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Geometry of Yang-Mills Field

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PREFACE

These Lectures Notes are an expanded version of the Fermi which I gave at the Scuola Normale in Pisa in June 1978. They material presented in the spring of 1978 in the Loeb Lectures at and the Whittemore Lectures at Yale. In all cases I was addressin audience of mathematicians and physicists and the presentation tailored accordingly. In writing up the lectures I have tried as f sible to keep this dual audience in mind, and the early chapters cular attempt to bridge the gap between the two points of view later chapters, where the material becomes more technical, there is of falling between two stools. On the one hand the mathemati may be unintelligible to the physicist, while the presentation may, matical standards, be lacking in rigour. This is a risk I have d taken. The initiated mathematician should be able to fill in m gaps by himself or by referring to other published papers. Phys have survived the early chapters may derive some benefit by bein to new mathematical techniques, applied to problems they are fam With this aim in mind I have throughout presented the mathem terial in a somewhat unorthodox order, following a pattern wh would relate the new techniques to familiar ground for physicis

The main new results presented in the lectures, namely the co of all multi-instanton solutions of Yang-Mills fields, is the culm several years of fruitful interaction between many physicists an maticians. The major breakthrough came with the observatio R. S. Ward that the complex methods developed by R. Penr « twistor programme » were ideally suited to the study of the equations. The instanton problem was then seen [4] to be equiv problem in complex analysis and finally to one in algebraic Using the powerful methods of modern algebraic geometry and t results of G. Horrocks and W. Barth it was not long before th was finally solved [2].

The first two chapters provide an introduction to the basic a statement of the problem and an explicit description of the solu next two chapters are devoted to the Penrose theory and its application to the Yang-Mills equations. In Chapter V I present Horrocks' construction in algebraic geometry which is equivalent via the Penrose theory to the explicit instanton construction of Chapter II. Chapter VI introduces the important mathematical tool of sheaf cohomology and relates it to physically interesting equations. There are a number of digressions which may help to make the material less mysterious and more understandable. Chapter VII is an account of the theorem of Barth [8] which shows that the Horrocks construction of Chapter II yields all relevant bundles and hence that the construction of Chapter II yields all instantons. Finally in Chapter VIII we discuss some other aspects and open problems concerning the Yang-Mills equations.

Although the presentation is somewhat discursive, and includes much background material, it is also reasonably complete from the mathematical point of view. The one point where the proof is only sketched is the identification in Chapter VI of the sheaf cohomology group $H^1(P_3, E(-2))$ with the solution space of an appropriate Laplace operator. A detailed account of this can be found in various forms in [18] [29] [36]. An alternative presentation of the whole instanton theory is contained in the papers of Drinfeld and Manin [16] [17] [18] [19], and mathematicians, particularly if they are proficient in algebraic geometry, may prefer to read these.

My acquaintance with the geometry of Yang-Mills equations arose from lectures given in Oxford in Autumn 1976 by I. M. Singer, and I am very grateful to him for arousing my interest in this aspect of theoretical physics. We have collaborated since on many topics in this area. I have also, over the past few years, greatly benefited from numerous discussions with R. Penrose concerning twistor theory and complex analysis. In developing the mathematical theory of instantons I have throughout worked in close collaboration with N. J. Hitchin, and these lectures embody the results of our joint efforts. I am in addition greatly indebted to my now numerous friends in the physics community who have helped to give me some small understanding of the fascinating mathematical problems facing elementary particle physics.

Finally I should express my thanks to the Accademia Nazionale dei Lincei and to the Scuola Normale for their invitation to deliver the Fermi Lectures and for their hospitality in Pisa.

CONTENTS

CHAPTER I

The Yang-Mills Lagrangian

1. – Physics background.

The aim of quantum field theory is broadly speaking to pu mentary particles on the same footing as photons. Whereas ph pear as the quanta of classical electromagnetic theory other el particles should arise by the quantization of appropriate class theories. In recent years gauge theories have appeared the most p candidates, and the Yang-Mills equation is the generalization of M equations (in vacuo). The circle group which embodies the phase Maxwell theory is generalized to a non-abelian compact Lie grou as $SU(2)$ or $SU(3)$, the choice of group being dictated by the en observed symmetries of elementary particles. The non-abelian nat leads to non-linearity for the Yang-Mills equations. This non-lin of course the source of great mathematical difficulties and the qua of non-abelian gauge theories is still in its infancy.

One recognized way of attempting to develop the quantum t to use the Feynman functional integral approach which involves ing exp (iS) where S is the action. If we analytically continue to i time, so that Minkowski space gets replaced by Euclidean 4-sp Euclidean action is a positive multiple of i and so the integrand becomes a decaying exponential whose maximum value occurs at nimum of the Euclidean action. It is reasonable therefore to ask determination of the classical field configurations in Euclidean spa minimize the action, subject to appropriate asymptotic conditions in These classical solutions are the « instantons » of the Yang-Mills and it will be the primary purpose of these lectures to show how all instantons. For further explanations of their physical significa ticularly in relation to tunnelling, I refer to [12] or [30]. From a v eral point of view one can also say that a thorough understandin classical equations is likely to be a pre-requisite for developing tl tum theory, and one may hope that important structural features will appear at the classical level.

If one were to search ab initio for a non-linear generalization of Maxwell's equation to explain elementary particles, there are various symmetry properties one would require. These are

(i) *external symmetries* under the Lorentz and Poincaré groups and under the conformal group if one is taking the rest-mass to be zero,

(ii) *internal symmetries* under groups like $SU(2)$ or $SU(3)$ to account for the known features of elementary particles,

(iii) *covariance* or the ability to be coupled to gravitation by working on a curved space-time.

Gauge theories satisfy these basic requirements because they are geometric in character. In fact on the mathematical side gauge theory is a well established branch of differential geometry known as the theory of fibre bundles with connection. It has much in common with Riemannian geometry which provided Einstein with the basis for his theory of general relativity. As is well known Einstein spent many years on a fruitless search for a unified field theory, a search which most physicists regarded as a chimera. If the current expectations of Yang-Mills theory are eventually fulfilled, it will in some measure justify Einstein's point of view that the basic laws of physics should all be combined in geometrical form.

Gauge theory first appeared in physics in the early attempt by H. Weyl [43] to unify general relativity and electro-magnetism. Weyl had noticed the conformal invariance of Maxwell's equations and sought to exploit this fact by interpreting the Maxwell field as the distortion of relativistic length produced by moving round a closed path. Weyl's interpretation was disputed by Einstein and never generally accepted. However after the advent of quantum mechanics with its all-important complex wave-functions it became clear that phase rather than scale was the correct concept for Maxwell's equations, or in modern language that the gauge group was the circle rather than the multiplicative numbers. Unfortunately, while scale changes could be fitted into Einstein's theory by replacing the metric with a conformal structure, there was no room for phase to be incorporated into general relativity. Rather the gauge theory had to be superimposed as an additional structure on space-time and the unification sought by Weyl then disappeared.

Non-abelian gauge theories were introduced in 1954 by Yang and Mills [32] and have been increasingly studied by physicists since that time. The relation with the mathematical theory of fibre bundles was either

ignored or considered irrelevant until comparatively recently, when perturbative aspects related to instantons have come to the fore. Mathe cally this involves global questions of fibre bundle theory incorporating topology and analysis, as opposed to the purely local theory of cla differential geometry. A great deal of modern geometry of a sophisti character is involved in dealing with such global problems and the techn developed by mathematicians are unfamiliar to physicists. One purp these lectures is to try to bridge the gap between mathematician physicists by explaining the relevant techniques as simply as possibl illustrating how they apply to the determination of instantons. Th that so many new mathematical tools are naturally involved with this lem may lead to some optimism concerning the ultimate aim of devel the quantized form of gauge theories.

2. – Gauge potentials and fields.

We shall now recall the data of a classical theory as understo physicists and then reinterpret them in geometrical form.

We begin by fixing a compact Lie group G, typically $SU(2)$ or S but not excluding yet the abelian group $U(1)$. We consider its Lie al $L(G)$, which for $SU(n)$ consists of skew-hermitian $n\times n$ matrices of zero. A gauge potential is then a set of functions $A_\mu(x)$ taking val $L(G)$, where $x=(x_1 \dots x_4)$ is a point of Minkowski or Euclidean spac $\mu=1, \dots, 4$ is a spatial index. Associated with this potential we als sider the operator

$$\nabla_\mu = \partial_\mu + A_\mu \tag{2.1}$$

where $\partial_\mu = \partial/\partial x_\mu$. This operator acts on a vector function $(f_1(x), \dots,$ whenever we are given an m-dimensional representation of G: for ex when $G=SU(n)$ we can take $m=n$ using the standard represent

Computing the commutator of ∇_μ and ∇_ν we get the *gauge fiel* given by

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

where the commutator $[A_\mu, A_\nu]$ is taken in the Lie algebra of G. T portant point to note is that for non-abelian G this commutator do vanish and so F is a non-linear function of A. For $G=U(1)$ howev term drops out and we get the usual linear relation between the fie vector potential that characterizes Maxwell theory.

The usual non-uniqueness of the potential has its counterpart in the general case in the form of *gauge transformations*. By definition a gauge transformation is a function $g(x)$ taking values in G and transforming the potential A_μ by the formula

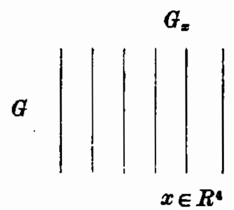
$$A_\mu \to g^{-1} A_\mu g + g^{-1} \partial_\mu g$$

which corresponds to sending ∇_μ into $g^{-1}\nabla_\mu$ (here we consider G as a group of matrices so that $\partial_\mu g$ is simply the differentiated matrix). The gauge field $F_{\mu\nu}$ then transforms by

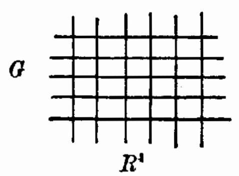
$$F_{\mu\nu} \to g^{-1} F_{\mu\nu} g \, .$$

It is important to observe that the A_μ transform inhomogeneously whereas the $F_{\mu\nu}$ transform homogeneously. In other words $F_{\mu\nu}$ is a vectorial (or tensorial) object whereas A_μ is an affine object (with no preferred zero).

Geometrically or mechanically we can interpret this data as follows. Imagine a structured particle, that is a particle which has a location at a point x of R^4 and an internal structure, or set of states, labelled by elements g of G. We then consider the total space P of all states of such a particle. In general we conceive of the internal spaces G_x and G_y for $x \neq y$ as not being identified and so we draw the picture of P as a collection of « fibres »



In the absence of any external field however we consider that all G_x can be identified to each other so that in addition to the vertical lines or fibres we can also draw horizontal lines (called sections) making the usual Cartesian type of grid



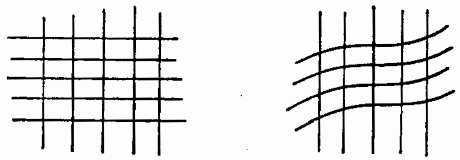
Now we imagine an external field imposed which has the effect of distor the relative alignment of the fibres so that no coherent identificatio possible between the G_x at different points. However we assume tha and G_y can still be identified if we choose a definite path in R^4 from x t In more physical terms we imagine the particle moving from x to y and rying its internal space with it. In Minkowski space such a motion w take place along the world line of the particle. This identification of fi along paths is called « parallel transport ». If we now imagine two diffe path joining x to y then there is no reason for the two different par transports to agree and they are assumed to differ by multiplication a group element, which could be viewed as a generalized « phase sh This phase shift is interpreted as produced by the external field. In geo trical terms it is viewed as the total « curvature » or distortion of the bundle over the region enclosed by the two paths.

If we now infinitesimalize this picture in Newtonian style we get infinitesimal parallel transport at a point x in a given direction. will be an infinitesimal shift A of the fibre G_x into the nearby fibre, an called a *connection*. The infinitesimal *curvature* F depends on two direct at x and takes values in the Lie algebra of G_x, *i.e.* it is an infinitesimal « p shift ». As usual the infinitesimal picture, that is the connection, ca integrated up to give the global picture of parallel transport along cur the two points of view are mathematically equivalent.

If we now compare this picture with the situation where we had no and all fibres G_x were coherently identified we can view parallel trans as a change of phase in a fixed copy of G, and the connection as an elen $A_\mu(x)$ of the Lie algebra depending on the point x and the μ-th direct Thus we recover the gauge potential of the physicists' language. Simil the curvature F becomes the gauge field $F_{\mu\nu}(x)$, taking values in the f Lie algebra of G. Thus the curvature F can be thought of as the distor produced by an external field, or it can be *identified* with the field wher think of a field of force as measured by its local effects. This identifica of fields with geometrical distortion is of course at the heart of Einst theory of gravitation. The difference here is that the distortion is not ta place in the geometry of space-time but in the geometry of some fictit state-space of internal structure super-imposed on space-time. This d rence makes the relevant geometry less obvious and historically, bot physics and in mathematics, the geometry of fibre-bundles came later t the geometry of space. It is significant however that both mathematic and physicists, each for their own reasons, were led to study these obj which in fact turn up naturally in a great variety of contexts.

Despite its later historical appearance the geometry of fibre bundle the type we have been describing is much simpler technically than the Riemann-Einstein geometry of space. This is because the relevant group of our theory was taken as a finite-dimensional group G whereas in Riemannian geometry we have to deal with the group of all coordinate transformations. To clarify this point we shall now return to our fibre bundles and re-examine their relation to gauge theory.

In order to describe our geometrical connection in algebraic terms we compare our parallel transport with the situation in the absence of a field. In this case we used a coherent identification of all the fibres G_x. Now it is important to emphasize that this coherence represents the absence of a field but the particular choice of coherent identification is at our disposal. A particular choice is called picking a gauge and a change from one choice to another is a gauge transformation. Pictorially we imagine two different

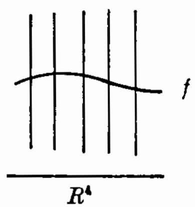


sets of horizontals for our fibre bundle and the change from one to the other is described by a function $g(x)$, taking values in G. No particular choice is regarded as being preferred (despite the appearance of the picture!). Once a gauge has been picked the connection and curvature can be written down in coordinate form. The group of gauge transformations plays a role analogous to that of coordinate transformations in Riemannian geometry. Since it is basically a simpler group the geometry of fibre-bundles is an easier theory: it is, in a definite sense, « less non-linear ».

It should be emphasized that a connection is a definite geometric object and is more primitive than the curvature. As a consequence one should consider the gauge potential as more primitive than the gauge field. This is borne out physically even in electro-magnetism by an experiment which shows that the field may be identically zero but physical effects are still detected due to the fact that parallel transport need not be trivial if the region of space is not simply-connected. The vanishing of curvature only gives information about parallel transport round very small closed paths. In physical terminology parallel transport in general is described by talking about non-integrable phase factors. Non-integrability locally refers to a non-vanishing field, whereas large scale non-integrability is topological in character (going round a wire for example) and may arise even for zero fields (outside the wire). Classically potentials were introduced as a mathematical

device to simplify the field equations and the ambiguity (or gauge free
in choice of potential was taken as an indication that the potential h
genuine physical meaning. The geometrical point of view shows tha
is too narrow an interpretation. The connection is a geometric objec
so the potential should be regarded as physical. The unphysical thi
the choice of gauge in which one chooses to describe the potential,
responding to the fact that our geometrical fibre bundle where the
nection sits has no natural horizontal sections. These remarks abou
predominant role of the potential will acquire more substance when we di
the field equations in the general non-abelian case.

So far we have talked only about fibre bundles in which the fibr
the group G. These are called principal fibre bundles by differential g
eters. However for applications one is usually interested in assoc
bundles in which the fibre is a vector space C^n corresponding to a repres
tion of G. Again the typical case is to take $G = U(n)$. The geometric pi
is essentially similar in that we consider a space (the vector bundl
fibred over R^4 so that the fibre E_x is thought of as a vector space depe
smoothly on x. Parallel transport from x to y is regarded as a unitary t
formation from E_x to E_y. Thus parallel transport in the principal b
gives rise to parallel transport in the vector bundle and the same a
to connection and curvature. In particular a section of the vector bu
namely a function $f(x)$ defined on R^4 and taking values in the variable v
space E_x is thought of in terms of its graph. A connection enables us to



this graph infinitesimally in a given direction of R^4. This shift is pre
the covariant derivative $\nabla_\mu f$. This is a geometric notion independe
any choice of gauge. Once we choose a gauge we can describe f algebrai
by a n-vector $(f_1(x), \dots, f_n(x))$ of ordinary functions and the covariant
vative is then given explicitly by formula (2.1). The curvature $F_{\mu\nu}$ de
as the commutator $[\nabla_\mu, \nabla_\nu]$ is again seen to be geometric in chara
appearing now as an (algebraic) operator on sections of the v
bundle.

3. – The field equations.

We come now to the field equations for a gauge theory which will generalize Maxwell's equations. Written in terms of the covariant derivative ∇_μ, and in units where the velocity of light is 1, the equations can be written in terms of commutators

$$[\nabla_\mu, [\nabla_\nu, \nabla_\sigma]] + [\nabla_\nu, [\nabla_\sigma, \nabla_\mu]] + [\nabla_\sigma, [\nabla_\mu, \nabla_\nu]] = 0 \tag{3.1}$$

$$[\nabla_\mu, [\nabla_\mu, \nabla_\nu]] = 0 \,. \tag{3.2}$$

In (3.2) we sum over μ and in Minkowski space the term corresponding to the time component has a minus sign (whereas the Euclidean analogue has all positive signs).

