P, Q$. Show $H^1(X, \mathbb{Z}_U) \neq 0$.

Proof. Using the exact sequence $0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to \mathbb{Z}_{\{P,Q\}} \to 0$ and computing cohomology gives $0 \to \Gamma(U, \mathbb{Z}_U) \to \Gamma(X, \mathbb{Z}_X) \to \Gamma(X, \mathbb{Z}_{\{P,Q\}}) \to H^1(X, \mathbb{Z}_U) \to H^1(X, \mathbb{Z}_X) \to \ldots$ $\mathbb{A}^1_k$ irreducible implies the constant sheaf $\mathbb{Z}_X$ is flasque, so $H^1(X, \mathbb{Z}_X) = 0$. $\Gamma(X, \mathbb{Z}_X) \cong \mathbb{Z}$ and $\Gamma(X, \mathbb{Z}_{\{P,Q\}}) \cong \mathbb{Z} \oplus \mathbb{Z}$ (since P and Q are distinct, we can assign independent values at the two points) then implies $H^1(X, \mathbb{Z}_U) \neq 0$, as desired. □

2.1b. More generally, let $Y \subseteq X = \mathbb{A}^n_k$ be the union of n+1 hyperplanes in suitably general position, and let $U = X - Y$. Show that $H^n(X, \mathbb{Z}_U) \neq 0$.

Proof. TODO □

2.2. Let $X = \mathbb{P}^1_k$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathcal{O} \to \mathcal{K} \to \mathcal{K}/\mathcal{O} \to 0$ of (II Ex. 1.21d) is a flasque resolution of $\mathcal{O}$. Conclude from (II Ex. 1.21e) that $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.

Proof. TODO □

4 Chapter IV Solutions

4.1 Section 1

1.1. Let X be a curve, and let $P \in X$ be a point. Then there exists a nonconstant rational function $f \in K(X)$, which is regular everywhere except at P.

Proof. Let X have genus g. Since X is dimension 1, there exists a point $Q \in X, Q \neq P$. Pick an $n > max\{g, 2g-2, 1\}$. Then for the divisor $D = n(2P - Q)$ of degree n, $l(K - D) = 0$ (1.3.4), so Riemann-Roch gives $l(D) = n + 1 - g > 1$. Thus there is an effective divisor D' such that $D' - D = (f)$. Since (f) is degree 0 (II 6.10), D' has degree n, so D' cannot have a zero of order large enough to kill the pole of D of order $2n$. f is regular everywhere except at P. Note we cannot control the zeros of f with this proof. □

1.2. Again let X be a curve, and let $P_1, P_2, \ldots, P_r \in X$ be points. Then there is a rational function $f \in K(X)$ having poles (of some order) at each of the P_i, and regular elsewhere.

Proof. We have to be careful. Multiplying functions from (Ex. 1.1) may result in zeroes cancelling poles. So proceed as follows: Fix a point Q distinct from the P_i, and consider the divisor $n(P_1 + P_2 + \ldots + P_r - (r-1)Q)$, with $n > max\{2g-2, g\}$. Then similar to (Ex. 1.1), we find an effective divisor D with $D + n(r-1)Q - nP_1 - nP_2 - \ldots - nP_r = (f)$. Again since (f) has degree 0 (II 6.10), degree of D is n. Since each P_i occurs with order $-n$ outside of D, either D cannot have a zero at any P_i large enough to cancel the pole, or $D = nP_i$ for some fixed i. In the former case we're done, in the latter use (Ex. 1.1) to get a principal divisor (g) with a pole at P_i. (fg) may have unwanted cancellation. Suppose (g) has zeroes of order n_j at some (or none, or all) P_j. Then for $\alpha > max\{n_j, 1\}$, (f^α) has poles of higher order at $P_j \neq P_i$. Then the principal divisor $(f^\alpha g)$ must have a pole at P_i, and the zeros of (g) cannot cancel the poles at the remaining P_j of (f^α), so this is the divisor. □

1.3. Let X be an integral, separated, regular, one-dimensional scheme of finite type over k, which is *not* proper over k. Then X is affine.

Proof. Embed $X \to \overline{X}$, the closure of X(TODO - explain). Then $\overline{X} = X \cup \{P_1, P_2, \ldots, P_r\}$. This is a finite set since (TODO). Then use (Ex. 1.2) to get a principal divisor (f) with poles at exactly the P_i, and this defines a morphism $f : \overline{X} \to \mathbb{P}^1_k$ which sends the P_i to ∞, and the rest of $\overline{X}$ to $\mathbb{A}^1$. $f(\overline{X}) \neq pt$ so (II 6.8) implies f is finite. (II Ex. 5.17b) says any finite morphism is affine, so $f^{-1}(\mathbb{A}^1) = X$ is affine. □

1.4. Show that a separated, one-dimensional scheme of finite type over k, none of whose irreducible components is proper over k, is affine.

Proof. (III Ex. 3.2) gives the scheme X is affine if and only if each irreducible component is affine. Then (III Ex. 3.1) implies each irreducible component Y is affine if and only if Y_{red} is affine. Since irreducible and reduced imply integral (II 3.1), we have reduced to the case X is integral, and then the result follows from (Ex 1.3). □

1.5. For an effective divisor D on a curve X of genus g, show that $dim|D| \leq degD$. Furthermore, equality holds if and only if $D = 0$ or $g = 0$.