These two equations involving the potential have a rather different character, in that the first is an identity (the Bianchi identity of differential geometry) and only the second, the *Yang-Mills equation*, imposes a condition on the potential. For $G = U(1)$ these equations written in terms of the field $F_{\mu\nu}$ are Maxwell's equations in vacuo. The first equation is then just the integrability condition on $F_{\mu\nu}$ which asserts that (at least locally) we can introduce a potential A_μ so that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

For non-abelian G it is not possible to write these equations in terms of $F_{\mu\nu}$ alone because the covariant derivatives ∇_μ explicitly involve the potential A_μ. This emphasizes once more the primary role of the potential as opposed to the field.

The Yang-Mills equation (3.2) is derived from a Lagrangian $\mathfrak{L}$ by integrating over R^4 a Lagrange density which is an invariantly defined quadratic expression in the curvature. For $G = U(n)$ or $SU(n)$ one puts (up to constant factor)

$$\mathfrak{L} = -\tfrac{1}{2}\int_{R^4} \mathrm{Trace}\,(F_{\mu\nu}F^{\mu\nu})\, dx_1\, dx_2\, dx_3\, dx_4 \,, \tag{3.3}$$

where $F^{\mu\nu}$ is obtained in the usual way from $F_{\mu\nu}$, raising indices by the standard metric tensor of Minkowski or Euclidean space and we sum over all μ, ν. Equations (3.2) are the corresponding Euler Lagrange equations. The minus sign was inserted in (3.3) so that in the Euclidean case we get

a positive Lagrangian, the point being that $-\mathrm{Trace}\,(AB)$ is positive defi
on the Lie algebra of $U(n)$. For other Lie groups one can either use an eml
ding in $U(n)$ or more intrinsically one replaces Trace (AB) by the Kil
form which is the standard invariant bilinear form on $L(G)$: the two meth
give the same answer up to a positive scalar multiple.

In the Euclidean case the Lagrangian can be viewed as the nat
L^2-norm of the curvature, that is the integral over R^4 of the sums of
squares of the absolute values of all its components in a standard ort
normal base. More invariantly one can rewrite this as follows. First we re
that a skew tensor $\alpha_{\mu\nu}$ corresponds to an exterior differential 2-form

$$\alpha = \tfrac{1}{2}\sum_{\nu,\mu} \alpha_{\mu\nu}\, dx^\mu \wedge dx^\nu = \sum_{\mu<\nu} \alpha_{\mu\nu}\, dx^\mu \wedge dx^\nu \,.$$

The dual 2-form $*\alpha$ defined relative say to the standard Euclidean met
is given by replacing α_{12} by α_{34} etc. with an appropriate minus sign if
indices involve an odd permutation. It satisfies $*^2 = 1$. The natu
L^2-inner product of 2-forms is then defined by

$$\langle \alpha, \beta \rangle = \int_{R^4} \alpha \wedge *\beta \,,$$

where $\wedge$ denotes the exterior multiplication and gives here an exteri
4-form, *i.e.* a volume form which can therefore be integrated. Passing n
to the curvature F which is a 2-form with values in the Lie algebra $L(G$
we define $*F$ in the same way, switching spatial indices and leaving untouch
the Lie algebra variables. Then we put

$$\|F\|^2 = \langle F, F \rangle = -\int_{R^4} \mathrm{Trace}\,(F \wedge *F) \tag{3.4}$$

and this is the invariant way of writing the Lagrangian.

Equation (3.1) asserts that the covariant derivative of F, skew-sym
metrized, is zero or symbolically

$$\nabla \wedge F = 0 \,. \tag{3.5}$$

Using the duality operator $*$ we see that (3.2) can then be written

$$\nabla \wedge *F = 0 \,. \tag{3.6}$$

Th...

clearly its invariance and covariance properties. First of all the equations clearly make sense on a curved (Riemannian) 4-space, since the $*$-operator uses only the infinitesimal duality. Secondly the $*$-operator on 2-forms in 4-space is conformally invariant in the sense that two metrics ds^2 and $\varrho(x)\,ds^2$ give the same $*$. Thus the Yang-Mills equation (and the Yang-Mills Lagrangian) depend only on the conformal structure of 4-space. This important property of Maxwell's theory is therefore preserved in the non-abelian case.

The apparent symmetry or duality between (3.5) and (3.6) is delusory as we have explained, although in Maxwell theory it reflects the duality between electricity and magnetism and attempts have been made to understand the non-abelian analogue. This is a deep question and the proper understanding of this duality is likely to be found only at the quantum level [22] [40]. However at the classical level we note an elementary consequence of (3.5) and (3.6), namely that (3.6) follows from the identity (3.5) if the field F satisfies one of the equations

(3.7) $$*F = F \qquad \text{(self-duality)}$$

(3.8) $$*F = -F \qquad \text{(anti-self-duality)}\,.$$

Thus we have here *first-order* non-linear equations for the potential which imply the second-order Yang-Mills equations. These equations have a particularly simple significance in the Euclidean case as we shall see in the next section. Note that the definition of $*$ involves an orientation of R^4 (an ordering of the coordinates $x_1, \ldots, x_4$) and that (3.7) and (3.8) switch when we reverse the orientation. Thus there is no essential mathematical difference between the two cases.

4. – Asymptotic conditions and topology.

We now restrict ourselves to the Euclidean 4-space so that the Yang-Mills Lagrangian $\mathfrak{L}$ is positive. It is natural to consider potentials for which the action $\mathfrak{L}$ is finite, so that the integral over R^4 converges. To achieve this we assume that the field F decays sufficiently fast as we go to infinity. If we work in a given gauge this simply means that $F_{\mu\nu}(x) \to 0$ sufficiently fast as $|x| \to \infty$. At first sight this might seem to require the gauge potential $A_\mu(x)$ to have similar decay together with its first derivatives. However, because of gauge freedom, all that is necessary is that, for large $|x|$, we can find a gauge transformation $g(x)$ so that the potential in the new

gauge should decay. This means that

(4.1) $$A_\mu(x) \sim g^{-1}(x)\,\partial_\mu g(x) \qquad \text{as } |x| \to \infty$$

where $\sim$ implies asymptotic behaviour including first derivatives. T
portant point is that the gauge transformation $g(x)$ need only be defi
large $|x|$. In fact it may be impossible to extend the definition of $g($
tinuously to the whole 4-space. To see this consider the restriction
to a sphere $|x| = R$ of large radius, and take for example $G =$
Then g gives a continuous map

$$g\colon S^3_R \to SU(2)$$

and both sides are topologically 3-spheres. Such a map has a well-
integer invariant, its degree k, which counts (with appropriate multi
and signs) the number of points $x \in S^3_R$ which map to a given genera
in $SU(2)$. The function g can be extended continuously to $|x| < R$ if a
if $k = 0$. The identity map (thinking of both spaces as the stand
has $k = +1$, and $k = -1$ corresponds to an orientation reversal.

An analogous and more easily visualized situation occurs for G
and R^4 replaced by R^2 in which case g becomes a map of the circle t
and the degree k is the « winding number ». Note however that if w
on R^4 with $G = U(1)$ there is no topological invariant since eve
tinuous map of S^3 to the circle can be deformed to a constant map.

For any simple non-abelian compact Lie group G a correspondin
holds, namely that continuous maps of S^3 into G have an integer top
invariant which classifies the map up to deformation. This intege
from the fact that every such G contains copies of $SU(2)$ (or SO
subgroups.

Thus in non-abelian gauge theories on R^4, potentials which are
totically flat, *i.e.* have fields asymptotic to zero, fall into distinct
indexed by the corresponding integer k. This is frequently referred
topological quantum number even though at this stage we are only
with classical fields.

In dealing with asymptotic properties various technical analytic
stions arise concerning the precise rate of decay. There is one natu
convenient definition of decay to take arising from conformal inva
We recall that stereographic projection of a sphere onto flat space is
a conformal map, relating the standard curved metric of the sphere
flat Euclidean metric. In particular the sphere S^4 (unit sphere in R
mapped conformally onto R^4.

Alternatively we can say that S^4 is the conformal compactification of R^4 obtained by adding a point at ∞. A potential on R^4 is then said to decay at ∞ if it extends to a potential on S^4. If the integer k is non-zero this means that we cannot describe our potential using a single gauge, we need one gauge in the finite region $|x| < R$ and another gauge near ∞ i.e. for $|x| > R$, the two gauges being related on $|x| = R$ by the gauge transformation $g(x)$ of degree k. This means that our fibre bundle P over S^4 is no longer the topological product $S^4 \times G$. In fact the topological theory of fibre bundles over spaces like S^4 which are not contractible tells us that in this case they are precisely classified by the same integer k. Thus the integer k which appeared in the asymptotic description on R^4 is now directly coded into the topology of the space P. For example if $G = SU(2)$ and $k = 1$, P turns out to be topologically the sphere S^7 which is quite different from the product $S^4 \times S^3$: this will be explained in more detail later. The potential or connection in our fibre bundle now has a well-defined curvature F on the whole of S^4 and the particular role of the base point $\infty \in S^4$ can now be ignored. Note that $F \to 0$ at ∞ on R^4 but F need not be zero on S^4 at the point ∞ because when viewed as a differential form one uses different coordinates on R^4 and S^4. Clearly the Lagrangian $\mathfrak{L} = \|F\|^2$ computed relative to the curved metric of S^4 is necessarily finite because S^4 is compact (we always assume enough local differentiability so that F is always continuous at least). Moreover because of the conformal invariance of the Yang-Mills functional $\mathfrak{L}$ takes the same value whether computed on S^4 or R^4.

Now on closed manifolds like S^4 there are well known theorems of global differential geometry which relate topological invariants to integral expressions in the curvature. One might expect such results because if the curvature is everywhere zero the connection is flat and (on a simply-connected space) one obtains a global gauge implying $k = 0$. In dimension 2 the classical theorem of Gauss expressing the Euler characteristic as the integral of the scalar curvature is the prototype of higher-dimensional generalizations. In our case the formula takes the form

$$8\pi^2 k = -\int_{S^4} \operatorname{Trace}(F \wedge F)\,, \tag{4.2}$$

for the group $SU(2)$. For other groups the same result holds but with a different normalization, the integer k being replaced by a suitable multiple: for precise details the reader may refer to [3]. We now have a topological constraint on F and this should be fed into the formula (3.4) for the Lagrangian. To do this it is convenient to decompose F under the action of * into

its self-dual part F^+ and its anti-self-dual part F^-:

$$F = F^+ \oplus F^-$$

so that $*F^+ = F^+$ and $*F^- = -F^-$ (recall $*^2 = 1$ for the positive case). Then (3.4) and (4.2) can be written

$$\mathfrak{L} = \|F^+\|^2 + \|F^-\|^2 \tag{3.4$'$}$$

$$8\pi^2 k = \|F^+\|^2 - \|F^-\|^2$$

from which we deduce that $\mathfrak{L} \geq 8\pi^2|k|$ and equality holds if and $*F = (\operatorname{sign} k)F$. Thus the special solutions (3.7) and (3.8) of the Mills equations correspond to the absolute minimum of the Lagr (assuming the value $8\pi^2|k|$) is attained. This argument is due to I et al. [10] who also showed that for $k = \pm 1$ the minimum is at and the term instanton has been coined for such solutions of the Yan equation. Later other solutions were discovered for all k [14] [31] an were called multi-instantons.

The general problem to which we shall address ourselves is the mination, in as explicit a fashion as possible, of all multi-instanto only for $SU(2)$ but for all compact classical groups. As we shall se problem admits of a surprisingly simple and complete answer, but the require a great deal of sophisticated mathematical machinery.

At this point we shall merely note that any gauge transform of a instanton is again a multi-instanton, and such solutions will be re as equivalent. In geometric terms this means that two fibre bundles with connections (satisfying $*F = \pm F$) which are isomorphic will b tified: they cannot be distinguished geometrically.

In the next chapter we shall describe explicitly how to write do most general multi-instanton for $SU(2)$, and we shall indicate how t to be generalized to other groups.

CHAPTER II

Description of Instantons

1. – Quaternions.

In R^2 it is well known that many formulae are simpler when written in terms of complex numbers. It was the discovery of Hamilton that for R^4 one can introduce a non-commutative extension of the complex numbers called quaternions, and it was Hamilton's hope that quaternions would turn out to be the natural tool for an algebraic description of the physical world. Although this was an exaggerated hope there is some merit in Hamilton's point of view as we shall see shortly. For the present however, we simply use quaternions as a convenient algebraic formalism that simplifies notation. This is particularly relevant to the group $SU(2)$ which as we shall see can be identified with the group of quaternions of unit norm, in the same way that $U(1)$ is the complex numbers of unit norm.

We begin by briefly recalling the definition and elementary properties of quaternions. Just as the complex numbers C are formed from the real numbers R by adjoining a symbol i with $i^2 = -1$, so the quaternions H (in honour of Hamilton) are formed from R by adjoining three symbols i, j, k satisfying the identities:

$$(1.1) \qquad \begin{aligned} &i^2 = j^2 = k^2 = -1 \\ &ij = -ji = k\,, \quad jk = -kj = i\,, \quad ki = -ik = j\,. \end{aligned}$$

Thus a general quaternion x is of the form

$$(1.2) \qquad x = x_1 + x_2 i + x_3 j + x_4 k$$

where x_1, x_2, x_3, x_4 are real numbers. The conjugate quaternion $\bar{x}$ is defined by

$$\bar{x} = x_1 - x_2 i - x_3 j - x_4 k$$

and conjugation is an anti-involution, i.e. $\overline{(xy)} = \bar{y}\bar{x}$. In virtue identity (1.1), one finds

$$x\bar{x} = \bar{x}x = \sum_{\mu=1}^{4} x_\mu^2$$

This quantity is denoted $|x|^2$ and is zero only for $x = 0$. If $x \neq 0$ a unique inverse x^{-1} given by

$$x^{-1} = \bar{x}/|x|^2\,.$$

The quaternions x with norm 1, *i.e.* $|x| = 1$, form therefore a multip group which is geometrically the 3-sphere

$$\sum_{\mu=1}^{4} x_\mu^2 = 1\,.$$

In analogy again with the complex numbers we refer to the compo in (1.2) as the real part of x and the remainder $x - x_1$ as the imagina

If we identify i with the usual complex number we can reg complex numbers C as contained in H (taking $x_3 = x_4 = 0$). Moreove quaternion x as in (1.2) has a unique expression.

$$x = z_1 + z_2 j\,, \quad \text{where } z_1 = x_1 + x_2 i \text{ and } z_2 = x_3 + x_4 i\,.$$

This identifies H with C^2. Now consider the quaternion multipl $x \to xg$ where $g = g_1 + g_2 j$, with $g_1, g_2 \in C$. Computing we find

$$xg = (z_1 + z_2 j)(g_1 + g_2 j) = z_1 g_1 - z_2 \bar{g}_2 + (z_1 g_2 + z_2 \bar{g}_1) j$$

so that the vector (z_1, z_2) is multiplied on the right by the 2×2 c matrix

$$\begin{pmatrix} g_1 & g_2 \\ -\bar{g}_2 & \bar{g}_1 \end{pmatrix}.$$

Thus, if we wish, we can identify the algebra of quaternions as a sub- of the 2×2 complex matrices in which i, j, k are the matrices

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

In particular the group $Sp(1)$ of quaternions of unit norm gets id with $SU(2)$ and its Lie algebra can be viewed as the pure imaginary quaternions with natural basis i, j, k.

We now identify R^4 with H via (1.2) and an $SU(2)$-potential will be given by functions $A_\mu(x)$ whose values are imaginary quaternions. It will be convenient if we go further and write

$$A(x) = \sum_{\mu=1}^{4} A_\mu(x)\, dx^\mu$$

so that $A(x)$ is a differential form with values in $\operatorname{Im}(H)$. Finally we shall consider the quaternion differential

$$dx = dx^1 + dx^2 i + dx^3 j + dx^4 k$$

and its conjugate

$$d\bar{x} = dx^1 - dx^2 i - dx^3 j - dx^4 k$$

just as in complex variable theory one uses $dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$. If $f(x)$ is any function of the quaternion variable x with quaternion values the expression

$$A(x) = \operatorname{Im}\{f(x)\, dx\} = \tfrac{1}{2}\{f(x)\, dx - d\bar{x}\, \overline{f(x)}\} \tag{1.3}$$

will represent an $SU(2)$-potential. Here $f(x)dx$ is computed formally, written as $\sum a_\mu dx^\mu$ with $a_\mu \in H$ and then $A_\mu = \operatorname{Im}(a_\mu)$. Note that, before taking the imaginary part we have a potential for the group H^* of all non-zero quaternions which is $SU(2)$ times a scale factor.

We shall also write the curvature F as an exterior 2-form

$$F = \sum_{\mu<\nu} F_{\mu\nu}\, dx^\mu \wedge dx^\nu = \tfrac{1}{2} \sum_{\mu,\nu} F_{\mu\nu}\, dx^\mu \wedge dx^\nu .$$

Then F can be computed from A by

$$F = dA + A \wedge A$$

where

$$dA = \sum_\mu dA_\mu \wedge dx^\mu = \tfrac{1}{2} \sum_{\mu,\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \wedge dx^\nu$$

and

$$A \wedge A = \sum_{\mu,\nu} A_\mu A_\nu\, dx^\mu \wedge dx^\nu = \tfrac{1}{2}[A_\mu, A_\nu]\, dx^\mu \wedge dx^\nu .$$

When A is written in the quaternionic form (1.3) we get a similar

$$F = \operatorname{Im}\{df \wedge dx + f\, dx \wedge f\, dx\} . \tag{1.4}$$

Taking imaginary parts commutes with formation of curvature be our remark above concerning the larger group H^*.

The use of the quaternion differentials dx and $d\bar{x}$ is also conve connection with the study of self-duality. To see this let us comput We get

$$dx \wedge d\bar{x} = (dx^1 + dx^2 i + dx^3 j + dx^4 k) \wedge (dx^1 - dx^2 i - dx^3 j - dx^4 k) =$$

$$= -2\{(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) i + (dx^1 \wedge dx^3 + dx^4 \wedge dx^2) j +$$

$$+ (dx^1 \wedge dx^4 + dx^2 /$$

The coefficients of i, j, k in this expression are precisely a basis for dual 2-forms, *i.e.* 2-forms ω with $^*\omega = \omega$. Hence $dx \wedge d\bar{x}$ is a 2-for values in the Lie algebra of $SU(2)$, which is self-dual. A similar com shows that $d\bar{x} \wedge dx$ is anti-self-dual.

2. – The basic instanton.

Using the quaternionic notation of the preceding section we sh exhibit the basic instanton with $k = +1$ (and the anti-instant $k = -1$).

Consider the $SU(2)$-potential A defined by

$$A(x) = \operatorname{Im}\left\{\frac{\bar{x}\, dx}{1 + |x|^2}\right\} = \frac{1}{2}\left\{\frac{\bar{x}\, dx - d\bar{x}\, x}{1 + |x|^2}\right\}. \tag{2.1}$$

The explicit components $A_\mu(x)$ can of course be read off from this formula, for example

$$A_1(x) = \frac{-x_2 i - x_3 j - x_4 k}{1 + |x|^2}, \qquad A_2(x) = \frac{x_1 i - x_4 j + x_3 k}{1 + |x|^2} .$$

Computing the curvature F of this potential as in (1.4) we get

$$F = \operatorname{Im}\left\{\frac{d\bar{x} \wedge dx}{1 + |x|^2} + \bar{x}\, d(1 + |x|^2)^{-1} \wedge dx + \frac{\bar{x}\, dx \wedge \bar{x}\, dx}{(1 + |x|^2)^2}\right\}. \tag{2.2}$$

Writing $|x|^2 = x\bar{x}$ the middle term in this expression gives

$$-\frac{\bar{x}\,dx \wedge \bar{x}\,dx}{(1+|x|^2)^2} - \frac{\bar{x}x\,d\bar{x} \wedge dx}{(1+|x|^2)^2}.$$

Substituting this in (2.2) and simplifying we get the purely imaginary expression:

(2.3) $$F = \frac{d\bar{x} \wedge dx}{(1+|x|^2)^2}$$

so that, as explained in § 1, F is anti-self-dual, i.e. $*F = -F$. As $|x| \to \infty$ we see from (2.1) that

(2.4) $$A(x) \sim \operatorname{Im}(x^{-1}dx) = \varphi(x)^{-1}d\varphi(x)$$

where $\varphi(x) = x/|x|$. This shows that A is asymptotically the gauge transform of 0 by the gauge transformation $g(x) = \varphi(x)$, or equivalently that if we apply the inverse gauge transformation $\varphi(x)^{-1}$ to A we get 0 asymptotically. On the unit sphere $|x| = 1$ in quaternion space we have $\varphi(x)^{-1} = \bar{x}$ and the map $x \to \bar{x}$ of S^3 to itself has degree -1. Thus (2.1) describes an anti-instanton.

Clearly if we replace x by $\bar{x}$ throughout we will obtain an instanton with potential and field given by

(2.5) $$A = \operatorname{Im}\left\{\frac{x\,d\bar{x}}{1+|x|^2}\right\}, \qquad F = \frac{dx \wedge d\bar{x}}{(1+|x|^2)^2}.$$

These formulae are of course just those of Belavin et al. [10] written in quaternionic notation.

To examine more carefully the behaviour of the anti-instanton (2.1) as $|x| \to \infty$ we change gauge by $\varphi(x)^{-1}$, corresponding to (2.4), and introduce the quaternion coordinate $y = x^{-1}$ around the point at ∞, regarded now as a point of S^4. Taking the imaginary part of the identity

$$x\left(\frac{\bar{x}\,dx}{1+|x|^2}\right)x^{-1} + x\,dx^{-1} = \frac{\bar{y}\,dy}{1+|y|^2}$$

shows that the anti-instanton extends to S^4 and has precisely the same form at ∞ as it has near 0. Similar calculations hold naturally for the instanton (2.5).

If we simply put $y = x^{-1}$ in (2.1) we get

(2.6) $$A(y) = -\operatorname{Im}\left\{\frac{dy\,y^{-1}}{1+|y|^2}\right\}$$

and this describes the anti-instanton in the « singular » or « asympt gauge, namely the gauge in which $A \to 0$ as $|y| \to \infty$, but which is b behaved at $y = 0$, where $A(y)$ is singular. As we have seen this singul can be removed by the appropriate gauge transformation, but the gular form (2.6) is useful in practice.

It is perhaps worth pointing out that all the above formulae hol complex numbers instead of quaternions, in which case we obtain gauge potentials and fields on R^2 or S^2. Self-duality no longer makes in dimension two but the 2-form F given by (2.3) is a multiple of the inva spherical area. In other words F is invariant under $SU(2)$ acting by tional linear transformations $x \to (ax+b)(cx+d)$ of the complex variab

Since the equations $*F = \pm F$ are conformally invariant it fo that any conformal transformation of S^4 into itself will convert the instanton into some other instanton. We recall now that, just as the p (*i.e.* orientation preserving) conformal group of S^2 is $SL(2, C)/\{\pm 1\}$ a via fractional linear transformations of a complex variable, so the p conformal group of S^4 is $SL(2, H)/\{\pm 1\}$ acting similarly on a quatern variable. Thus the transformations $x \to axb$ with x, a, b all quaternions $a \neq 0$, $b \neq 0$ generate the rotation group $SO(4)$ together with scale ch $x \to x + c$ gives translations while $x \to 1/x = \bar{x}/|x|^2$ gives a proper i sion (i.e. inversion together with a compensating reflection $x \to \bar{x}$ to re orientation). Because of the non-commutativity of the quaternions proper conformal transformation of S^4 can be written either using multiplication or using right multiplication, i.e.

$$x \to (ax+b)(cx+d)^{-1}$$

or

$$x \to (x\gamma+\delta)^{-1}(x\alpha+\beta).$$

Taking quaternionic conjugates interchanges these two ways of represe the conformal transformations. More precisely let S, T denote the transformations above with $(\alpha, \beta, \gamma, \delta) = (\bar{a}, \bar{b}, \bar{c}, \bar{d})$ and let C denote jugation $x \to \bar{x}$, then $T = CSC$. Note that C, being a reflection, is a proper conformal transformation.

We return now to the basic anti-instanton (2.1) and we apply conf transformations to it. Recall that, up to a gauge transformation, (2 preserved by the inversion $x \to x^{-1}$. It is evidently unchanged by $x \to ax$ with $|a| = 1$ and $x \to xa$ produces only a constant gauge transformation. Thus it is essentially invariant by $SO(4)$. In fact it is invariant up to gauge transformations by the larger group $SO(5)$ which may be viewed here as $Sp(2)/\{\pm 1\}$ where $Sp(2) \subset SL(2, H)$ is the compact subgroup leaving norms fixed. This verification is best left till later when the proper geometric interpretation of (2.1) will make this invariance evident. To get new anti-instantons therefore we should use elements representing $SL(2, H)$ modulo $Sp(2)$. Such elements are naturally given by the transformations

$$x \to \mu(x - b) \tag{2.7}$$

where μ is a positive real scalar and b is a quaternion.

These parameters can be regarded as parametrizing $SL(2, H)/Sp(2)$, the space of quaternion norms on H^2 with volume 1, by associating to (μ, b) the positive self-adjoint matrix

$$\begin{pmatrix} \mu & b \\ b^* & \nu \end{pmatrix}$$

with $\mu\nu - |b|^2 = 1$.

As μ, b vary the transformation (2.7) applied to (2.1) generates a 5-parameter family of anti-instantons with « centre » b and « scale » μ. From (2.3) we see that the field density is a maximum at the centre and its strength there is μ^2. This shows that no two members of our family can be gauge equivalent. The more difficult result, which will emerge much later, is that every anti-instanton (i.e. $k = -1$) is gauge equivalent to one of our family.

It will be convenient to apply inversion to (2.7) (and a sign change) to (2.7) to get

$$x \to \lambda(b - x)^{-1} . \tag{2.8}$$

This transformation applied to (2.3) gives us the general anti-instanton in an asymptotic gauge as explained before. We can also apply it to (2.5) to generate the family of all instantons in an asymptotic gauge.

We now come to the more difficult question of constructing multi-instantons for larger values of k. For this purpose we introduce the space H^k consisting of column vectors u with quaternion components u_α ($\alpha = 1, \ldots, k$), and we define an $SU(2)$-potential on the space $H^k = R^{4k}$ by a formula quite similar to (2.1), namely

$$A(u) = \text{Im}\left\{\frac{u^* du}{1 + |u|^2}\right\} \tag{2.9}$$

where u^* stands for the transposed conjugate of the column vector u

and $u^* du$ stands for the matrix product, so that

$$u^* du = \sum_{\alpha=1}^{k} \bar{u}_\alpha du_\alpha$$

and $|u|^2 = u^* u = \sum_\alpha |u_\alpha|^2$ is the Euclidean norm.

Note that (2.9) restricts to (2.1) on each coordinate axis (all ı except $\alpha = \beta$) and is unchanged by the group Sp (k) acting on H^k. it restricts to (2.1) on any one-dimensional H-subspace of H^k. It has fore a high degree of symmetry and we shall in due course explain its etrical significance. For the present we regard (2.9) as simply an au formula used to construct potentials on $H = R^4$ by using suitable fun $u = f(x)$, i.e. maps $f : H \to H^k$. Given any such f we substitute in (2. obtain the potential

$$A_f(x) = \text{Im}\left\{\frac{f^*(x)\, df(x)}{1 + |f(x)|^2}\right\} = \text{Im}\left\{\sum_\alpha \frac{\bar{f}_\alpha(x)\, df_\alpha(x)}{1 + |f(x)|^2}\right\} . \tag{2.10}$$

For our function $f(x)$ we now take the matrix analogue of (2.8), com with a conjugation (to give instantons rather than anti-instantons), n

$$u(x) = [\lambda(B - x)^{-1}]^* . \tag{2.11}$$

Here B is a symmetric $k \times k$ matrix of quaternions, λ is a row $(\lambda_1, \ldots, \lambda_k)$ of quaternions and x stands for the scalar quaternion xI wl is the unit $k \times k$ matrix. For $k = 1$ the parameters were arbitrary e that λ had to be invertible. In the general case however the parameter will have to satisfy algebraic constraints as follows:

(I) $B^*B + \lambda^*\lambda$ is a *real* $k \times k$ matrix

(II) For every $x \in H$ the equations

$$(B - x)\xi = 0 , \quad \lambda\xi = 0 \text{ with } \xi \in H^k \text{ imply } \xi = 0 .$$

Condition (I) asserts that the coefficients of i, j, k in the $k \times k$ quate matrix $B^*B + \lambda^*\lambda$ all vanish. This gives a system of quadratic rel on the coefficients of B and λ.

Condition (II) is a non-degeneracy or open condition which can al formulated as saying that the $(k + 1) \times k$ matrix $\begin{pmatrix} \lambda \\ B - x \end{pmatrix}$ has ma rank k for all $x \in H$. Condition (I) is the crucial algebraic one whicl ensure that the potential $A_{\lambda,B}(x)$ defined by substituting (2.11) in (2.10) is self-dual. Condition (II) will ensure the solution is non-degenerate and in particular that the points x for which $(B-x)$ is singular give singularities of the potential which can be removed by a gauge transformation.

In principle, given gauge potentials $A_{\lambda,B}(x)$ defined as above it is a direct matter of computation to verify that the resulting field F satisfies $*F=-F$. However the computations can be carried out more elegantly once we have explained the geometrical meaning of some of the formulae. This will be done in the next section. Much more difficult is the proof that our construction gives all self-dual fields. This requires the introduction of quite new ideas and techniques which will be explained in subsequent chapters.

The matrix transformation (2.11) includes, as a specially simple case, the obvious choice of a diagonal matrix B (with diagonal entries $b_1, \ldots, b_k \in H$) and real positive scalars $\lambda_1, \ldots, \lambda_k$. Provided the b_i are all distinct, conditions (I) and (II) will be satisfied. The resulting k-instanton therefore looks like a superposition of k instantons with scales λ_i and centres b_i. These special solutions were discovered by 't Hooft and others [14] [31]. The general solution cannot however be put into this form (even after a conformal transformation).

If we replace the vector $\lambda=(\lambda_1, \ldots, \lambda_k)$ in (2.11) by $q\lambda$ where q is a quaternion of unit norm then the resulting potential A given by (2.9) is unaltered. Similarly if we replace λ by λT and B by $T^{-1}BT$, where T is a (real) orthogonal $k\times k$ matrix, then the potential gets conjugated by the constant matrix T which simply gives a gauge transformation. Note that both these alterations of (λ, B) preserve the conditions (I) and (II) above. We shall in due course prove that no other transformations of the parameters (λ, B), except those just described, give gauge equivalent potentials. Thus the main theorem to be proved can be stated as follows [2] [18] [19]

THEOREM. *Every k-instanton for $SU(2)$ arises from parameters (λ, B) satisfying (I) and (II), the potential being given in an asymptotic gauge by formula (2.9) where $u(x)$ is defined by (2.11). The potentials defined by (λ, B) and (λ', B') are gauge-equivalent if and only if $\lambda'=q\lambda T$, $B'=T^{-1}BT$ with $q\in Sp(1)$ and $T\in O(k)$.*

It is a simple matter to count the number of effective parameters involved in our construction. The initial data of a pair (λ, B) involves

$$4k+4\cdot\tfrac{1}{2}k(k+1)=2k^2+6k$$

real parameters. The number of real equations involved in (I) is $3k(k-1)/2$ while the groups $Sp(1)$ and $O(k)$ have dimensions 3 and $\frac{1}{2}k(k-1)$ respect-

ively. Computing naively we find

$$2k^2+6k-\frac{3k(k-1)}{2}-3-\tfrac{1}{2}k(k-1)=8k-3$$

as the number of effective parameters. This checks with the calcul infinitesimal variation methods [3] [37]. If we start from a 't Hooft (B diagonal, λ real) and consider small perturbations of the soluti parameters can be interpreted as follows. Each b_i has 4 parame each $\lambda_i\in H$ has 4 parameters, giving $8k$ in all: we subtract 3 becau action of $Sp(1)$. However we cannot satisfy condition (I) with B and the λ_i not real. What happens is that condition (I) requires non- terms to enter in B so as to cancel the imaginary contributions Near a 't Hooft solution one can actually solve by power series e: for these off-diagonal terms so that $(b_1 \ldots b_k, \lambda_1 \ldots \lambda_k)$ modulo $S\!p$ local parameters for the space of k-instantons. However these are n parameters and the global structure (even topologically) of this « space is quite complicated (see [5]).

For the group $Sp(n)$ (consisting of $n\times n$ quaternion matrice also preserve the norm of H^n) there is an entirely analogous solutio multi-instanton problem. We replace (λ, B) by (Λ, B) where Λ is matrix satisfying the analogues of (I) and (II), namely

$(\mathrm{I})_n$ $B^*B+\Lambda^*\Lambda$ is real

$(\mathrm{II})_n$ $(B-x)\xi=0$ and $\Lambda\xi=0$ imply $\xi=0$.

We then define the $k\times n$-matrix U as a function of $x\in H$ by

$$U(x)=[\Lambda(B-x)^{-1}]^* . \tag{2.12}$$

Finally the potential $A(x)$ is defined in terms of $U(x)$ by a formula izing (2.9):

$$A(x)=\sigma U^* dU\sigma+\sigma^{-1}d\sigma \tag{2.13}$$

where $\sigma=(1+U^*U)^{-\frac{1}{2}}$ is now a self-adjoint $n\times n$-matrix. No when $n=1$, σ is a real scalar and (2.13) coincides with the alt form (2.9).

3. – Geometrical interpretation.

There is a geometrical way to construct potentials which is ver but appears not to be familiar to physicists. Roughly speaking struction is analogous to the way in which Riemannian metrics on a manifold can be constructed by embedding the manifold in a Euclidean space and considering the induced metric. Historically this is of course the way Riemannian metrics first arose (e.g. surfaces in 3-space).

We recall that a vector bundle E over a space X is a family of vector spaces E_x parametrized (continuously) by $x \in X$: for example if X is an n-dimensional manifold its tangent spaces form such a vector bundle with $E_x \cong R^n$. We say that E is embedded in the trivial bundle $X \times R^N$ if each E_x is embedded in R^N, the embedding varying continuously (or differentiably) with x. For example if X is embedded as a manifold in R^N its tangent spaces (translated to the origin) get identified with subspaces of R^N. When E is embedded in $X \times R^N$ a section of E, namely a function $f(x)$ taking its values in E_x can be viewed as a function with values in R^N. We can then form its partial derivatives $\partial_\mu f$, but these need no longer take values in the sub-spaces E_x. However if we let P_x be any linear transformation on R^N varying smoothly with x and projecting onto E_x (i.e. $P_x^2 = P_x$) we can put

$$\nabla_\mu f = P \partial_\mu f \tag{3.1}$$

and we get a covariant derivative defined on E. If E is the tangent bundle of a manifold X and P is orthogonal projection ∇_μ is just the usual cova-riant derivative of Riemannian geometry (given by the Levi-Civita connection).