Proof. Since D is effective, the subspaces $|K - D| \subseteq |K|$ imply $l(K - D) \leq l(K) = g$ (1.3.3). Using $dim|D| = l(D) - 1$ (pg. 295), Riemann-Roch gives $dim|D| = degD + l(K - D) - g \leq degD + l(K) - g = degD$. Equality thus follows if $l(K - D) = l(K)$. Clearly we have equality for $D = 0$, and if $g = 0$ then $degK = -2$ and since D is effective $l(K - D) = 0 = g$.

Conversely, suppose we have equality, and suppose $D \neq 0$. $P \in SuppD$ gives $|K - D| \subseteq |K - D - P| \subseteq |K|$ TODO

□

1.6. Let X be a curve of genus g. Show there is a finite morphism $f : X \to \mathbb{P}^1_k$ of degree $\leq g + 1$.

Proof. Pick any closed point $P \in X$. Let $D = (g + 1)P$ be a divisor. Riemann-Roch gives $l(D) = (g + 1) + 1 - g + l(K - D) = 2 + l(K - D)$, or $l(D)/geq2$. So there exists an effective divisor (TODO) , and (II 6.8) gives □

1.7.

Proof. TODO □

1.8.

Proof. TODO □

1.9.

Proof. TODO □

1.10. Let X be an integral projective scheme of dimension 1 over k, which is locally a complete intersection, and has $p_a = 1$. Fix a point $P_0 \in X_{reg}$. Imitate (1.3.7) to show that the map $P \to \mathcal{L}(P - P_0)$ gives a one-to-one correspondence between the points of X_{reg} and the elements of the group $Pic^0 X$. This generalizes (II, 6.11.4) and (II, Ex. 6.7).

Proof. Let D be any divisor of degree 0. We need to show there exists a unique point $P \in X_{reg}$ such that $D \sim P - P_0$. Since X satisfies (Ex. 1.9d), we find that $l(D + P_0) - l(K - D - P_0) = 0 + 1 - 1 = 1$. Applying (Ex. 1.9d) to the divisors 0 and K, we get $l(0) - l(K) = deg 0 + 1 - p_a$ and $l(K) - l(0) = deg K + 1 - p_a$. Combining we find that $deg K = 2p_a - 2$, so in our problem $deg K = 0$, thus $deg(K - D - P_0) < 0$ and $l(K - D) = 0$, giving $l(D + P_0) = 1$. Then $dim|D - P_0| = 0$, so there exists a unique effective divisor linearly equivalent to $D + P_0$. Since the degree is 1, this divisor must be a single point $P \sim D + P_0$, or $D \sim P - P_0$. Note that (Ex. 1.9 c,d) implies $P \in X_{reg}$. □

4.2 Section 2

2.2. Classification of Curves of Genus 2. Fix an algebraically closed field k of characteristic $\neq 2$.

(a). If X is a curve of genus 2 over k, the canonical linear system $|K|$ determines a finite morphism $f : X \to \mathbb{P}^1$ of degree 2 (Ex. 1.7). Show that it is ramified at exactly 6 points, with ramification index 2 at each one. Note that f is uniquely determined, up to automorphism of $\mathbb{P}^1$, so X determines an (unordered) set of 6 points of $\mathbb{P}^1$, up to an automorphism of $\mathbb{P}^1$.

Proof. Hurwitz gives $2(4-2) = 2(0-2) + degR$ which implies $degR = 6$. Since $degf = 2$, each ramification index $e_P \leq 2$, so each point in the support of R occurs with at most degree 1. Thus the morphism is ramified at exactly 6 points, with ramification index 2 at each point. □

(b). Conversely, given six distinct elements $\alpha_1, \ldots, \alpha_6 \in k$, let K be the extension of $k(x)$ determined by the equation $z^2 - (x-\alpha_1)\ldots(x-\alpha_6)$. Let $f : X \to \mathbb{P}^1$ be the corresponding morphism of curves. Show that $g(X) = 2$, the map f is the same as the one determined by the canonical linear system, and f is ramified over the six points $x = alpha_i$ of $\mathbb{P}^1$, and nowhere else. (Cf. (II, Ex. 6.4)).

Proof. TODO □

(c). Using (I, Ex. 6.6), show that if P_1, P_2, P_3 are 3 distinct points of $\mathbb{P}^1$, then there exists a unique $\varphi \in Aut\mathbb{P}^1$ such that $\varphi(P_1) = 0, \varphi(P_2) = 1, \varphi(P_3) = \infty$. Thus in (a), if we order the six points $x = \alpha_i$ of $\mathbb{P}^1$, and then normalize by sending the first three to $0, 1, \infty$, respectively, we may assume that X is ramified over $0, 1, \infty, \beta_1, \beta_2, \beta_3$, where $\beta_1, \beta_2, \beta_3$ are three distinct elements of $k, \neq 0, 1$.

Proof. □

(d). Let Σ_6 be the symmetric group on 6 letters. Define an action of Σ_6 on the sets of three distinct elements of $k, \neq 0, 1$ as follows: reorder the set $0, 1, \infty, \beta_1, \beta_2, \beta_3$ according a given element $\sigma \in \Sigma_6$, then renormalize as in (c) so that the first three become $0, 1, \infty$ again. Then the last three are the new $\beta_1', \beta_2', \beta_3'$.