In general (3.1) just corresponds to a $GL(n, R)$-connection or potential, but if we impose additional structures, preserved by P, we can get potentials for appropriate subgroups such as $O(n)$, $U(m)$ or $Sp(l)$ ($n = 2m$ or $n = 4l$ respectively). Thus to get $Sp(1)$ potentials we would consider quaternionic lines in H^k and use orthogonal projection.

Choosing a gauge for the bundle E will give rise to linear maps $u_x: R^n \to R^N$ whose image is just $E_x \subset R^N$. If inner products are fixed throughout so that u is an orthogonal gauge then the orthogonal projection P_x onto E_x is given by $P = uu^*$, while $u^*u = 1$. To compute the covariant deriva-tive ∇ in the gauge u we put $f = ug$ where g is now a function on X with values in R^n and find

$$\nabla(ug) = uu^* d(ug) = u\{dg + u^*(du)g\}$$

showing that the gauge potential A is given by

$$A = u^* du \quad \text{or} \quad A_\mu = u^* \partial_\mu u . \tag{3.2}$$

Note that u is here an $(N \times n)$ matrix of functions so that A_μ is a matrix. If $N = n$ then (3.2) asserts that A is gauge equivalent corresponding to the fact that $E_x = R^N$ does not really depend on ever for $N > n$ (3.2) gives interesting potentials. The formula for tl field is

$$F = du^* du + u^* du \wedge u^* du \tag{3.3}$$

or

$$F_{\mu\nu} = \partial_\mu u^* \partial_\nu u - \partial_\nu u^* \partial_\mu u + [u^* \partial_\mu u, u^* \partial_\nu u] .$$

For many purposes it is unnecessary to pick a gauge since we ca work directly inside the larger R^N space which has a natural basis. illustrate this by deriving an alternative expression for the field, e directly in terms of the projection operator P and the complement jection $Q = 1 - P$. Let us take the $GL(N, R)$-potential B definec

$$B = Q\,dQ . \tag{3.4}$$

Computing the covariant derivative $\nabla_B = d + B$ on functions lie in E i.e. satisfying $Pf = f$ or $Qf = 0$, we see that

$$\nabla_B f = df + Q(dQ)f = df - Q^2 df = P\,df = \nabla f$$

where ∇ is the covariant derivative on E defined by (3.1). Thus ∇_B e to all R^N-valued functions.

Now differentiating $Q^2 = Q$ we get $Q\,dQ + dQ\,Q = dQ$ and so

$$B^2 = Q\,dQ \wedge Q\,dQ = Q(dQ - Q\,dQ) \wedge dQ = 0 .$$

Hence the field F_B of ∇_B is given simply by

$$F_B = dQ \wedge dQ . \tag{3.5}$$

Restricting this to E we see that the field F corresponding to the c derivative (3.1) is given by

$$F = P\,dQ \wedge dQ\,P = P\,dP \wedge dP\,P . \tag{3.6}$$

The components $F_{\mu\nu}$ of F are here linear transformations on the im If we choose an orthogonal gauge u and take $P = uu^*$ then (3.6)

$$u\{u^* d(uu^*) \wedge d(uu^*)u\}u^*$$

and the term in brackets gives

$$du^* \wedge du + u^* du \wedge u^* du + u^* du \wedge du^* u + du^* u \wedge du^* u \, .$$

The last two terms cancel, since $u^* du = - du^* u$, and so we find again formula (3.3).

The bundle E when embedded in R^N has a complementary bundle $E^\perp$ given by the image of Q. The relationship between E and $E^\perp$ is symmetrical with the roles of P and Q being interchanged. If $v\colon R^{N-n} \to R^N$ is an orthogonal gauge for $E^\perp$ so that $Q = vv^*$ we can compute the field F of E from (3.6) by

$$F = P\, d(vv^*) \wedge d(vv^*) P \tag{3.7}$$

$$= P\, dv \wedge dv^* P$$

the other terms dropping out since $Pv = 0$.

If v is simply a linear, but not orthogonal, gauge these formulae get modified slightly. We take the polar decomposition $v = \omega\varrho$ where $\varrho^2 = v^* v$ and $Q = \omega\omega^* = v\varrho^{-2}v^*$. Substituting in (3.6) for Q we get

$$F = P\, dv \varrho^{-2} dv^* P \tag{3.8}$$

All these formulae hold unchanged if we replace the real numbers by complex numbers or quaternions. In the quaternion case if we want to consider all our matrix operators as left operators then we should regard H^N as a right vector space, i.e. a scalar quaternion q acts on a quaternion vector ξ by $\xi \to \xi\bar{q}$.

We now apply this general construction with $X = S^4 = P_1(H)$, the quaternion projective line. The calculations which follow are similar to those in [13] and [15]. We consider a point of $P_1(H)$ as given by homogeneous coordinates (x, y) with $x, y \in H$ and scalar multiplication on the right so that (x, y) and (xq, yq) denote the same point. Now let

$$v(x, y) = Cx + Dy \tag{3.9}$$

be a $(k+n) \times k$ matrix of quaternions, C and D being constant matrices (independent of the scalar quaternion variables x, y). We now assume

(3.10) $v(x, y)$ has maximal rank for all $(x, y) \neq (0, 0)$.

The columns of $v(x, y)$ then span a subspace of H^{n+k} having dimension k

and depending only on the ratio xy^{-1}, i.e. on the point of S^4. The ort
complement is then a subspace $E_{(x,y)}$ of dimension n. We now g
vector bundle E over S^4 the covariant derivative induced from
orthogonal projection as in (3.1). The field or curvature can then be c
by one of the above formulae. If we restrict to $R^4 \subset S^4$ where $y \neq$
we can take affine coordinates $(x, 1)$ and $v(x) = v(x, 1)$ then gives
gauge for $E^\perp$. Substituting for v in (3.8) gives the following ex
for the field F:

$$F = PC\, dx \varrho^{-2} d\bar{x} C^* P \tag{3.11}$$

where $\varrho^2 = v^* v = (\bar{x}C^* + D^*)(Cx + D)$.

If we now assume that

$$(\bar{x}C^* + D^*)(Cx + D) \tag{3.12}$$

is a *real* matrix for all $x \in H$ then the term ϱ^{-2} in (3.11) commutes
scalar quaternion dx and shows that F involves only the self-dual
sion $dx\, d\bar{x}$. Hence we have verified that (3.9), subject to conditio
and (3.12), gives rise to a multi-instanton for the group $Sp(n)$. N
the vector bundle E on which the self-dual potential is defined has t
of v as its orthogonal complement $E^\perp$.

It is now easy to verify that the instanton number, i.e. the to
invariant, of E is precisely k. Since this invariant is additive f
sums it is equivalent to check that $E^\perp$ has invariant $-k$. But $E^\perp$ i
finition, a direct sum of k quaternion line-bundles correspondin
k basic vectors of H^k (i.e. the columns to v). Each of these line-bu
be identified with the standard line-bundle over $S^4 = P_1(H)$ wh
ciates to (x, y) the one-dimensional subspace of H^2 consisting of sc
tiples of (x, y). This has invariant ± 1 depending on conventions
case the sign must be -1 so that the invariant of the self-dual
becomes the positive integer k.

If we want to write down explicitly a gauge potential A corre
to the matrix $v(x)$ we must first pick an orthogonal gauge for the b
i.e. a $(k+n) \times n$ matrix $u(x)$ such that

$$u^* v = 0 \qquad u^* u = 1 \, . \tag{3.13}$$

Then $A = u^* du$ as in (3.2). Note that conditions (3.13) are pre
we replace u by ug where $g(x) \in Sp(n)$ and this produces a gene
transformation on A.

The matrices u, v can be put into a sort of normal form which gives more explicit formulae although this will introduce « apparent singularities ». To obtain these normal forms we first decompose $v(x, y)$ into blocks as follows

$$v(x, y) = \begin{pmatrix} C_0 x + D_0 y \\ C_1 x + D_1 y \end{pmatrix}$$

where C_0, D_0 are $n \times k$ and C_1, D_1 are $k \times k$ matrices. Since v is assumed by (3.10) to have maximal rank we may, after a change of the (x, y) variables (a conformal transformation of S^4), assume C_1 non-singular. Replacing v by RvS where S is a real $k \times k$ matrix and $R \in Sp(n + k)$ we can then take $C_1 = -I$ (I the unit $k \times k$) matrix and $C_0 = 0$.

Now putting $y = 1$ we get $v(x)$ in the form

(3.14) $$v(x) = \begin{pmatrix} \Lambda \\ B - xI \end{pmatrix}$$

where Λ is an $n \times k$ matrix and B is a $k \times k$ matrix, both have quaternion entries but are independent of the quaternion variable x. Condition (3.12) is equivalent to requiring both

(3.15) $$\Lambda^* \Lambda + B^* B \text{ is real}$$

(3.16) $$B^* x + \bar{x} B \text{ is real for all } x \in H .$$

But (3.16) is easily seen to be equivalent to requiring

(3.17) $$B \text{ is a symmetric matrix} .$$

We now take u in the form $u = \begin{pmatrix} -I \\ U \end{pmatrix} \sigma$ where I is the unit $n \times n$-matrix, U is an $k \times n$-matrix and σ is a self-adjoint $n \times n$-matrix. Equations (3.13) become

(3.18) $$-\Lambda + U^*(B - xI) = 0 \qquad \sigma^{-2} = I + U^* U .$$

Except for singularities at points x where $B - xI$ is singular we can solve for U:

(3.19) $$U^* = \Lambda (B - xI)^{-1} .$$

Substituting $u = \begin{pmatrix} -I \\ U \end{pmatrix} \sigma$ in (3.2) we find the gauge potential given in

terms of U and $\sigma = (1 + U^* U)^{-\frac{1}{2}}$ by

(3.20) $$A = \sigma U^* dU \sigma + \sigma^{-1} d\sigma .$$

Equations (3.19) and (3.20) are precisely those given in section 2. W
thus verified that the formulae of section 2 do indeed give k-instanton
have also explained why the singularities of section 2 are only « appa
and due to the particular choice of gauge.

Geometrically our construction of multi-instantons can also be form
as follows. On the Grassmannian $G_{k,n}(H)$ of n-dimensional subspa
H^{n+k} the standard vector bundle with fibre H^n has a standard conn
(induced by orthogonal projection) and our instanton connections
are induced by suitable maps $f\colon S^4 \to G_{k,n}(H)$. Equation (3.19) desc
explicitly in terms of appropriate coordinates in the two manifold
standard connection on $G_{k,n}(H)$ is automatically invariant under the
$Sp(n + k)$. In particular for $n = k = 1$ this shows that the basic ins
on S^4 is invariant under $Sp(2) \cong \mathrm{Spin}(5)$.

CHAPTER III

The Penrose Twistor Space

1. – Complex projective 3-space.

Our attack on the instanton problem will rest on complex analytic methods which are part of the general twistor theory of R. Penrose [35]. Very roughly Penrose's programme consists in re-interpreting physical space-time data in terms of corresponding data in a space of 3 complex variables. This space is the (projective) twistor space and the transformation of data can be called the twistor transform. The Penrose theory when continued to Euclidean 4-space can be developed in a slightly different way. In this section we shall describe the basic geometrical picture and comment on it from a variety of viewpoints.

As in previous sections we shall use the quaternions H and we shall identify S^4 with $P_1(H)$, the projective line over the quaternions. We shall use left scalars so that (q_1, q_2) and $(\lambda q_1, \lambda q_2)$ represent the same point of $P_1(H)$. Taking conjugates (which reverses the orientation) would convert to right scalars.

We identify the complex numbers C with the subfield of H generated by 1 and i, and H then becomes identified with C^2 by writing quaternions in the form $z_1 + z_2 j$ with $z_1, z_2 \in C$. Similarly H^2 gets identified with C^4. Now consider the complex projective 3-space $P_3(C)$, the space parametrizing complex lines (through 0) in C^4. If to each complex line we associate the quaternion line it generates, we get a map

$$P_3(C) \to P_1(H)\,. \tag{1.1}$$

In terms of homogeneous coordinates this map is given by

$$(z_1, z_2, z_3, z_4) \to (z_1 + z_2 j, z_3 + z_4 j)\,.$$

If we fix a quaternion line (a copy of C^2) then all complex lines in it form a copy of $P_1(C) = S^2$. Thus (1.1) is a fibre bundle with fibre $P_1(C)$.

Left multiplication by j induces a transformation σ on $P_3(C)$ whi anti-linear (i.e. anti-holomorphic in local complex coordinates) and sat $\sigma^2 = 1$. In homogeneous coordinates

$$\sigma(z_1, z_2, z_3, z_4) = (-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3)\,. \tag{1.2}$$

Clearly σ preserves the fibration (1.1), acting trivially on $P_1(H)$, and a as the anti-podal map on each fibre (on S^2). We shall consider σ as def a « real structure » for $P_3(C)$ which is different from the usual real stru given by just conjugating all coordinates. This terminology is standa algebraic geometry and means that, for some suitable algebraic embe of $P_3(C)$ in $P_N(C)$, σ will be given by conjugating, in the usual way th ordinates of $P_N(C)$. As a lower-dimensional example consider a single *I* fibre with its anti-podal σ. This can be embedded in $P_2(C)$ as a conic equation $\omega_1^2 + \omega_2^2 + \omega_3^2 = 0$. Note that this has no real points, corres ing to the fact that σ has no fixed points.

Although σ on $P_3(C)$ has no fixed points it does have fixed lines; are precisely the fibres of (1.1). We call these the *real lines*. Thus S^4 ap as the parameter space of all the real lines.

Now we recall the famous Klein representation of all lines of *1* Given two distinct points (z_α), (ω_α) of $P_3(C)$ we introduce the Pl coordinates

$$p_{\alpha\beta} = z_\alpha \omega_\beta - z_\beta \omega_\alpha$$

which give a skew-symmetric matrix characterizing the line joinin and (ω). The six homogeneous variables $p_{12}, p_{13}, p_{14}, p_{23}, p_{34}, p_{42}$ satisf quadratic identity

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0 \tag{1.3}$$

(this expression is the square root of $\det(p_{\alpha\beta})$). Thus the parameter of all lines in $P_3(C)$ is the complex 4-dimensional quadric $Q_4 \subset P_5(C)$ by equation (1.3). A real structure on $P_3(C)$ induces a real structure c For the standard real structure on $P_3(C)$ the real structure of Q_4 is giv conjugating the $p_{\alpha\beta}$ and so Q_4 has (1.3) as its real equation: this is a dratic form of type (3, 3), i.e. having 3 plus signs and 3 minus signs diagonalization. For our real structure given by σ on $P_3(C)$ we get ferent real form of Q_4 corresponding to the signature (5, 1). To verif note that (1.2) has the following effect on the $p_{\alpha\beta}$

$$p_{12} \to -\overline{p_{12}}\,, \quad p_{34} \to -\overline{p_{34}}\,, \quad p_{13} \to \overline{p_{24}}\,, \quad p_{14} \to -\overline{p_{23}}\,.$$

Hence the six quantities

$$X_1 = ip_{12}\,, \qquad X_2 = ip_{34}\,, \qquad X_3 = p_{12} + p_{34}\,,$$

$$X_4 = i(p_{13} - p_{24})\,, \quad X_5 = i(p_{14} + p_{23})\,, \quad X_6 = p_{14} - p_{23}$$

are conjugated normally by (1.2). Rewriting equation (1.3) in terms of the coordinates $X_1, \ldots, X_6$ we get

$$4X_1X_2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = 0 \tag{1.4}$$

confirming that the signature is (5, 1). Moreover the real points of (1.4), representing S^4, are indeed given by the affine equation (taking $X_1 - X_2 = 1$)

$$X_3^2 + X_4^2 + X_5^2 + X_6^2 + (X_1 + X_2)^2 = 1\,.$$

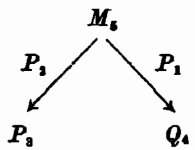
Reverting to the complex geometry of the Klein representation we recall that there are two families of projective planes lying on Q_4. If the equation of Q_4 is written

$$Y_1^2 + Y_2^2 + Y_3^2 = Z_1^2 + Z_2^2 + Z_3^2$$

then the planes are given by the equations $Y = AZ$ where $A \in O(3, C)$. The two families depend on the sign of $\det A$. One family corresponds to points of $P_3(C)$ and the other to planes of $P_3(C)$ or equivalently to points of the dual $P_3(C)$. This means that as a line l in $P_3(C)$ varies through a given point A its representative point $L \in Q_4$ varies in a plane α on Q_4. Similarly if l varies in a plane B its representative L varies in a plane β on Q_4, β being of the opposite family to α.

With our real structure σ every α-plane contains a unique real point, namely the intersection $\alpha \cap \sigma(\alpha)$. This corresponds to the real line in $P_3(C)$ joining A to $\sigma(A)$, and so describes once more the map $P_3(C) \to S^4$.

All this can be conveniently summarized by introducing the correspondence space $M_5 \subset P_3 \times Q_4$ consisting of « incident » pairs (A, L), i.e. A lies on the line l (or equivalently L lies in the plane α). We then have two fibre maps



with fibres P_2, P_1 as indicated. All structures here are complex alg

If we now pick our real structure σ then this defines

(i) the real subspace S^4 of Q_4

(ii) a section $s: P_3 \to M_5$ which picks out the unique point P_2-fibre which is real (regarded as a subspace of Q_4).

Thus the inverse image of S^4 in M_5 gets identified via (ii) witl and so we recover our basic fibration $P_3(C) \to S^4$. Note that wher projection $M_5 \to P_3$ is complex analytic the section $s: P_3 \to M_5$

2. – Lie groups.

The Klein representation can also be looked at from the point of Lie groups. The group $SL(4, C)$, of complex 4×4 matrices of minant 1, acts on $P_3(C)$ by projective transformations, and hence on the space of lines i.e. the quadric Q_4. But the group of projective formations in P_5 keeping Q_4 fixed is just the complex orthogonal $O(6, C)$ modulo $\{\pm 1\}$ and so we get a homomorphism

$$SL(4, C) \to O(6, C)/\{\pm 1\} \tag{2.1}$$

with kernel the fourth roots of unity. Since these two groups ha same dimension it follows that (2.1) is a local isomorphism or tha

$$SL(4, C) \cong \mathrm{Spin}\,(6, C)\,. \tag{2.2}$$

This is one of the coincidences that happen in low dimensions betw various classical groups.

If we start from the group Spin (6, C) then the isomorphism (2 plies that Spin (6, C) has a representation on C^4. This is one of the h representations. The other half-spin representation is on the dual

These complex Lie groups have many real forms, and the lo morphism (2.1) leads in particular to the following local isomorph real Lie groups:

(i) $SL(4, R) \sim SO(3, 3)$

(ii) $SU(2, 2) \sim SO(4, 2)$

(iii) $SL(2, H) \sim SO(5, 1)$

(iv) $SU(4) \sim SO(6)$.

Case (ii) arises from Minkowski space and is the one mainly studied in the Penrose theory. Case (iii) arises from Euclidean space and is the one that concerns us. The action of $SL(2, H)$ on $P_3(C)$ preserves the fibration (1.1) and induces the conformal group action on S^4. In other words the conformal group of S^4 acts naturally on the fibration (1.1).

Taking maximal compact subgroups of (iii) we get the local isomorphism

$$Sp(2) \sim SO(5) .$$

Thus $Sp(2)$ is the group of automorphisms of the fibration (1.1) which preserves in addition the natural metrics on $P_3(C)$ and on S^4. Since $Sp(2)$ acts transitively on $P_3(C)$ we can express $P_3(C)$ as a homogeneous (coset) space:

$$P_3(C) = \frac{Sp(2)}{U(1)\times Sp(1)} .$$

Since

$$S^4 = \frac{SO(5)}{SO(4)} = \frac{Sp(2)}{Sp(1)\times Sp(1)}$$

we see that the fibration (1.1), expressed in terms of compact Lie groups becomes

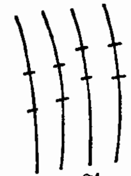
$$\text{(2.3)} \qquad \frac{Sp(2)}{U(1)\times Sp(1)} \to \frac{Sp(2)}{Sp(1)\times Sp(1)}$$

with fibre $Sp(1)/U(1) = S^2$. This description shows clearly that, starting from S^4 we have essentially two different choices to produce $P_3(C)$ depending on which $Sp(1)$ factor we pick. These two choices switch when we reverse orientation on S^4 and lead to dualizing $P_3(C)$.

The fibration (2.3) is also related to the basic $Sp(1)$-instanton on S^4. As explained in Chapter II the basic instanton can be thought of as the quaternion line-bundle over $P_1(H)$ with its connection induced from the fixed space H^2. An equivalent description is to say that the principal $Sp(1)$-bundle of the instanton is the fibration

$$\text{(2.4)} \qquad \frac{Sp(2)}{Sp(1)} \to \frac{Sp(2)}{Sp(1)\times Sp(1)} .$$

The connection can now be described in terms of horizontals to



S^4

If we give all spaces their natural metrics, inherited from the bi metric of the compact group $Sp(2)$, we can choose the orthogon fibres as horizontals. This gives a connection admitting $Sp(2)$- symmetry group. Moreover this is the *unique* choice that has thi because any invariant choice of horizontals must, at each point, be under the action of the isotropy group $Sp(1)$, which acts by in representations in the vertical (fibre) and horizontal directions we have the adjoint representation of $Sp(1)$ and horizontally we h representation of $Sp(1) \cong SU(2)$). Since the instanton connecti $SO(5)$ as symmetry group it follows that our connection must be t ton or anti-instanton, depending on which $Sp(1)$ factor we choo left hand side of (2.4).

Thus $P_3(C)$ is naturally the quotient of the principal bun $Sp(1)$-instanton by the action of $U(1)$. Equivalently $P_3(C)$ can b from the instanton, considered as $H = C^2$ bundle over S^4, by each fibre by the corresponding projective space $P_1(C)$. This way ing $P_3(C)$ from S^4 can be generalized to other 4-manifolds as explai

3. – Complex coordinates in R^4.

The introduction of the Penrose space $P_3(C)$ as a tool to stud on S^4 (or R^4) can be motivated in the following way. It is very w that many classical problems in 2 real variables (x, y) are best introducing the single complex variable $z = x + iy$, and then powerful methods of holomorphic function theory. When we pa to R^4 one might naively try a similar device by introducing tw variables

$$z_1 = x_1 + ix_2 , \quad z_2 = x_3 + ix_4 .$$

There is one immediate objection to such a method, namely tha closely tied to the particular coordinate system and therefore i to give significant results. For example a permutation of the four coordinates x_1, x_2, x_3, x_4 would lead to new complex variables not well related to the first choice. In R^2 on the other hand provided we have fixed the metric and orientation the complex structure is unambiguous. The complex number i is given by rotation through $\pi/2$ in the positive sense.

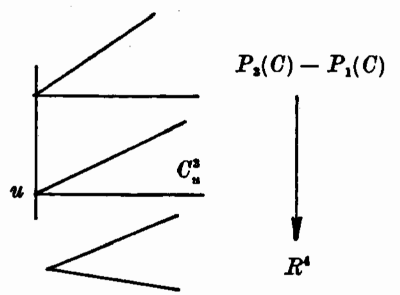
Since there is no natural choice of complex structure on R^4 we can try to consider simultaneously *all* choices which are compatible with the metric and orientation. Effectively we have to define i as a proper orthogonal transformation with $i^2 = -1$. If we have made one such choice then transforming (conjugating) by elements of $SO(4)$ will produce all other choices. Moreover transforming by elements of $U(2)$ leaves the first choice unaltered. Hence the set of all complex structures is naturally parametrized by the coset space

$$\frac{SO(4)}{U(2)} = \frac{SU(2)\times SU(2)}{U(1)\times SU(2)} = S^2 .$$

Thus to each $u \in S^2$ we have a complex structure on R^4, namely an isomorphism $R^4 \cong C^2$.

If we now want to introduce complex variable methods in R^4 we need 3 complex variables (u, z_1^u, z_2^u): the first variable u tells us which complex structure to use and the next two are the complex coordinates themselves. Since the coordinates z_1, z_2 depend on u the situation is a little delicate. The Penrose picture clarifies this geometrically as we shall now explain.

For simplicity we shall now work only over $R^4 \subset S^4$ and so we remove the «point at ∞» in S^4 and correspondingly we remove the projective «line at ∞» in P_3 that lies over it in the fibration (1.1). Then we get a fibration $P_3(C) - P_1(C) \to R^4$. Projective planes in $P_3(C)$ which meet in the «line at ∞» become «parallel» affine planes in $P_3(C) - P_1(C)$. Thus we have the following picture for our fibration:



Over a given point (say the origin) of R^4 we have the fibre $P_1(C) =$
ametrized by u. The plane of our parallel system through u is a c
and under the projection gets identified with R^4. This is the way in
gets the complex structure corresponding to u. The fact that this
structure is changing with u means that the vertical identification
the different C^2_u does not preserve the complex structure. This ca
pressed differently by noting that the «horizontal» projection $P_3(C)$ -
$\to P_1(C)$ mapping C^2_u into u is that of a complex vector bund
is not isomorphic to the product $P_1(C)\times C^2$, although the underl
vector bundle is isomorphic to $P_1(C)\times R^4$. This can happen
bundles over $S^2 = P_1(C)$ are topologically classified by maps of t
torial S^1 into the group of the bundle and the fundamental gro
our case are:

$$\pi_1(U(2)) \cong Z \text{ (integers)} \qquad \pi_1(SO(4)) \cong Z_2 \text{ (integers mod 2}$$

and our bundle corresponds to the integer 2 which gives zero in π

This topological fact therefore lies behind the «linkage» bet
complex coordinates z_1^u, z_2^u and the complex parameter u. Working
in the space $P_3(C) - P_1(C)$ we can locally introduce three ind
(«unlinked») complex variables and use these instead of $(u, z_1^u,$
the Penrose space enables us to introduce 3 complex variables in
way to study problems in R^4.

The same picture applies infinitesimally for the complete
$P_3(C) \to S^4$. At each point $u \in P_3(C)$ we can consider the tangent
and the subspace L_u of tangents to the fibre through u. The quoti
T_u/L_u is then a complex vector space which by projection gets
identified with the real tangent space to S^4 at the point below u.

Note in particular that, since complex manifolds have a natura
tion, S^4 inherits a natural orientation from its description as the ba
fibration $P_3(C) \to S^4$. This is the orientation which we fix (the
convention is adopted in [3]).

CHAPTER IV

Holomorphic Bundles

1. – Holomorphic and unitary gauges.

In Chapter I we explained that gauge theories had a direct differential-geometric interpretation in terms of fibre bundles with connection. In this chapter we shall encounter holomorphic fibre bundles and since these are unfamiliar to physicists and have somewhat different features we begin with some elementary remarks designed to explain the essential points.

Let us begin by considering gauge theory for the non-compact group $GL(n, C)$. Geometrically it will be convenient to consider the representation on C^n and discuss complex vector bundles. Thus over our base space X (e.g. S^4) we consider a vector bundle E with fibre C^n, the fibre at $x \in X$ being denoted by E_x. Intuitively we consider E_x as a vector space varying continuously with x. A linear gauge for E means a choice of basis of E_x varying continuously with x. A gauge transformation is then the function $g(x) \in GL(n, C)$ which provides the change of basis at each point x.

In the definition of a vector bundle it is always assumed that local gauges always exist, so that E is locally isomorphic to the product $X \times C^n$. A global gauge does not necessarily exist since E need not be globally isomorphic to the product. The instanton bundles already provide examples of this situation with $X = S^4$ and $n = 2$.

If we assume that each fibre E_x has a positive inner product, varying continuously with x, then we can consider unitary gauges in which the basis at each x consists of an orthonormal base. A change from one unitary gauge to another is then described by a gauge transformation $g(x) \in U(n)$. Local unitary gauges always exist and global unitary gauges exist if and only if a global linear gauge exists, since the topological properties of $GL(n, C)$ are essentially carried by $U(n)$.

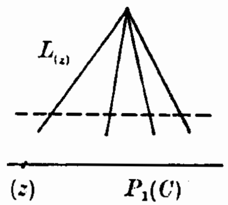
Instead of fixing the inner product and defining unitary gauges in terms of it we can reverse the procedure by choosing one linear gauge, decreeing

that this is orthonormal, and considering all gauges obtained b
unitary gauge transformations.

If X is a differentiable manifold then it is natural to requi
word continuous be replaced throughout by (sufficiently) dif
This introduces no essential differences.

If X is replaced by a complex analytic manifold Z, for exan
then we can introduce the notion of a holomorphic, or comple
vector bundle. Naively we now think of the fibres E_z as va
morphically with $z \in Z$. A holomorphic gauge is then a basis
varying holomorphically with z and a holomorphic gauge transf
given by a holomorphic function $g(z)$ with values in $GL(n, C)$.
morphic structure can be defined by fixing one gauge, decreeing
holomorphic, and then allowing as new holomorphic gauges onl
tained by applying holomorphic gauge transformations.

The analogy and difference between the holomorphic and u
is fairly clear. The important point to note is that unitary g
formations are defined by a point-wise restriction on values wh
morphic gauge transformations cannot be defined this way. No
two types of gauge transformations are diametrically opposed sin
formation which is both unitary and holomorphic is necessaril
To illustrate these basic ideas let us consider a simple exampl
base space $P_1(C)$ and associate to each point $(z) \in P_1(C)$ the c
$L_{(z)} \subset C^2$ which that point parametrizes, namely all muntiples



It is clear intuitively that $L_{(z)}$ varies holomorphically with (z).
holomorphic gauge is given by intersecting $L_{(z)}$ with any affine
(not through 0) (see dotted line in figure): for example the line z_2
gauge is well-defined except at the point $(1, 0) \in P_1(C)$. Simil
gives a gauge except at $(0, 1)$. The gauge transformation bet
two is given by the function z_1/z_2 and this is of course a holomo
tion on $P_1(C)$ outside the two points $(1, 0)$ and $(0, 1)$.

If we give C^2 its natural inner product each fibre $L_{(z)}$ inheri product and we can introduce unitary gauges. Note again that the linear holomorphic gauges above are definitely not unitary.

If we associate to each (z) the orthogonal complement $L^{\perp}_{(z)}$ we again get a vector bundle with unitary structure. However the process of taking orthogonal complements involves complex conjugation and so the vector spaces $L^{\perp}_{(z)}$ do *not* vary holomorphically with (z). Thus L is a holomorphic vector bundle but $L^{\perp}$ is not. This is a major difference between the unitary and holomorphic theories.

Although taking orthogonal complements is not a holomorphic process we can instead form the quotient space $N_{(z)} = C^2/L_{(z)}$. This does again form a holomorphic line-bundle N, a holomorphic gauge coming from any holomorphic function on $P_1(C)$ with values in C^2. If we take linear duals the bundle N' appears as a sub-bundle of $P_1(C) \times (C^2)'$. In general if E is any holomorphic vector bundle, its linear dual E' is again a holomorphic vector bundle.

If we replace $P_1(C)$ by $P_n(C)$ then again we get a holomorphic line bundle L. The quotient C^{n+1}/L is now a holomorphic C^n-bundle over $P_n(C)$.

So far we have just holomorphic or unitary bundles with no further structure. Now we come to the question of connections. In the case of a linear vector bundle E with group $GL(n, C)$ a connection can be given by the covariant derivative ∇. The components ∇_μ, relative to coordinates x_μ in the base manifold, act on sections of the vector bundle, i.e. functions $f(x) \in E_x$. In a given linear gauge

$$\nabla f = df + Af\,,$$

where $A = \sum A_\mu dx^\mu$ is the gauge potential.

If E has a unitary structure and we require ∇ to be compatible with unitarity (i.e. the corresponding parallel transport preserves length) then in any unitary gauge $A^* = -A$.

Before proceeding to the holomorphic case let us review a few elementary definitions used in complex manifold theory. If $(z_1 \dots z_n)$ are local complex coordinates on Z we introduce the formal differentials $dz_\alpha, d\bar{z}_\alpha$ defined by

$$dz_\alpha = dx_\alpha + i\,dy_\alpha\,, \qquad d\bar{z}_\alpha = dx_\alpha - i\,dy_\alpha$$

where $z_\alpha = x_\alpha + iy_\alpha$. The total differential

$$df = \sum_{\alpha=1}^{n}\left(\frac{\partial f}{\partial x_\alpha}\,dx_\alpha + \frac{\partial f}{\partial y_\alpha}\,dy_\alpha\right)$$

can be decomposed into two parts

$$df = d'f + d''f$$

where $d'f$ involves only the dz_α and $d''f$ involves only the $d\bar{z}_\alpha$. ! composition is independent of the choice of local holomorphic coo and the equation $d''f = 0$ is the Cauchy-Riemann equation, chara holomorphic functions.

If now E is a holomorphic vector bundle over Z and ∇ is a c derivative, we can write

$$\nabla = \nabla' + \nabla''\,.$$

If we put

$$\nabla = \sum(\nabla_\alpha dx_\alpha + \tilde{\nabla}_\alpha dy_\alpha)$$

then $\nabla' = \sum \nabla'_\alpha dz_\alpha$, $\nabla'' = \sum \nabla''_\alpha d\bar{z}_\alpha$ where

$$\nabla'_\alpha = \tfrac{1}{2}(\nabla_\alpha - i\tilde{\nabla}_\alpha)$$

$$\nabla''_\alpha = \tfrac{1}{2}(\nabla_\alpha + i\tilde{\nabla}_\alpha)\,.$$

We shall say that ∇ is compatible with the holomorphic structure $\nabla'' f = 0$ for every holomorphic section f of E. This is equivalent to r that, in any holomorphic gauge, the gauge potential A, when dec as $A = A' + A''$, has $A'' = 0$. Such a potential is said to be of ty In general a differential form is said to be of type (p, q) if it invol the dz_α and q of the $d\bar{z}_\alpha$.

The link between connections on unitary and holomorphic bu provided by the following simple and well-known result:

PROPOSITION (1.1). *Let E be a holomorphic vector bundle with c structure. Then there is a unique connection compatible with both s i.e. such that*

(i) *in every unitary gauge the gauge potential A satisfies A*

(ii) *in every holomorphic gauge $A'' = 0$.*

The curvature F of this unique connection is of type (1, 1).

PROOF. – First pick a holomorphic gauge and choose $A'' = 0$ to sa Now transform to a unitary gauge by the gauge transformation g transformed potential is $B = B' + B''$, then $B'' = d''g \cdot g^{-1} + A'' =$

is determined by g. Finally to satisfy (i) we must take $B' = -(B'')^*$. This uniquely determines B and hence fixes the potential. Computing the field F in the holomorphic gauge we have

$$F = dA' + [A', A'] = \{d'A' + [A', A']\} + d''A'$$

exhibiting only terms of type $(2, 0)$ and $(1, 1)$. But unitarity implies that, in any gauge, $F^* = -F$, and so the component of type $(2, 0)$ is also zero. Thus F is of type $(1, 1)$.

Proposition (1.1) has an important converse which can be stated as follows:

THEOREM (1.2). *Let E be a complex vector bundle, with a unitary structure, over a complex manifold. Let E have a unitary connection whose curvature is of type* $(1, 1)$. *Then there is a unique holomorphic structure on E such that the connection is that given by* (1.1).