Proof. Nothing to do. □

(e). Summing up, conclude that there is a one-to-one correspondence between the set of isomorphism classes of genus 2 over k, and triples of distinct elements $\beta_1, \beta_2, \beta_3$ of $k, \neq 0, 1$, modulo the action of Σ_6 described in (d). In particular, there are many non-isomorphic curves of genus 2. We say that curves of genus 2 depend on three parameters, since they correspond to the points of an open subset of $\mathbb{A}_k^1$ modulo a finite group.

Proof. □

2.5. Automorphisms of a Curve of Genus ≥ 2. Prove the theorem of Hurwitz that a curve X of genus $g \geq 2$ over a field k of characteristic 0 has at most $84(g-1)$ automorphisms. We will see later (Ex. 5.2) or (V Ex. 1.11) that the group $G = AutX$ is finite. So let G have order n. Then G acts on the function field $K(X)$. Let L be the fixed field. Then the field extension $L \subseteq K(X)$ corresponds to a finite morphism of curves $f : X \to Y$ of degree n.

(a). If $P \in X$ is a ramification point, and $e_P = r$, show that $f^{-1}f(P)$ consists of exactly n/r points, each having ramification index r. Let $P_1, \ldots, P_s$ be a maximal set of ramification points of X lying over distinct points of Y, and let $e_{P_i} = r_i$. Then show that Hurwitz's theorem implies that

$$(2g-2)/n = 2g(Y) - 2 + \sum_{i=1}^{s}(1 - 1/r_i).$$

Proof. Suppose $P \in X$ is ramified as in the statement, and let $f(P) = Q \in Y$. The local ring $B = \mathcal{O}_{Y,Q}$ has field of fractions L, and under characteristic 0 the extension $L \subseteq K(X)$ is separable (and finite already). Let $A = \overline{B}$ be the integral closure of B in $K(X)$. Then the points over Q correspond to the dvrs in $K(X)$ that lie over A. Each of these dvrs is just A localized at some maximal ideal in A. We want to show G acts transitively over these dvrs, so for any two primes P_i and P_j we want an element $\sigma \in G$ such that $\sigma(P_i) = P_j$. Clearly each element in B is fixed by G, and since A is the integral closure of B, B is the fixed field of A under the action of G. This follows since A is the set of elements in $K(X)$ that are integral over B, so any element in $A - B$ is not fixed by G, and any element in A fixed under G is in L, hence also in B. Then from an easy problem in Atiyah-MacDonald Ch.5, #12,13, it follows that G acts transitively, so there is a curve automorphism taking any ramified point P_i over Q to any other ramified point P_j over Q.

Thus each point has the same index $e_{P_i} = r$, and there has then to be n/r of them since f is tamely ramified at each point ($chark = 0$), and (II, 6.9) gives $n = degf degQ = degf^*Q = deg(\sum e_{P_i} P_i)$, but since each $e_{P_i} = r$, this implies there are n/r points over Q.

Then Hurwitz's thoerem gives $(2g-2)/n = 2g(Y)-2+1/n\sum_{P\in X}(e_P-1)$. Since there are n/r_i points over Q_i, each with ramification index r_i, we get $1/n\sum_{P\in X}(e_P-1) = 1/n\sum_{i=1}^{s}(n/r_i)(r_i-1) = \sum_{i=1}^{s}(1-1/r_i)$, finishing the problem. □

(b). Since $g \geq 2$, the left hand side of the equation is > 0. Show that if $g(Y) \geq 0, x \geq 0, r_i \geq 2, i = 1, \ldots, s$ are integers such that

$$2g(Y) - 2 + \sum_{i=1}^{s}(1 - 1/r_i) > 0,$$

then the maximum value of this expression is $1/42$. Conclude that $n \leq 84(g-1)$. See (Ex. 5.7) for an example where this maximum is achieved.

Proof. Let $g_Y = g(Y)$. Since each term $1-1/r_i$ is of the form $n/(n+1)$, we get that $(*)s > \sum_{i=1}^{s}(1-1/r_i) \geq s/2$. If $g_Y \geq 2$, then $2g_Y - 2 + \sum_{i=1}^{s}(1-1/r_i) \geq 2 + s/2/geq2$. If $g_Y = 1$, then the expression is $\geq s/2$, so is greater than $1/2$. If $g_Y = 0$, then guessing gives $1/2 > 1/6 = -2 + 1/2 + 1/2 + 1/2 + 2/3 > 0$ corresponding to $s = 4, r_1, r_2, r_3 = 2, r_4 = 3$. So any smaller value must have $g_Y = 0$. If $s \geq 5$, then the sum is $\geq -2 + 5/2 = 1/2 > 1/6$, so $s \leq 4$. The only solution with $s = 4$ smaller than the one above giving $1/6$ is $-2 + 1/2 + 1/2 + 1/2 + 1/2 = 0$, no solution at all. Any other changes make the value larger than $1/6$. If $s < 3$, $(*)$ gives that the expression is < 0. Thus any better solution than the $1/6$ must have $s = 3$. So we want to minimize $h(a, b, c) = a/(a+1) + b/(b+1) + c/(c+1) - 2$, where a, b, c are positive integers and the we require $h > 0$. WLOG assume $a \geq b \geq c$. Then checking possible cases: $h(1, 1, c) < 0$ for all c, so any solution has at most 1 entry of 1. Checking higher cases (note the expression value increases as we pick larger numbers!): $h(1, 2, 2) < 0, h(1, 2, 3) < 0, h(1, 2, 4) < 0, h(1, 2, 5) = 0, h(1, 2, 6) = 1/42$, corresponding to $r_1 = 2, r_2 = 3, r_3 = 7$. Any value $h(1, 2, c), c > 6$ must be larger. So we check $h(1, 3, 3) = 0, h(1, 3, 4) = 1/20$, and again there can be no better solution with $h(1, 3, c), c > 4$. Continuing, $h(1, 4, 4) = 1/10$, so there is no better solution with $h(1, b, c)$. Next $h(2, 2, 2) = 0, h(2, 2, 3) = 1/12, h(2, 3, 3) = 1/6$ so there is no better one starting with $a = 2$. $h(3, 3, 3) = 1/4$, and any higher starting a value will do worse, so the minimum $1/42$ occurs at $g_Y = 0, s = 3, r_1 = 2, r_2 = 3, r_3 = 7$. Using part (a) gives that $1/n(2g-2) \geq 1/42$ which is $84(g-1) \geq 1/42$. $\square$

4.3 Section 3

3.1. If X is a curve of genus 2, show that a divisor D is very ample $\iff$ $degD \geq 5$.

Proof. Cor 3.2 gives $degD \geq 5 \Rightarrow D$ very ample. For the other direction, assume D very ample. Suppose $degD < 5$. Then 3.1(b) implies $l(D-P-Q) = l(D) - 2 \Rightarrow l(D) \geq 2$. But $l(D) = 2 \Rightarrow dim|D| = 1$, so there is a closed immersion $X \hookrightarrow \mathbb{P}^1$, so X is either a point or $\mathbb{P}^1$ (II, 6), contradicting $g(X) = 2$. Thus $l(D) > 2$.

Thus there exists an effective divisor $D' \in |D|$, and since $|D'| = |D|$ (proof: $\alpha \in |D| \Rightarrow \alpha - D = (f), D - D' = (g) \Rightarrow D' - D + \alpha - D = (fg) \Rightarrow |D| \subseteq |D'|$. The other way is similar.) we can apply (Ex. 1.5) to D' to get $dim|D| = dim|D'| \leq degD' = degD$. If $D' = 0$ then $l(D) = l(D') = 1$, contradiction. Since $g \neq 0$ the inequality is strict (Ex. 1.5), so $1 < l(D) - 1 = dim|D| < degD$ and $degD > 2$. So we check cases:

If $degD = 3$, then D nonspecial implies (Riemann-Roch) $l(D) = 2$, a contradiction. If $degD = 4$, then D nonspecial implies (Riemann-Roch) $l(D) = 3$, so $dim|D| = 2$, and D gives a closed immersion $X \hookrightarrow \mathbb{P}^2$, but since the genus of a plane curve (I Ex. 7.1) is $1/2(d-1)(d-2)$ which can never be 2, this too is a contradiction. Thus $degD \geq 5$. □

3.2. Let X be a curve of degree 4.

(a). Show that the effective canonical divisors on X are exactly the divisors $X.L$, where L is a line in $\mathbb{P}^2$.

Proof. Genus of a plane curve is $g = 1/2(d-1)(d-2)$ so degree 4 gives $g(X) = 3$. $degK = 2g - 2 = 4$. For a fixed line $L \in \mathbb{P}^2$ set $D = X.L$. Bezout's theorem says D is 4 points counted with multiplicity, so $degD = 4$. Riemann-Roch gives $l(D) - l(K_D) = 4 + 1 - 3 = 2$. Since it takes exactly 2 points of X to define any line L, $dim|D| = 2$ (TODO - better proof), so $l(D) = 3$, and $l(K - D) = 1$. As in proof of (Ex. 3.1), (Ex. 1.5) gives that $dim|K - D| \leq deg(K - D) = 0$, and since $|K - D| \neq \emptyset$, (why? TODO), $g \neq 0$ gives equality, so $K - D \sim 0$, or $k \sim D$. □

(b). If D is any effective divisor of degree 2 on X, show that $dim|D| = 0$.

Proof. Take a line L defined by 2 distinct points of D if possible, else if $D = nP$ use (I Ex. 7.3) to get the unique line with proper intersection multiplicity. Let $D' = X.L$, a degree 4 effective divisor by Bezout's theorem. Then by proof of (a), $dim|D'| = 2$, and 3.3.5 implies D' very ample (why-todo). $D' - D$ is 2 points, and theorem 3.1b gives $dim|D| = dim|D' - (D' - D)| = dim|D'| - 2 = 0$. $\square$

(c). Conclude X is not hyperelliptic.