This theorem is essentially a consequence of the Newlander-Nirenberg integrability theorem for complex structures [33]. The basic idea is that the holomorphic structure of E is defined by taking the solutions of the equation $\nabla'' f = 0$ as its holomorphic sections. Here ∇'' denotes the $(0, 1)$ component of the covariant derivative of the connection. The condition on the curvature implies that $[\nabla''_\alpha, \nabla''_\beta] = 0$, where $\nabla'' = \sum_\alpha \nabla''_\alpha d\bar{z}_\alpha$, and the z_α are local complex coordinates on our manifold. The integrability theorem of [33] then implies that the equation $\nabla'' f = 0$ has locally enough solutions to provide a basis for E. For further details we refer to [3]: see also [25].

2. – Twistor interpretation of instantons.

In this section we shall show how to interpret the self-duality equations for a Yang-Mills field on S^4 in terms of complex analysis on the twistor space $P_3(C)$.

As an algebraic preliminary we shall need to understand the significance of the equations $^*\omega = \pm\omega$ for a 2-form ω on R^4 in terms of complex co-ordinates. We recall that once we have introduced complex coordinates, identifying R^4 with C^2, a 2-form ω can be expressed in terms of its type decomposition:

(2.1) $$\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$$

We ask how this decomposition corresponds to the *-decompositi

(2.2) $$\omega = \omega^+ + \omega^-$$

according to the eigenvalues ± 1 of *. This is just a question algebra and it can be viewed in terms of group representations. Equa corresponds to decomposing a representation of $SO(4)$ into two ir pieces (each of dimension 3), while (2.1) corresponds to decompo representation under the subgroup $U(2)$. The $(2, 0)$ and $(0, 2)$ co have dimension one while the 4-dimensional representation in t has a further decomposition:

(2.3) $$\omega^{1,1} = \omega_0^{1,1} + \alpha\,.$$

Here α is a multiple of the $(1, 1)$ form μ corresponding to the metric and $\omega_0^{1,1}$ is the « primitive » part, orthogonal to μ. The 3-di representation of $U(2)$ on this primitive part is easily seen to be i and so this must coincide with one of the two irreducible pieces But the form μ corresponding to the metric is self-dual. Hence have

(2.4) $$\omega^- = \omega_0^{1,1}\,.$$

In particular this shows that the space Ω^- of ω with $^*\omega = -$ type $(1, 1)$ for all complex structures (compatible with metric an tion). The converse is also true, because the space

$$V = \bigcap_u \Omega_u^{1,1}$$

of 2-forms on R^4 which are of type $(1, 1)$ for all complex struc invariant under $SO(4)$, contains Ω^- but is not the whole space: it m for coincide with Ω^-.

Thus we have established the following algebraic lemma.

LEMMA (2.5). *A 2-form ω on R^4 is anti-self-dual if and only type* $(1, 1)$ *for all compatible complex structures.*

We shall now apply this lemma to a 2-form ω on S^4 which we tain a 2-form $\tilde\omega$ on $P_3(C)$. This form $\tilde\omega$ is purely horizontal, i. if α or β is a fibre direction. To compute the horizontal part of $\tilde\omega$ a we note the interpretation, explained in the preceding section, to which u parametrizes (infinitesimally) complex structures on S^4 at the point below. From this and our Lemma we deduce

PROPOSITION (2.6). *A 2-form ω on S^4 is anti-self-dual if and only if its lift $\tilde{\omega}$ to $P_3(C)$ is of type* (1, 1).

Note that this proposition is a purely local one, valid for any open set U in S^4 and its counterpart $\tilde{U}$ in $P_3(C)$.

Finally we consider a complex vector bundle E on S^4 with a unitary structure and connection. Let F be its curvature. If we lift E to give a bundle $\tilde{E}$ with connection on $P_3(C)$ its curvature is just the lift $\tilde{F}$ of F. Applying (2.6) to the matrix coefficients of F we deduce

PROPOSITION (2.7). *A vector bundle E on S^4 with unitary structure and connection has anti-self-dual curvature if and only if the lifted bundle $\tilde{E}$ on $P_3(C)$ with the lifted connection has curvature of type* (1, 1).

Using (2.7) and theorem (1.2) we deduce the important result

(2.8) $$E \text{ anti-self-dual} \to \tilde{E} \text{ holomorphic}.$$

We plan now to make this statement more precise by specifying which holomorphic bundles on $P_3(C)$ arise in this way and how one gets back from $\tilde{E}$ to E.

Note first of all that, restricted to any fibre P_x of $P_3(C) \to S^4$, $\tilde{E}$ is holomorphically trivial, a basis of E_x giving rise to a holomorphic basis or gauge of $\tilde{E}|P_x$. Conversely this shows that E_x can be uniquely defined as the space of holomorphic sections of $\tilde{E}|P_x$.

We turn next to consider the unitary structures on E and $\tilde{E}$. The unitary structure on E can be given by an anti-linear isomorphism $\tau: E \to E^*$ such that $(u, \tau v)$ is a positive hermitian form (E^* denotes the dual of E). Passing to $\tilde{E}$ we use τ to define a lifting $\tilde{\tau}$ of the conjugation σ on $P_3(C)$, namely we define a commutative diagram

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{\tau}} & \tilde{E}^* \\ \downarrow & & \downarrow \\ P & \xrightarrow{\sigma} & P \end{array} .$$

The map $\tilde{\tau}$ is anti-holomorphic, i.e. if we give $\tilde{E}^*$ its opposite complex structures it becomes a holomorphic isomorphism. This follows from the fact that $\tilde{\tau}$ preserves the unitary structure and connection and hence (by the uniqueness part of Theorem (1.2)) it preserves the holomorphic structures.

Restricting to the fibre P_x our map $\tilde{\tau}$ induces a similar map
morphic sections of $\tilde{E}$ and in this way we recover τ_x. Lifting back
the unitary structure of $\tilde{E}$ which together with the holomorphic
yields, by (1.1), a unique connection of type (1, 1). Our final step i
that this connection descends to a connection on E: by (2.7) this v
sarily be anti-self-dual. The condition that the connection on $\tilde{E}$
from $P_3(C)$ to S^4 is that the curvature should be purely horizontal
$F_{\alpha\beta} = 0$ if α is a vertical direction (i.e. along the fibres). If α and β
vertical this is clear because, restricted to a fibre, $\tilde{E}$ is trivial holom
and unitarily. The stronger statement, when only α is vertica
from the triviality of $\tilde{E}$ restricted to the *first formal neighbour*
fibre. More concretely this triviality means that we can pick a ga
near a fixed fibre P_x, which is holomorphic and unitary on P_x a
first derivatives normal to P_x. The verification of this triviality
of $\tilde{E}$ is best postponed until later (see Chapter VI, § 3) since it
complex analytical machinery. We note in passing that $\tilde{E}$ is not
to second derivatives unless the whole curvature vanishes.

To summarize our results in a convenient form we shall mak
lowing definitions. Let V be a holomorphic vector bundle over P
an anti-linear isomorphism $p: V \to V^*$ covering σ on $P_3(C)$ such

$$(u, pv) = \overline{(v, pu)} \qquad v \in V_{(x)}, \quad u \in V_{\sigma(x)}$$

will be called a *real form* on V. If V is further assumed to be tri
real lines of $P_3(C)$ then p induces a non-degenerate hermitian fo
space of holomorphic sections of V restricted to any real line. If
mitian form is positive we say that our real form is *positive*. T
bundles V, W with real forms are called isomorphic if there is a
analytic isomorphism from V to W commuting with p.

Our conclusion can now be stated as follows:

THEOREM (2.9). *There is a natural* (1-1) *correspondence bet*

(i) *anti-self-dual $U(n)$-potentials over S^4 (up to gauge equiva*

(ii) *holomorphic vector bundles with fibre C^n over $P_3(C)$ with real form (up to isomorphism).*

REMARKS.

1) This theorem is purely local in character. It holds for
set U of S^4 and its counterpart $\tilde{U}$ on $P_3(C)$.

2) In a purely complex form this theorem is originally due to R. S Ward [42]. We shall later discuss Ward's proof. The present version is stated briefly in [4] and elaborated in [3].

3) It should be emphasized that in (ii) most of the information is already contained in the holomorphic vector bundle. The positive real form when it exists is generally unique.

4) Both in (i) and (ii) there is an integer topological invariant. In (i) it is the anti-instanton number, while in (ii) it is the second Chern class. These integers are equal (cf. [3]).

Theorem (2.9) can easily be generalized to the orthogonal and symplectic groups. We shall consider the symplectic case in detail, the orthogonal case is quite similar except for sign changes.

Recall that the compact symplectic group $Sp(n)$ is the group of norm-preserving automorphisms of the quaternion vector space H^n. It can be identified with the subgroup of $U(2n)$ commuting with the action of the quaternion j on $H^n = C^{2n}$. Alternatively it is the sugbroup of $U(2n)$ commuting with the skew bilinear form (,) on C^{2n} defined by $(u, jv) = \langle u, v\rangle$ where $\langle,\rangle$ is the hermitian inner product. Hence an $Sp(n)$-potential can be represented geometrically by a vector bundle E with fibre C^{2n} having a unitary connection, together with an isomorphism $\alpha\colon E \to E^*$ which is skew i.e. $(u, \alpha v) = -(v, \alpha u)$ and preserves connection (E^* being endowed with the connection inherited from E by duality).

Suppose now we have an $Sp(n)$-potential on S^4 which is anti-self-dual. Then using (2.9) for $U(2n)$ we see first of all that $\tilde{E}$ on $P_3(C)$ is holomorphic. Moreover the isomorphism $\alpha\colon E \to E^*$ induces a holomorphic isomorphism $\tilde{\alpha}\colon \tilde{E} \to \tilde{E}^*$ which is also skew. Combining $\tilde{\alpha}$ with the anti-linear isomorphism $\sigma\colon \tilde{E} \to \tilde{E}^*$ given by (2.9) we obtain an anti-linear isomorphism $\tilde{E} \to \tilde{E}$ covering σ on $P_3(C)$. We shall denote this map on $\tilde{E}$ also by σ: it satisfies $\sigma^2 = -1$, and is compatible with $\tilde{\alpha}$ or equivalently with the skew form on $\tilde{E}$ defined by $\tilde{\alpha}$. Thus an anti-self-dual $Sp(n)$-potential on S^4 corresponds to a holomorphic vector bundle $\tilde{E}$, with fibre C^{2n}, over $P_3(C)$ which has two further structures:

(i) a holomorphic non-degenerate skew form on $\tilde{E}$

(ii) an anti-linear map $\sigma\colon \tilde{E} \to \tilde{E}$ lifting σ on $P_3(C)$ such that $\sigma^2 = -1$ and compatible with the skew form i.e. $(\sigma u, \sigma v) = \overline{(u, v)}$.

Moreover the bundle $\tilde{E}$ is holomorphically trivial on all real lines of $P_3(C)$ and the hermitian form induced by $(u, \sigma v)$ on sections of $\tilde{E}$, restricted to a real line, is positive definite.

In the special case of $Sp(1) \cong SU(2)$ condition (i) reduces to logical constraint, namely that the first Chern class $c_1(\tilde{E})$ shoul The non-degenerate skew form is then unique up to a constan Moreover the anti-linear map $\sigma\colon \tilde{E} \to \tilde{E}$ is unique (unless $k = 0$) two such σ differ by a holomorphic automorphism of $\tilde{E}$ and (as we later) $\tilde{E}$ has no automorphisms except scalars: the condition $\sigma^2 =$ positivity of the hermitian form then uniquely fix σ. We thus re version of the $SU(2)$ theorem given in [4].

3. – Bundles over $P_1(C)$.

In order to familiarize ourselves with holomorphic vector bu shall now describe the simplest case, namely over the complex proje

The basic example of a complex line bundle (fibre C^1) is, as we h given by the family $L_{(z)}$ of complex lines in C^2, parametrized by responding point $(z) \in P_1(C)$. This standard bundle is topologic trivial, and can be described by two holomorphic gauges near z $z = \infty$ on P_1 (z being a non-homogeneous coordinate), with the gau formation from one to the other being multiplication by z. More the bundle $L^n = L \otimes \ldots \otimes L$ (n times) is similarly given by t transformation z^n. Taking $L^{-1} = L^*$ (the dual) we can also allow a negative integer.

One can now ask if there are any further holomorphic line bu isomorphic to some L^n. Suppose for example we construct a bundle as a gauge transformation any function $f(z)$ which is holomorphic zero near the equator $|z| = 1$. First of all we can define the t invariant

$$k = \frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}\, dz$$

the integral taken over $|z| = 1$. Suppose first $k = 0$, then we hav defined holomorphic function

$$g(z) = \log f(z)$$

in some annulus $r < |z| < R$. Take the Laurent expansion

$$g(z) = \sum_{-\infty}^{\infty} a_n z^n$$

and write it in the form

$$g(z) = g_0(z) - g_\infty(z)$$

where g_0 is holomorphic in $|z| < R$ and g_∞ is holomorphic in $r < |z|$, i.e. g_0 involves terms with $n<0$ and g_∞ those with $n>0$. This decomposition is therefore unique up to additive constants. Exponentiating we get

$$f(z) = f_0(z) f_\infty(z)^{-1}$$

which we can interpret as saying that the holomorphic bundle given by $f(z)$ is holomorphically trivial. The terms $f_0(z)$ and $f_\infty(z)$ allow us to change our original holomorphic gauges so that they now coincide near the equator and so provide a global holomorphic gauge. Thus if $k = 0$ the bundle is holomorphically trivial and a similar argument tells us in general that

$$f(z) = f_0(z) z^k f_\infty(z)^{-1} \tag{3.1}$$

and so the bundle is holomorphically equivalent to L^k. A slight generalization of this argument, allowing more than two local gauges, leads to the conclusion that every holomorphic line-bundle on $P_1(C)$ is isomorphic to some L^k.

Passing now to vector bundles with fibre C^n we can construct obvious examples as direct sums:

$$E = L^{k_1} \oplus L^{k_2} \oplus \ldots \oplus L^{k_n}. \tag{3.2}$$

A much subtler theorem, proved in various versions over the years by Hilbert, G. D. Birkhoff, Grothendieck [26] and others asserts that every holomorphic vector bundle over $P_1(C)$ is isomorphic to such a direct sum of L^{k_i} and the integers $(k_1, \ldots, k_n)$ are unique up to permutation. In more concrete terms this implies in particular a matrix version of (3.1), namely if $f(z)$ is a holomorphic function defined in the annulus $r < |z| < R$, taking values in $GL(n, C)$, then we can write $f(z)$ in the form

$$f(z) = f_0(z) \Lambda(z) f_\infty(z)^{-1} \tag{3.3}$$

where $\Lambda(z)$ is the diagonal matrix with entries z^{k_i} and $f_0(z), f_\infty(z)$ are holomorphic functions with values in $GL(n, C)$ defined for $|z| < R$ and $r < |z|$ respectively.

It is important to note that although the exponents k_i in (3.2) are holomorphic invariants of the bundle E they are *not* topological invariants, only

the sum $k = \sum k_i$ is a topological invariant. Again, in more conc
if in (3.3) $f(z)$ is holomorphic in z and depends continuously on
parameter t, then the factorization (3.3) cannot in general be
tinuous in t. The integers k_i will depend on t in a semi-continuo
Namely as $t \to t_0$ the differences $|k_i(t) - k_j(t)|$ can suddenly inc
particular this implies that the trivial bundle, with all $k_i = 0$, is st
small deformations.

A simple geometrical example will illustrate the way in whi
tegers k_i can suddenly jump. Consider the case $n = 2$, and assum
topological invariant $k_1 + k_2 = 0$. Instead of vector bundles wi
we can equally well work with projective bundles with fibre
trivial bundle is $P_1 \times P_1$, while an example of a non-trivial bundle c
ing to integers $(1, -1)$ is provided by the family of generators of
cone in P_3 (we regard the vertex of the cone as a different poi
generator). If we now take a general quadric in P_3 it can be iden
$P_1 \times P_1$ (cf. a real hyperboloid). By continuously altering the
of the quadratic form we can end up with a singular quadric i.

On $P_1(C)$ therefore all holomorphic vector bundles are known.
cular they are all algebraic, i.e. in suitable gauges the gauge trans
are rational functions of z. Moreover this algebraic structure is
unique. This is a special case of a general theorem of Serre [39]
plies to all complex algebraic varieties in projective space of any
and in particular to projective spaces themselves. However the
complete classification available over $P_1(C)$ does not extend to
gebraic varieties.

One way to study holomorphic (or algebraic) vector bundles
jective space $P_m(C)$ (e.g. $m = 3$ which is our case) is to consider t
tion of the bundle to all the projective lines in $P_m(C)$. The semi-
property mentioned above tells us that over the «general line»
set of integers $(k_1, \ldots, k_n)$ but there will in general be some exce
jumping lines where the differences $|k_i - k_j|$ jump upwards. If
line all k_i are 0 (i.e. the restricted bundle is trivial) then this mu
for the general line. According to Theorem (2.9) we see that the h
vector bundles on $P_3(C)$ arising from anti-instantons on S^4 have
erty, since every real line has all $k_i = 0$. The jumping lines ar
and correspond to points in the complexification of S^4. Serre's t
sures us that, in a suitable gauge, our anti-self-dual gauge pote
will be given by *rational functions* in the coordinates. The com
of these rational functions account for the jumping lines.