Proof. (TODO - check thoroughly) X hyperelliptic implies there is a finite morphism $f : X \to \mathbb{P}^1$ of degree 2. To give such a morphism is equivalent to giving a base point free linear system $|D|$ such that $dim|D| = 1$. So for $D' \in |D|, degD' = 2$. But by part (b) $0 = dim|D'| = dim|D|$, a contradiction. So X is not hyperelliptic. $\square$

3.3. If X is a curve of genus ≥ 2 which is a complete intersection (II Ex. 8.4) in some $\mathbb{P}^n$, show that the canonical divisor K is very ample. Conclude that a curve of genus 2 can never be a complete intersection in any $\mathbb{P}^n$. Cf. (II. 7.8.5)

Proof. Let $X = \bigcap H_i$ be the intersection of hypersurfaces as in exerciseII8.4. Using the same exercise, $\omega_X \cong \mathcal{O}_X(\sum d_i - n - 1)$ for $d_i = degH_i$. Since $dim\Gamma(X, \omega_X) = g \geq 2$, ω_X must have global sections, so $m = \sum d_i - n - 1 \geq 0$. $m \neq 0$, since then $g = 1$, a contradiction. Thus $\omega_X \cong \mathcal{O}_X(m)$ with $m > 0$. Composing $X \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ where the second inclusion is the m-uple embedding gives $\mathcal{L}(K) \cong \omega_X \cong i^*\mathcal{O}_{\mathbb{P}^N}(1)$. Thus K is very ample by definition. Note that the m-uple embedding $i : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ gives $\mathcal{O}_X(m) \cong i^*\mathcal{O}_X(1)$.

By (Ex. 1.7a), if $g = 2$ $|K|$ defines a morphism $X \to \mathbb{P}^1$ so K is not very ample (this cannot be an embedding), thus X cannot be a complete intersection. $\square$

3.4. Let X be the d-uple (I Ex. 2.12) embedding of $\mathbb{P}^1$ in $\mathbb{P}^d$, for any $d \geq 1$. We call X the *rational normal curve of degree* d in $\mathbb{P}^d$.

(a). Show that X is projectively normal, and that its homogeneous ideal can be generated by elements of degree 2.

Proof. TODO - was not hard... use (I Ex. 2.12) $\square$

(b). if X is any curve of degree d in $\mathbb{P}^n$, with $d \leq n$, which is not contained in any $\mathbb{P}^{n-1}$, show that in fact $d = n, g(X) = 0$, and x differs from the rational normal curve of degree d only by an automorphism of $\mathbb{P}^d$. Cf. (II. 7.8.5).

Proof. Take a hyperplane H in $\mathbb{P}^n$. Then $H.X$ consists of d points (counted with multiplicity). These points span a hyperplane of dimension $d-1$. If $d < n$, then we can add any other points on X until a hyperplane of dimension $n-1$ is spanned, but this new hyperplane contains H, so H must itself have intersected X in $n-1$ points, contradicting $d < n$. Thus $d = n$. The argument of (II. 7.8.5) gives $g = 0$ and the rest of the problem. (TODO - write up) $\square$

(c). In particular, any curve of degree 2 in any $\mathbb{P}^n$ must be a conic in some $\mathbb{P}^2$.

Proof. Since $d = 2$ we must have $n = 2$ by part b. $\square$

(d). A curve of degree 3 in any $\mathbb{P}^n$ must either be a plane cubic curve, or the twisted cubic curve in $\mathbb{P}^3$.

Proof. If $n < d$, since $n = 1$ is impossible, we must have $n = 2$. Then elliptic curves exist in $\mathbb{P}^2$ so this case is possible. Otherwise by part b the curve must be the twisted cubic in $\mathbb{P}^3$. $\square$

3.5. Let X be a curve in $\mathbb{P}^3$, which is not contained in any plane.

(a). If $O \notin X$ is a point, such that projection from O indices a birational morphism φ from X to its image in $\mathbb{P}^2$, show that $\varphi(X)$ *must* be singular.

Proof. First note that since X is contained in no plane, $\varphi(X)$ is contained in no line in $\mathbb{P}^2$ since φ is projection from a point. So assume $Y = \varphi(X)$ is not singular. Then X is isomorphic to Y since they are birational. (I, 6). Associate X with its image to simplify (confuse?) notation. Use the twisted exact sequence (for $n = 2$ and $n = 3$) $0 \to \mathcal{I}_X(1) \to \mathcal{O}_{\mathbb{P}^n}(1) \to \mathcal{O}_X(1) \to 0$, where $\mathcal{I}_X$ is the ideal sheaf defining $X \subseteq \mathbb{P}^3$, and compute the long exact sequence of cohomology. For $n = 2$ or 3 we have that $H^0(\mathbb{P}^3, \mathcal{I}_X(1)) = 0$. Indeed, if there are any global sections of degree 1, there is a linear polynomial in the ideal defining X, and then X is contained in a plane ($n = 3$) or a line ($n = 2$), both contradictions. Thus we have $0 \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(X, \mathcal{O}_X(1)) \to H^1(\mathbb{P}^3, \mathcal{I}_X(1)) \to \ldots$. (Note the last map of H^0's is not surjective in general. Take something *not* projectively normal, like the quadratic embedding $(s : t) \hookrightarrow (s^4 : s^3t : st^3 : t^4)$.) Since the first term has dimension $n+1$, we get for $n = 3$ that $dim H^0(\mathbb{P}^3, \mathcal{O}_X(1)) \geq 4$. In case $n = 2$ if we show that $dim H^1(X, \mathcal{I}_X(1)) = 0$, then we get $dim H^0(\mathbb{P}^2, \mathcal{O}_X(1)) = 3$, a contradiction. (Note this assumes that these $\mathcal{O}_X(1)$'s are the same over $\mathbb{P}^2$ and $\mathbb{P}^3$, since the sheaf is the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$. They are the same sheaf by **Theorem** II 6.17 $\mathbb{P}^2 \hookrightarrow \mathbb{P}^3$, gives (TODO...)).