If E is a holomorphic vector bundle on $P_3(C)$ trivial on general
up to a constant matrix, E has a unique global holomorphic g

any general line (since holomorphic functions on P_1 are constant). However if we take a triangle formed by three coplanar general lines there is no reason why the three holomorphic gauges on these lines should be consistent. One can think of a trivial holomorphic bundle on P_1 as having a distinguished connection or parallel transport. Then, going round the triangle formed by three coplanar lines, parallel transport may not return to the identity. Thus we get a version of curvature for E purely out of the holomorphic structure and the compactness of the projective lines. This « curvature » is a global notion, associated with each general triangle, but we can obviously infinitesimalize it to derive something more like the differential-geometric curvature.

From this point of view it is now natural to introduce the space which parametrizes all lines in $P_3(C)$. As explained in Chapter III this space is the Klein quadric Q_4 in $P_5(C)$. The « general » lines of P_3, i.e. lines for which E is trivial, correspond to an open set U of Q_4. We can now construct a holomorphic vector bundle $\mathcal{E}$ over U by defining the fibre $\mathcal{E}_L$ to be the space of holomorphic sections of $E|l$, where $L \in U$ represents the line $l \subset P_3$. All the lines l through a point A correspond to points L of a plane α in Q_4 and the vector spaces $\mathcal{E}_L$ for $l \in \alpha \cap U$ can all be naturally identified with E_A. This gives $\mathcal{E}$ a flat connection along all α-planes through L. These α-planes generate the tangent cone to Q_4 at L and so we get components for a connection along all these directions in the tangent cone. But any holomorphic function on this cone which is homogeneous of degree one on each generating line automatically extends (uniquely) to a linear function on the whole tangent space. This is an elementary geometric property of quadric cones (in any number of dimensions); cf. Chapter VI, § 3. Hence $\mathcal{E}$ has a holomorphic connection, flat along the α-planes. On the other hand on the other system of planes on Q_4, corresponding to planes of P_3, the connection need not be flat. This corresponds to the other notion of curvature on P_3 described above. A connection for a bundle on Q_4 which is flat along α-planes precisely corresponds to anti-self-duality for the curvature. This is essentially Ward's approach, passing from holomorphic bundles on P_3 to holomorphic bundles with holomorphic anti-self-dual connection on an open set of Q_4.

If we now want to consider unitary structures and work on $S^4 \subset Q_4$ we would impose appropriate conjugations throughout. The only drawback of this approach is that, in order to pass from S^4 to a neighbourhood in Q_4 and apply the above argument, we have to assume that our bundle on S^4 is *real-analytic*. Our treatment in Section 2, based on the Newlander-Nirenberg integrability theorem, had the advantage of requiring only differentiability. Analyticity, and eventually rationality of the solutions, is then an automatic consequence.

As mentioned in Section 2 (and will be proved in Chapter V
morphic bundle E on $P_3(C)$ which is trivial on a given line l is
on the first neighbourhood of l. Thus a basis of E along l can b
formally up to first derivatives normal to l. This extension ess
just the connection on $\mathcal{E}$ at the point L defined by Ward's app
fact the property of quadric cones used in Ward's construction
related to the extension property of E: both can be formulated i
the vanishing of a certain sheaf cohomology group (cf. Chapte

CHAPTER V

Construction of Algebraic Bundles

1. – The linear complex.

In this chapter we shall give an algebraic construction for vector bundles on $P_3(C)$ which, by the results of Chapter IV, will correspond to anti-instantons. We begin in this section by considering in some detail the basic anti-instanton (for $k = -1$ and $G = SU(2)$). The algebraic geometry in this case is classical and is, in traditional language, the geometry of a « linear complex » of lines in $P_3(C)$. It will turn out that this special case is an illuminating preliminary of the general construction and so it repays careful study.

If we start again from our fibration $P_3(C) \to S^4$ and if we use the standard metrics of S^4 and of $P_3(C)$ we can obviously construct a vector bundle over $P_3(C)$ with fibre C^2 by taking the horizontal tangent vectors on $P_3(C)$, i.e. vectors orthogonal to the fibre direction. Now we recall that taking orthogonal complements is not a holomorphic process so that we would not expect this horizontal vector bundle to be holomorphic. However the vertical bundle, from which we start, is itself not holomorphic since its definition involves the « real structure » of $P_3(C)$. In fact it turns out that the horizontal vectors *do* form a holomorphic bundle (while the complementary verticals do not). The restriction of this bundle to any real line is clearly the normal bundle of the line in $P_3(C)$. This is not trivial but becomes so after tensoring with the standard line bundle L on $P_3(C)$. We then end up with a holomorphic vector bundle which satisfies the conditions of Theorem (2.9) of Chapter IV and corresponds to the basic anti-instanton on S^4.

The preceding construction used the metric and real structure of $P_3(C)$ or equivalently the metric and quaternion structure of $C^4 = H^2$. From these structures we can also extract the natural skew-form and this is essentially what is needed to produce the algebraic vector bundle. If $L_{(z)} \subset C^4$ is the line corresponding to the point $(z) \in P_3(C)$ we consider its annihilating or

polar space $L^\circ_{(z)}$ with respect to the (non-degenerate) skew-form mension of $L^\circ_{(z)}$ is 3 and, since the form is skew-symmetric, $L_{(z)} \subset L$ the quotient space

$$E_{(z)} = L^\circ_{(z)}/L_{(z)}$$

is a 2-dimensional vector space depending algebraically on $(z) \in P_3(C)$. In other words E is an algebraic vector bundle wit over $P_3(C)$. Moreover the skew-form induces a non-degenerate on E so that its structure group reduces to $SL(2, C)$.

In projective terms $L^\circ_{(z)}$ corresponds to a projective plane in sing through the point (z). Such a point-plane correspondence is called a null-correlation and the lines that lie in such a plane through its corresponding point form the associated « linear These projective lines correspond in C^4 to the 2-dimensional isot spaces of the skew-form, *i.e.* those $C^2 \subset C^4$ on which the skew-for tically zero. If the skew-form in C^4 is given by the skew- 4×4 matrix a_{ij} the line $(x)\,(y)$ belongs to the linear complex $\sum y_\alpha a_{\alpha\beta} x_\beta = 0$ or equivalently

$$\sum a_{\alpha\beta} p_{\alpha\beta} = 0$$

where $p_{\alpha\beta} = x_\alpha y_\beta - x_\beta y_\alpha$ are the Plücker coordinates of the line.

We shall now show that the lines of this linear complex ar the jumping lines of the bundle E. First assume that the line (not belong to the linear complex. This means that $L_{(x)}$ does not and hence that $L^\circ_{(x)} \cap L^\circ_{(y)} = R$ is a 2-dimensional subspace of for all (z) on the line $(x)\,(y)$, lies in $L^\circ_{(z)}$ but does not contain $L_{(z)}$. that E, restricted to the line $(x)\,(y)$ is a trivial bundle. Conversel show that E is not trivial when restricted to a line l of the linea The line l corresponds to an isotropic 2-dimensional subspace W so for all $(z) \in l$ we have the inclusions

$$L_{(z)} \subset W \subset L^\circ_{(z)} \subset C^4 . \tag{1.1}$$

Hence E_z contains $W/L_{(z)}$, i.e. $E|l$ contains the line-bundle $W/$ denotes the trivial bundle with fixed fibre W). Topological con then show that $W/L \cong L^*$. Since L^* has holomorphic sections it cannot be a sub-bundle of a trivial bundle and so $E|l$ is not fact $E|l$ corresponds to the integers $(1, -1)$ in the general cla

We come now to consider real structures so that, by the Chapter IV, we can construct anti-instantons on S^4.

If we take the standard skew-form $(,)$ on $C^4 = H^2$ this is related to j and the positive inner product $\langle , \rangle$ by

$$(1.2) \qquad \langle u, v\rangle = (u, jv)$$

and in particular j preserves the skew-form in the sense that

$$(1.3) \qquad (ju, jv) = \overline{(u, v)} .$$

(1.2) implies that $L^\circ_{(z)} = L^\perp_{(jz)}$ where $\perp$ denotes the orthogonal space with respect to the inner product, and so for any $(z) \in P_3$ we have an orthogonal decomposition:

$$(1.4) \qquad C^4 = L_{(z)} \oplus R_x \oplus L_{(jz)}$$

where $R_x = L^\circ_{(z)} \cap L^\circ_{(jz)}$ and x represents the real line $l_x = \{(z)(jz)\}$.

This shows first that a real line l_x for $x \in S^4$ is never a jumping line for the algebraic bundle $E = L^\circ/L$ on $P_3(C)$. It also shows that j induces on E a real structure, i.e. an anti-linear map $\sigma\colon E \to E$ covering σ on $P_3(C)$ satisfying $\sigma^2 = -1$. Finally this real structure is positive, meaning that $(u, \sigma v)$ induces a positive inner product on the sections of E restricted to any real line. This last statement follows from the fact that, restricted to the real line $(z)(jz)$, the sections of E can be identified with R and the inner product is just that coming, as in (1.2), from the inner product of C^4.

Thus the bundle E on $P_3(C)$ together with its real structure σ gives precisely the data which, according to Chapter IV, corresponds to an anti-instanton on S^4. The bundle R on S^4 has as fibre at the point x the vector space R_x in the decomposition (1.4) with its natural inner product. It remains to specify the connection it inherits. According to Chapter IV the connection is uniquely determined by the condition that, on $P_3(C)$, it should be compatible with both the unitary and holomorphic structures. Now quite generally if $V \subset W$ are holomorphic vector bundles with compatible unitary structures then the canonical connection on V is induced by orthogonal projection from that of W. This is clear because a holomorphic section f of V is also holomorphic in W and so satisfies $\nabla''_W f = 0$ (∇_W being the canonical covariant derivative of W), hence $P\nabla''_W f = 0$ where P is orthogonal projection from W to V. Since this property plus unitarity characterizes the canonical connection we must have $P\nabla_W = \nabla_V$. Dualizing, the same applies to holomorphic quotient bundles. Applying these observations first to the sub-bundle $L^\circ \subset P_3 \times C^4$ and then to the quotient bundle E of L° we see that the canonical connection on E coincides with the connection induced on $L^\circ \cap L^\perp$ from the trivial connection of $P_3 \times C^4$.

From (1.4) we see that $L^\circ_{(z)} \cap L^\perp_{(z)} = R_x$ and so the connection that induced from $S^4 \times C^4$ by orthogonal projection.

In quaternionic notation the decomposition (1.4) can also l

$$C^4 = R_x \oplus R_x^\perp$$

where $x \in S^4 = P_1(H)$ and $R_x^\perp$ is the quaternionic line in H^2 pa by x. Thus R and $R^\perp$ are the standard quaternionic line-bundl discussed in Chapter II and the connection we have given R is t one explained in Chapter II. With our present sign conventioı anti-instanton while $R^\perp$ is an instanton. Both are clearly acted compact symplectic group $Sp(2) \cong \mathrm{Spin}\,(5)$.

If we fix the metric on S^4 as we have done here then we get a uı instanton bundle R. However by applying a conformal transfor will get new anti-instantons. The space of moduli is then $SL(2$ and parametrizes the metrics on S^4 in its standard conformal Alternatively the moduli space can be viewed as the interior o ball in R^5 (the hyperbolic 5-space). This unit ball can be though interior component of the real part of $P_5(C) - Q_4$, the compl space. Note that the exterior component corresponds to a gauge with real singularities on an equatorial 3-sphere (the polar sect external point).

2. – The Horrocks construction.

We shall now give an algebraic construction for $SL(2, C)$ bu $P_3(C)$ which generalizes the linear complex bundle described in This construction is due to G. Horrocks and has been further s W. Barth and K. Hulek [8]. We shall return to it again in Chа

The basic idea can be expressed as follows. In section 1 our arose as the quotient L°/L. We now look for a generalization ir is allowed to be k-dimensional. The details are as follows.

We fix complex vector spaces V, W of respective dimensioп k and we suppose V endowed with a non-degenerate skew-form.

$$(2.1) \qquad A(z)\colon W \to V$$

be a linear map depending linearly on $z = (z_1, z_2, z_3, z_4)$. Thı $= \sum_{i=1}^{4} A_i z_i$ where each $A_i\colon W \to V$ is a constant linear map.

$U_z = A(z)W \subset V$ be the image space and we assume:

(2.2) for all $z \neq 0$, the space U_z is k-dimensional and isotropic for the skew-form on V.

With this assumption we have $U_z \subset U_z^\circ$, the polar space, and hence $E_z = U_z^\circ / U_z$ is a vector space depending algebraically on $(z) \in P_3(C)$. Since $\dim U_z = k$, $\dim U_z^\circ = 2k + 2 - k = k + 2$ and so $\dim E_z = 2$. Moreover E_z inherits a non-degenerate skew-form and so E is an algebraic vector bundle over $P_3(C)$ with group $SL(2, C)$.

Note that, when $k = 1$, U_z is one-dimensional and automatically isotropic. For $k > 1$ however the isotropic condition, which can be written in matrix form as

$$A(z)^t J A(z) \equiv 0 \tag{2.3}$$

where A^t is the transpose and J is the matrix of the skew-form, gives a system of quadratic equations on the coefficients of the 4 matrices A_1, A_2, A_3, A_4. The number of coefficients considerably exceeds the number of equations so that we certainly get solutions.

The assumption that $\dim U_z = k$ for all $z \neq 0$ implies that for any two distinct points (x) and (y) in P_3

$$U_x \cap U_y = 0 . \tag{2.4}$$

To see this we note first that the family of all U_z gives a k-dimensional vector bundle over P_3 isomorphic to the sum of k copies of L, each copy arising from a basis vector of W. Now any non-zero vector in $U_x \cap U_y$ would give rise to a holomorphic section (not identically zero) of U restricted to the line l joining (x), (y) in P_3. But $U|l$ is isomorphic to k copies of $L|l$ and this bundle has no holomorphic section besides zero (recall that, over any $P_m(C)$, a holomorphic section of L^* is a linear form in the coordinates, a section of L^{-n} is a homogeneous polynomial of degree n, and therefore L has only the zero section).

Now let us try to identify the jumping lines of the bundle E on P_3. Consider first a line l joining points (x) (y) such that

$$U_x^\circ \cap U_y = 0 . \tag{2.5}$$

This implies that

$$R = U_x^\circ \cap U_y^\circ \tag{2.6}$$

is a 2-dimensional complement to U_z in U_z° for all points (z) on the line l. This shows that $E|l$ is trivial and so l is not a jumping line. Suppose now

that (2.5) is false then the space R given by (2.6) has non-zero in
with U_y. Let v be a non-zero vector in $R \cap U_y$, then accordin
$v \notin U_x$ and so v defines an algebraic section of $E|l$ which is zero at y
zero at x. This shows that $E|l$ is not trivial and so l is a jumping l
the jumping lines are precisely the lines for which (2.5) fails.

To introduce reality conditions we now assume that we have
linear map σ acting on W, V with $\sigma^2 = +1$ on W and $\sigma^2 = -$
Moreover we require that σ preserve the skew-form on V and be
the corresponding hermitian form

$$\langle u, v \rangle = (u, \sigma v) \tag{2.7}$$

is positive. Our linear transformation $A(z)$ is now assumed to b
tible with σ, i.e.

$$\sigma\{A(z)\omega\} = A(\sigma z)(\sigma\omega) \tag{2.8}$$

where $\sigma(z)$ is, as before, multiplication by the quaternion j.

Condition (2.7) implies that

$$U_{\sigma z} = \sigma\{U_z\} \tag{2.9}$$

and (2.7) implies that

$$U_z^\circ = U_{\sigma z}^\perp . \tag{2.10}$$

Hence we get an orthogonal decomposition:

$$V = U_z \oplus R_x \oplus U_{\sigma z} \tag{2.11}$$

where $R_x = U_z^\circ \cap U_{\sigma z}^\circ$ depends only on the point $x \in S^4$ paramet
real line $(z)(\sigma z)$ of P_3. This shows in particular that the real lin
a jumping line for E and that E inherits a real structure from
properties required, as in Chapter IV, to give an anti-instant
Exactly as for the case $k = 1$ we see that the anti-instanton bu
is R with the connection induced by orthogonal projection fro

The instanton number of the bundle R over S^4 is $-k$. This fol
the fact that its orthogonal complement is topologically equival
sum of k copies of the basic 1-instanton bundle.

To generalize from $Sp(1)$ to $Sp(n)$ all we have to do in the
construction is to take V to have dimension $2k + 2n$. For the gr
we have to take V of dimension $2k + n$ and we have to alter the si
we require V to have a non-degenerate symmetric bilinear form and
on V and $\sigma^2 = -1$ on W. In all cases the instanton number is

We can deal with $U(n)$ by regarding it in the standard way as the subgroup of $SO(2n)$ commuting with J where $J^2 = -1$. We therefore require that V, W should both possess a complex linear automorphism J with $J^2 = -1$, $\sigma J = -J\sigma$. Moreover J is assumed to preserve the inner product on V or equivalently $(Ju, Jv) = -(u, v)$. Finally the linear transformation $A(z)\colon W \to V$ is required to commute with J. Then the decomposition (2.10) is J-invariant. Hence the bundle R has an action of J and (since J preserves inner products) the canonical connection of R is preserved by J. Hence the connection of R, which is anti-self-dual, reduces from $SO(2n)$ to $U(n)$.

Notice that we could proceed one stage further and regard $Sp(n) \subset SO(4n)$ and so describe $Sp(n)$ solutions as $SO(4n)$ solutions with extra structure. However the more direct approach to $Sp(n)$ used earlier is more economical.

Returning now to the symplectic case we see that two triples (W, V, A) (W', V', A') which are isomorphic, in the obvious sense that we have isomorphisms $W \cong W'$, $V \cong V'$ commuting with σ and the skew-forms on V, V', and taking A to A', will give rise to an isomorphism between the bundles R and R' preserving connections. Thus isomorphic triples in the linear algebra sense yield gauge equivalent solutions of the anti-instanton problem. As we shall prove later the converse is also true.