For $n = 2$, the curve is a Cartier divisor, so by **Theorem** II 6.9, $\mathcal{I}_X \cong \mathcal{L}(-D)$. By **Theorem** II 6.17 $\mathcal{L} \in Pic\mathbb{P}^n \Rightarrow \mathcal{L} \cong \mathcal{O}_X(n), n \in \mathbb{Z}$. **Theorem** II 6.4 gives $D \sim dH$, so we combine to get $sheafIX \cong \mathcal{L}(-D) \cong \mathcal{L}(-dH) \cong \mathcal{O}_{\mathbb{P}^2}(-d)$. Twisting, $\mathcal{I}_X(1) \cong \mathcal{O}_{\mathbb{P}^2}(1-d)$. $d = 1 \Rightarrow Y \cong \mathbb{P}^1$, but Y not in any line gives a contradiction. Thus $d > 1$, and $H^1(\mathbb{P}^2, \mathcal{I}_X(1)) \cong H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 0$ by **Theorem** II 5.1b. □

(b). If X has degree d and genus g, conclude that $g < \frac{1}{2}(d-1)(d-2)$.

Proof. Use **Theorem** 3.10 to get $O \notin X$ such that projection from O gives a birational morphism to the image, with at most nodes as singularities. Then part a implies there exists at least one node, and 3.4.1 with $r \geq 1$ gives the result. □

(c). TODO

Proof. □

3.6. *Curves of degree 4*

(a). If X is a curve of degree 4 in some —PSn, show that either

(1) $g = 0$, in which case X is either the rational normal quartic in $\mathbb{P}^4$ (Ex. 3.4) or the rational quartic curve in $\mathbb{P}^3$ (II. 7.8.6), or

(2) $X \subseteq \mathbb{P}^2$, in which case $g = 3$, or

(3) $X \subseteq \mathbb{P}^3$ and $g = 1$.

Proof. (1) If $g = 0$ then X is isomorphic to $\mathbb{P}^1$. $X \subsetneq \mathbb{P}^2$, since in $\mathbb{P}^2$ $g = \frac{1}{2}(d-1)(d-2) = 3$. So using (Ex. 3.4b), X is in $\mathbb{P}^3$ or $\mathbb{P}^4$, and using the same exercise $X \subsetneq \mathbb{P}^3$ implies X is the rational normal curve in $\mathbb{P}^4$, which has degree 4. So it remains to show there exists a curve of genus 0 and degree 4 in $\mathbb{P}^3$. This is given by (II. 7.8.6).

(2) $X \subseteq \mathbb{P}^2$ implies $g = 3$ by the usual formula.

(3) Assume $X \subseteq \mathbb{P}^3$. (Ex. 3.5b) gives that $g < 3$. and $g \neq 0$ since that is covered in case (1). Taking a hyperplane H and using Riemann-Roch gives $l(H) - l(K - H) = degH + 1 - g$. $degH = 4$ since a degree 4 curve. If $g = 2$ then $degK = 2g - 2 = 2$, and then $l(K - H) = 0$ by (. 1.3.4). Then $l(H) = 4 + 1 - 2 = 3$, but $l(H) = dimH^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) = 4$, a contradiction. Thus $g \neq 2$. (III Ex. 5.6) gives that a curve of type 2,2 has $g = 1$. Since the degree of a curve of type (a, b) is $a + b$ (think through the embedding

and the intersection with a hyperplane - (a,b) corresponds to a lines in one direction and b lines in the other.). Thus curves with $g = 1$ and degree 4 exist in $\mathbb{P}^3$. $\square$

(b). In the case $g = 1$, show that X is a complete intersection of two irreducible quadric surfaces in $\mathbb{P}^3$ (I Ex. 5.11).

Proof. Taking a twisted sequence with the ideal sheaf $\mathcal{I}_X$ gives $0 \to \mathcal{I}_X(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_X(2) \to 0$. Then $dim H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = (\begin{matrix} 3+2 \\ 2 \end{matrix}) = 10$ (the k dimension of the space of degree 2 monomials in 4 variables). $dim H^0(X, \mathcal{O}_X(2)) = 8$ since this is two hyperplanes intersecting the degree 4 curve X. Thus $dim H^0(\mathbb{P}^3, \mathcal{I}_X(2)) \geq 2$. So X is contained in at least 2 irreducible quadratic hypersurfaces. (This is also interesting for degree 3 hypersurfaces, etc...). (TODO - why complete intersection?) $\square$

3.7. In view of **Theorem** 3.10, one might ask conversely, is every plane curve with nodes a projection of a nonsingular curve in $\mathbb{P}^3$? Show that the curve $xy + x^4 + y^4 = 0$ (assume $char k \neq 2$) gives a counterexample.

Proof. The only singularity is a node at (0,0). Suppose this curve X is the projection of a nonsingular curve $\tilde{X} \subseteq \mathbb{P}^3$. Then 3.11.1 gives $g = \frac{1}{2}(d-1)(d-2) - r = \frac{1}{2}(3)(2) - 1 = 2$, but no such curve exists by (Ex. 3.6a). $\square$

3.8. We say a (singular) integral curve in $\mathbb{P}^n$ is *strange* if there is a point which lies on all the tangent lines at nonsingular points of the curve.

(a). There are many singular strange curves, e.g., the curve given parametrically by $x = t, y = t^p, z = t^{2p}$ over a field of characteristic $p > 0$.

Proof. The curve is clearly singular at $(0:0:0:1)$. (TODO) Since the curve is given parametrically, the tangent direction at a point is $\frac{\partial}{\partial t}(t, t^p, t^{2p}) = (1,0,0)$. At a point $(x_0 : y_0 : z_0 : 1)$ on the curve, this tangent line is the intersection of the hyperplanes $y = y_0 w$ and $z = z_0 w$. Thus in $\mathbb{P}^3$, every tangent line goes through $(1:0:0:0)$, the point at infinity. $\square$

(b). Show, however, that if $char k = 0$, there aren't even any singular strange curves besides $\mathbb{P}^1$.

Proof. (TODO-) Idea is to reprove 3.4,3.5,3.9, but the dimension of the tangent space in 3.5 may be very large, so only consider the tangent space of the nonsingular points on the curve. Needs some more work. □

3.9. Prove the following lemma of Bertini: is X is a curve of degree d in $\mathbb{P}^3$, not contained in any plane, then for almost all planes $H \subseteq \mathbb{P}^3$ (meaning a Zariski open subset of the dual projective space $(\mathbb{P}^3)^*$), the intersection $X \cap H$ consists of exactly d distinct points, no three of which are collinear.

Proof. 3 points are collinear $\iff$ H contains a multisecant of X. There are strictly less than d distinct points $\iff$ H contains a tangent line of X. The proof of 3.5 shows $TanX$, the tangent space of X, is closed and $dimTanX \leq 2$. TODO □

3.10. Generalize the statement that "not every secant is a multisecant" as follows. If X is a curve in $\mathbb{P}^n$, not contained in any $\mathbb{P}^{n-1}$, and if $char k = 0$, show that for almost all choices of $n-1$ points $P_1, P_2, \ldots, P_{n-1}$ on X, the linear space L^{n-2} spanned by the P_i does not contain any further points of X.

Proof. Let the degree of X be d. Then by (Ex. 3.4b) if $d \leq n$ we have $d = n$ and X differs from the n-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^n$. TODO

So assume $d > n$.

□

4.4 Section 4

(TODO)

4.5 Section 5

5.1. Show that a hyperelliptic curve can never be a complete intersection in any projective space.

Proof. Every hyperelliptic curve has genus ≥ 2 (Ex. 1.7), and every complete intersection has very ample canonical divisor (Ex. 3.3). But the canonical divisor of a hyperelliptic curve is not very ample by **Theorem** 5.2, so it cannot be a complete intersection. □

5.6. Show that a nonsingular plane curve of genus 5 has no g_3^1. Show that there are nonhyperelliptic curves of genus 6 which cannot be represented as a nonsingular plane curve.

Proof. (TODO) □

4.6 Section 6

6.1. A rational curve of degree 4 in $\mathbb{P}^3$ is contained in a unique quadric surface Q, and Q is necessarily nonsingular.

Proof. $X \cong \mathbb{P}^1 \Rightarrow g(X) = 0$. $X \subseteq \mathbb{P}^2 \Rightarrow g = \frac{1}{2}(d-1)(d-2)$, but degree 4 and genus 0 contradict. Thus $X \not\subseteq \mathbb{P}^2$.

Consider the sequence $0 \to \mathcal{I}_X(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_X(2) \to 0$. Take cohomology and dimension. $dimH^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = \left(\begin{array}{c} 3+2 \\ 2 \end{array} \right) = 10$. $\mathcal{O}_X(2)$ corresponds to degree 2 hypersurfaces, so any intersection with X is a divisor consisting of $4 \times 2 = 8$ points, so $degD = 8$. Since $8 > 2g - 2 = -2$, D is nonspecial, and Riemann-Roch then gives $l(D) = 8 + 1 - 0 = 9$, so $dimH^0(X, \mathcal{O}_X(2)) = l(D) = 9$, thus from the exact sequence $dimH^0(\mathbb{P}^3, \mathcal{I}_X(2)) \geq 1$. So X is contained in a quadratic surface Q. The intersection of 2 quadratic surfaces is a complete intersection of degree 4, and since X has degree 4, if X is contained in the intersection, it would be a complete intersection and have genus (II Ex. 8.4) $g = \frac{1}{2} \times 4(2+2-4)+1 = 1$, contradicting $g(X) = 0$. Thus X lies on a unique quadratic surface. Then by (Ex. 3.6) X is the rational quartic curve in $\mathbb{P}^3$, so Q is nonsingular. (TODO - not clear?) $\square$

6.2. A rational curve of degree 5 in $\mathbb{P}^3$ is always contained in a cubic surface, but there are such curves not contained in any quadric surface.

Proof. Again, $X \cong \mathbb{P}^1$, so $g(X) = 0$. As above, consider the sequence $0 \to \mathcal{I}_X(3) \to \mathcal{O}_{\mathbb{P}^3}(3) \to \mathcal{O}_X(3) \to 0$. Degree 3 hypersurfaces in $\mathcal{O}_X(3)$ intersected with X give divisors D of degree $3 \times 5 = 15$, which makes D nonspecial ($15 > 2g-2 = -2$). Riemann-Roch gives $l(D) = 15+1-0$, so $dimH^0(X, \mathcal{O}_X(3)) = 16$. $DimH^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) = \left(\begin{array}{c} 3+3 \\ 3 \end{array} \right) = 20$, so again $dimH^0(\mathbb{P}^3, \mathcal{I}_X(3)) \geq 4$, and X lies on a cubic surface. To get a curve not contained in any quadratic, use the idea in (. II)7.8.6. We embed $\mathbb{P}^1$ as a degree 5 curve as $\varphi : (s:t) \hookrightarrow (s^5 : s^4t : st^4 + \alpha s^2t^3 : t^5) = (x : y : z : w)$ for some $\alpha \in k^*$. This is degree 5 since it forms a basis for a linear subspace $V \subseteq \Gamma(X, \mathcal{O}_X(5))$. It is easy to check it has no degree 2 relations by checking degrees of the s and t involved. For fun it satisfies the cubic $x^2w - yz^2 + yzw - xw^2 = 0$. To check it is an embedding, we use **Theorem** II 7.3. Either x or w has to be nonzero. $w \neq 0 \Rightarrow x/y = s$, separating points. $x \neq 0 \Rightarrow y/x = t/s$, separating points.

Since $Y = \varphi(X)$ is dimension 1, the tangent space is dimension 1 if Y is nonsingular, so there are no tangents to separate. So we only need to check nonsingularity. (TODO- easy?) $\square$

6.3. A curve of degree 5 and genus 2 in $\mathbb{P}^3$ is contained in a unique quadric surface Q. Show that for any abstract curve X of genus 2, there exists embeddings of degree 5 in $\mathbb{P}^3$ for which Q is nonsingular, and there exists other embeddings of degree 5 for which Q is singular.

Proof. Noting $\mathcal{O}_X(2)$ corresponds to degree 2 hypersurfaces, and that degree of such a (nonspecial) divisor is 10 $0 \to \mathcal{I}_X(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_X(2) \to 0$ gives $dimH^0(\mathbb{P}^3, \mathcal{I}_X(2)) \geq 1$. So X lies in a quadric surface Q. X cannot lie on 2, since $degQ_1 \cup Q_2 = 4$, and X could have degree at most 4. (why - TODO). So X lies on the unique quadric. (TODO- rest...) $\square$

6.4. Show there is no curve of degree 9 and genus 11 in $\mathbb{P}^3$

Proof. First we show any such curve X must lie on a quadratic surface. Consider the sequence $0 \to \mathcal{I}_X(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_X(2) \to 0$. $dimH^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 10$, so if $dimH^0(X, \mathcal{O}_X(2)) < 10$ then $dimH^0(\mathbb{P}^3, \mathcal{I}_X(2)) \geq 1$, so X lies on a quadratic surface. $\mathcal{O}_X(2)$ corresponds to degree 2 hypersurfaces, which will intersect the curve X in $9 * 2 = 18$ points, so $deg|\mathcal{O}_X(2)| = 18$. For $\mathcal{O}_X(2)$ nonspecial, Riemann-Roch gives $dimH^0(X, \mathcal{O}_X(2)) = 18 + 1 - 11 = 9$, so X will lie on a quadratic hypersurface. For $\mathcal{O}_X(2)$ special, and effective divisor D in the linear system given by $\mathcal{O}_X(2)$ must have **Theorem** 5.4 $dim|D| \leq \frac{1}{2}degD = 9$, so again X will lie on a quadratic hypersurface.

Then suppose X lies on a nonsingular quadratic hypersurface of type (a, b). Then by 6.4.1, $d = 9 = a+b$ and $g = 11 = ab-a-b+1$. Substituting, $11 = a(9-a) - a - (9-a) + 1$, or $0 = a^2 - 9a + 19$, which has no integer solution. Thus X cannot lie on a nonsingular quadratic hypersurface.

X cannot lie in the product of two hyperplanes, since it will then either be a line and have genus 0, or it will be in a plane, and then contradicts $g = \frac{1}{2}(d-1)(d-2) = 28 \neq 11$.

The only case left is X lies on a quadratic cone, but then 6.4.1 again gives $d = 2a + 1 \Rightarrow a = 4$, and then $g = a^2 - a \Rightarrow g = 16 - 4 = 12 \neq 11$.

Thus no curve exists in $\mathbb{P}^3$ of degree 9 and genus 11. $\square$

5 Chapter V Solutions

5.1 Section 1

6 Miscellaneous

First, please do not copy and post this file elsewhere, since I will update it occasionally, and would like versions currently distributed to be up to date. If you want to link to it from your webpage that is fine.

I solved many of the problems in Hartshorne's book Algebraic Geometry while studying for an advanced topics exam. After working for several months on this book, I decided to start learning LaTeX, so I decided to TeX many of my solutions to help both tasks. Hopefully I'll go back and put lots of earlier solutions in here as I find time, especially since this format allows text searching. I have most problems from chapter 1 done (and there are also some short solutions floating around the net I found later), and I have many solutions from chapters 2 and 3. I am currently working on chapter 4 problems, and plan to do chapter 5 after that. So if I have time and inspiration I'll make these available also.

Problems marked TODO I have done, but the solution needs cleaned and typed.

Thanks to Razvan Veliche, another Purdue graduate student, who helped solve many of these problems. Also thanks to Aijneet Dhillon, another grad student, and recently graduated Jaydeep Chipalkatti (spelling?? - sorry) for their help.

Note that many of the formatting and table of contents is experimental until I get my macros designed well. I hope to get all this ironed out soon.

This file was prepared on July 28, 1999. E-mail any corrections or suggestions to clomont@math.purdue.edu.

Chris Lomont, Purdue graduate student

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