3. – Quaternionic formulae.

In this section we shall show how the construction of the preceding section, for $Sp(n)$, can be formulated in terms of quaternions. We then obtain the explicit formulae given in Chapter II.

We recall that we have identified the space C^4 with H^2 in such a way that the map σ on C^4 is given by left multiplication on H^2 by the quaternion j. Similarly the vector space V of dimension $2k + 2n$ can be viewed as a left quaternion vector space of dimension $k + n$ with j given by the anti-linear map σ.

The vector space W, of complex dimension k, has σ with $\sigma^2 = 1$ and so can be viewed as the complexification of the real vector space W_R left fixed by σ. Then $C^4 \otimes_C W \cong H^2 \otimes_R W_R$ and σ corresponds again to multiplication by j on the quaternion vector space $H^2 \otimes_R W_R$.

The linear map $A(z)\colon W \to V$ can now be viewed as a map

$$A\colon H^2 \otimes_R W_R \to V$$

which is quaternion linear. This corresponds to the compatibility of $A(z)$ with σ.

If we take a real basis of W_R and an orthogonal H-basis of V, ε
gets identified with H^{k+n}, then A is described by two matrices C, I
ternions. The row-vectors of C are the image under A of $(1, 0) \otimes 1$
tors of W_R and D is similarly defined replacing $(1, 0)$ by $(0, 1)$ in
use matrices as right multipliers here since our scalars act on th
that C, D are $k \times (k + n)$ matrices. Regarded as a matrix functi
pair (x, y) of quaternions in H^2 we then have

$$A(x, y) = xC + yD\,. \tag{3.1}$$

The non-degeneracy condition on A is just that this matrix have
rank for all $(x, y) \neq (0, 0)$. Finally the isotropy condition (2.2) is ec
using (2.10), to saying that for any two row-vectors u, v of the ma
u and iu are orthogonal to jv and $j(iv)$. This means that

$$\mathrm{Re}\,(uv^* j) = \mathrm{Re}\,(uv^* k) = \mathrm{Re}\,(uv^* i) = 0$$

or equivalently that the quaternion uv^* is real. Hence (2.2) is equi

(3.2) for all $(x, y) \in H^2$, the matrix $(xC + yD)(C^* \bar{x} + D^* \bar{y}$

is real.

We thus recover the description given in Chapter II, Section
that we have transposed our matrices since we are now consid
action by scalars. For this reason we now get anti-instantons i
the instantons of Chapter II.

CHAPTER VI

Linear Field Equations

1. – Bundles and sheaf cohomology.

The theory of holomorphic bundles is intimately related to the cohomology theory of sheaves. Although this cohomology theory arises in various contexts, and can be developed independently of bundle theory, it is rather natural from our point of view to explain it in terms of bundles. The main thing to emphasize just now is that cohomology is a linear or abelian theory and it first arises from abelian gauge groups and, at a later stage, from solvable groups.

We shall begin, as in Chapter IV, § 3, by considering holomorphic bundles over $P_1(C)$. This simple case provides a good illustration of the general theory and ties up with the classical function theory of one complex variable. In addition this special case occupies a key role in our applications, since $P_1(C)$ appears as the fibre of our basic fibration $P_3(C) \to S^4$.

In Chapter IV, § 3, we explained how a holomorphic line-bundle over $P_1(C)$ given by the holomorphic gauge-transformation $f(z)$, near the equator, could be reduced, as in (3.1), to the standard form z^k. In particular when $k = 0$ we could take $g(z) = \log f(z)$ and using the Laurent expansion

$$g(z) = \sum_{-\infty}^{\infty} a_n z^n \qquad r < |z| < R \tag{1.1}$$

we could write $g(z)$ in the form

$$g(z) = g_0(z) - g_\infty(z) \tag{1.2}$$

with $g_0(z)$ holomorphic for $|z| < R$ and $g_\infty(z)$ holomorphic for $r < |z|$. Suppose now we ask instead whether we can write $g(z)$ in the form

$$g(z) = z^m g_0(z) - g_\infty(z) \tag{1.3}$$

where m is some fixed integer, and g_0, g_∞ are as before holomor[
0, ∞ respectively. Clearly if $m < 1$ this is possible while for $m > 2$
sible if and only if the coefficients of (1.1) satisfy:

$$a_1 = \ldots = a_{m-1} = 0 .$$

In other words the functions $z, \ldots, z^{m-1}$ in general provide obstru
the solubility of (1.2). Alternatively the space of all holomorp
tions $g(z)$, modulo those which can be written in the form (1.3), h
sion $m-1$ and a basis is represented by $z, \ldots, z^{m-1}$. This is our first
with a *sheaf cohomology group*: it is usually denoted by $H^1(P$
$H^1(P_1, \mathcal{O}(-m))$. The reasons for these notations will be explaine

Equation (1.2) asserts that a holomorphic bundle over $P_1(C)$ wit
(the additive group) is holomorphically trivial: by exponentiation
responds to bundles with group C^* (the multiplicative group) i
the topological invariant $k = 0$. We can also consider C as th
subgroup of translations in $SL(2, C)$, i,e. matrices of the form

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

Then (1.2) says that holomorphic bundles for this group over
trivial. Geometrically a bundle for this matrix group correspo
2-dimensional holomorphic vector bundle E over $P_1(C)$ having a
bundle N such that N and the quotient E/N are both trivial. The
of (1.2) then asserts that N always possess a holomorphic com
Recall that, unlike unitary bundles, holomorphic complements d
cessarily exist. In fact (1.3) corresponds to a slightly more gene
tion of this type. To see this note that eq. (1.3) is equivale
matrix equation

$$\begin{pmatrix} 1 & g \\ 0 & z^m \end{pmatrix} = \begin{pmatrix} 1 & g_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^m \end{pmatrix} \begin{pmatrix} 1 & g_\infty \\ 0 & 1 \end{pmatrix}^{-1}. \tag{1.4}$$

If we regard the left-hand side as the gauge transformation, nea
defining a 2-dimensional vector bundle E over $P_1(C)$, then E has
sub-bundle N with $E/N \cong L^{-m}$, and solubility of (1.4) is equival
existence of a holomorphic complement to N. Thus for $m > 2$ su
plement does not generally exist. More precisely any such bund
an element of the vector space $H^1(P_1, L^m)$ and a complement e
if this element is zero. More generally $H^1(P_1, L^m)$ classifies the ext
of the trivial bundle by L^{-m}, two such extensions being regarded as e

if there is an isomorphism between the bundles which is the identity on the sub-bundle and quotient bundle.

After this introduction we shall now give the precise definition of sheaf cohomology groups on any compact complex manifold X, e.g. $P_3(C)$. We begin with the simplest case, corresponding to (1.2). Instead of just the 2 open sets $|z| < R$ and $r < |z|$ which cover $P_1(C)$ we now have to allow a finite number of open sets $\{U_\alpha\}$ to cover X: for $P_3(C)$ we could take the 4 sets given by $z_\alpha \neq 0$ $(\alpha = 1, 2, 3, 4)$. A holomorphic 1-cochain is then a collection of holomorphic functions $g_{\alpha\beta}$ defined in $U_\alpha \cap U_\beta$ (conventionally $g_{\beta\alpha} = -g_{\alpha\beta}$). Such a 1-cochain is called a 1-cocycle if it satisfies the transitivity condition

$$g_{\alpha\beta} = g_{\alpha\gamma} + g_{\gamma\beta} \quad \text{in } U_\alpha \cap U_\beta \cap U_\gamma . \tag{1.5}$$

It is called a 1-coboundary if there exist holomorphic functions h_α defined in U_α so that

$$g_{\alpha\beta} = h_\alpha - h_\beta \quad \text{in } U_\alpha \cap U_\beta . \tag{1.6}$$

Every 1-coboundary is evidently a 1-cocycle but the converse need not be true. We measure the effect by considering the quotient space H^1 of 1-cocycles modulo 1-coboundaries. If the open sets U_α are sufficiently small and well-chosen (technically all finite intersections should be domains of holomorphy) H^1 is independent of the covering. It is called the first cohomology group of H with coefficients in the sheaf of holomorphic functions and is denoted by $H^1(X, \mathcal{O})$.

We cannot pursue cohomology theory in any detail, for which we refer to standard texts, and we confine ourselves for the present to a few further comments.

1) For $P_1(C)$ we used just 2 open sets, hence there was only one $g_{\alpha\beta}$ and every 1-cochain was automatically a 1-cocycle.

2) $H^1(X, \mathcal{O})$ always classifies holomorphic bundles over X with group C or equivalently extensions of the trivial line-bundle by itself.

3) For every integer $q > 0$ one can define $H^q(X, \mathcal{O})$: $H^0(X, \mathcal{O})$ consists of global holomorphic functions on X (which are constant for compact X). The cohomology groups are linked together by exact sequences.

4) If X is not compact similar definitions work but one must allow infinite coverings which are locally finite.

5) For X compact algebraic, as for $P_m(C)$ for any m, c
groups can be computed by using only rational functions. This
the main results of Serre [38].

It is implicit in our notation that other coefficients than the h
functions $\mathcal{O}$ can be used to form cohomology groups. Thus in the ca
the cohomology group we denoted by $H^1(P_1, L^m)$ uses local h
sections of the holomorphic line-bundle L^m. Clearly the definit
of H^1 goes through unaltered for any X and any holomorphic lin
or even vector bundle E over X. The cohomology group $H^1(X,$
viewed as classifying extensions of the line-bundle L by the t
bundle. More generally holomorphic vector bundles E of any
which have a given sub-bundle E_1 and quotient $E/E_1 = E_2$ ar
by $H^1(X, E_2^* \otimes E_1)$ where E_2^* is the dual bundle of E_2.

If X is compact all H^q of the type considered here are *finite-d*
vector spaces. This ceases to be true for non-compact X. Fo
$H^0(X, \mathcal{O})$ consists of all holomorphic functions on X. Example
being infinite-dimensional will occur naturally in our applicati
shall see in the next section.

2. – Linear aspects of the Penrose transform.

The main result of Chapter III was that solutions of the ant
Yang-Mills equations on S^4 converted, via the Penrose transfor
holomorphic bundles on $P_3(C)$. There is a parallel twistor interpr
solutions of certain important *linear* differential equations. We
in this section that they correspond to elements of appropriate s
mology groups.

We begin by considering the case of U(1) Yang-Mills theory.
there are no global anti-self-dual solutions on S^4, solutions cert
in R^4 corresponding to solutions of the Euclidean Maxwell equat
2-form ω: $d\omega = 0$, $*\omega = -\omega$. From Chapter III we know that
respond to holomorphic line-bundles on $P_3(C) - P_1(C)$. If we ignor
and work with complex-valued 2-forms the holomorphic line-
unrestricted, except that it must be topologically trivial. As ex
Section 1 such a line-bundle is then described by an element of
mology group $H^1(P_3 - P_1, \mathcal{O})$. Thus this group corresponds
Penrose transformation to the solutions of the anti-self-dual Max
tions on R^4. This is the type of correspondence which we propose
As we shall see all the cohomology groups $H^1(P_3 - P_1, \mathcal{O}(-m))$
in a similar way to solutions of other linear differential equatio

Before leaving the case of Maxwell fields we should comment on one special topological feature, related to the bundle interpretation, which does not generalize to the other cases. If we work with some open set $U \subset R^4$ and the corresponding open set $\tilde{U} \subset P_3(C)$ then holomorphic line-bundles on $\tilde{U}$ which are trivial on the fibres of $P_3(C) \to S^4$ correspond to anti-self-dual C^*-potentials on U. If U is R^4 or a contractible open set then the homology of $\tilde{U}$ is generated by the fibres and the line-bundle on $\tilde{U}$ is then topologically trivial. However if U has 2-dimensional homology, e.g. if $U = R^4 - R^1$ then the bundle on $\tilde{U}$ need not be topologically trivial and the corresponding bundle (with connection) on U is then also non-trivial. We cannot in such a case take the logarithm of the gauge transformations to reduce from C^* to C. Thus the geometrical correspondence between bundles given in Chapter III contains more information than the correspondence described above between $H^1(\tilde{U}, \mathcal{O})$ and solutions of the anti-self-dual Maxwell equations on U.

The cohomology groups $H^1(\tilde{U}, \mathcal{O}(-m))$ fall naturally into two families according as $m > 2$ or $m < 2$. The case $m = 2$ is in a sense the most basic and the situation is essentially symmetric about this case, as we shall see. For example $H^1(\tilde{U}, \mathcal{O}(-4))$ will turn out to correspond to solutions of the self-dual Maxwell equations on U.

We begin therefore with the case $m = 2$ and we take any element $\Phi \in H^1(\tilde{U}, \mathcal{O}(-2))$. If P_x is the fibre of $P_3(C) \to S^4$ over the point $x \in U \subset S^4$ we can restrict Φ to P_x to get an element

$$\varphi_x \in H^1(P_x, \mathcal{O}(-2)).$$

As explained in Section 1

$$\dim H^1(P_x, \mathcal{O}(-2)) = 1$$

so that if we fixed, in some standard way, a basis of this space φ_x would become a scalar function of x, defined for $x \in U$. If $U \subset R^4$ then $\tilde{U} \subset P_3(C) - P_1(C)$ has a natural map to $P_1(C)$, projecting along parallel planes, and this identifies all P_x with $x \in U$. Thus in this case φ_x gives a scalar function φ defined in U. Analogy with the Maxwell fields suggests that φ should satisfy a differential equation and the only candidate with the necessary invariance properties is the Laplace operator of R^4. In fact the correspondence $\Phi \leftrightarrow \varphi$ establishes an isomorphism between $H^1(\tilde{U}, \mathcal{O}(-2))$ and the space of solutions of the Laplace equation in U. When $U = R^4$ both spaces are acted on by the 11-parameter group of conformal motions of R^4 (the Euclidean motions together with scalar magnification about an origin) and the isomorphism is compatible with these actions. In particular the subspace

of $H^1(P_3(C) - P_1(C), \mathcal{O}(-2))$ which is homogeneous of degree scale change, corresponds to homogeneous polynomials of degree which satisfy the Laplace equation. We shall indicate how this can be verified.

Once we have fixed an origin in R^4, $P_3(C) - P_1(C)$ can be view normal bundle N of the corresponding projective line. The bund is trivial or equivalently $N \cong L^* \otimes S^-$, where $S^- \cong C^2$ is a fixe space. As explained in [3] (where a different notation is used), naturally identified with one of the half-spin spaces of R^4. Hol functions on N which are homogeneous, in each fibre, of degree sheaf over P_1 isomorphic to the sections of $L^n \otimes \mathcal{T}^n(S^-)$, where $\mathcal{T}$ the polynomials of degree n (so $\mathcal{T}^n(S^-) \cong C^{n+1}$). Since we want to $H^1(N, L^2) = H^1(N, \mathcal{O}(-2))$ we tensor with L^2 and deduce a ma

$$H^1(P_1, L^{n+2}) \otimes \mathcal{T}^n(S^-) \to H^1(P_3 - P_1, \mathcal{O}(-2))_n \tag{2.1}$$

where the suffix n on the right denotes the part of the cohomolo is homogeneous of degree n. By general theorems one can deduce is actually an isomorphism. Now we saw in Section 1 that H^1 was of dimension $n + 1$. With a little more care (see section 4) naturally identify it with $\mathcal{T}^n(S^+)$, where S^+ is the other half-spin sp Now $S^+ \otimes S^- \cong R^4 \otimes C$, the complexification of R^4. Hence (2.1) sh the space of polynomials φ corresponding to elements $\Phi \in H^1(P_3 - P_1$, consists precisely of the image of

$$\mathcal{T}^n(S^+) \otimes \mathcal{T}^n(S^-) \to \mathcal{T}^n(C^4)\,. \tag{2.2}$$

The classical invariant theory of $SO(4)$ then shows that the imag is just the space of harmonic polynomials; for example when $n =$ side of (2.2) has dimension 9 while the right side has dimension 10, c the extra polynomial $r^2 = \sum_1^4 x_i^2$.

The preceding piece of algebra can be used as a basis for th correspondence $\Phi \leftrightarrow \varphi$ by associating to each of Φ, φ a power seri sion in homogeneous terms. Appropriate analytic questions of cor have then to be examined but these can be dealt with by standard Note that such convergence questions are irrelevant when we wo whole of S^4 since all spaces are then finite-dimensional.

If we work on the whole of S^4 the different fibres P_x cannot all tified (the bundle $P_3(C) \to S^4$ being topologically non-trivial) and care has to be taken in the interpretation of the correspondenc

We must now view φ as a section of the line-bundle W over S^4 whose fibre W_x is the one-dimensional space $H^1(P_x, \mathcal{O}(-2))$. One can show that W^4 is the volume of density bundle on S^4: thus φ is the fourth root of a density. The Laplace operator has a conformally invariant counterpart [34] which acts on such fourth-roots of densities and $\Phi \leftrightarrow \varphi$ is an isomorphism between $H^1(P_3, \mathcal{O}(-2))$ and the space of solutions of this conformally invariant Laplace equation on S^4. Written in terms of the standard metric of S^4 this operator is $\Delta + R/6$, where Δ is the Laplace-Beltrami operator d^*d of S^4 and R is the scalar curvature [34], which vanishes on R^4 but is positive on S^4.

Since $\Delta \geqslant 0$ as an operator, $\Delta + R/6 > 0$ and so the equation $(\Delta + R/6)\varphi = 0$ has no global solutions on S^4 except $\varphi = 0$. This checks with the well-known result of algebraic geometry that $H^1(P_3, \mathcal{O}(-2)) = 0$: in fact the same is true for all $\mathcal{O}(-m)$.

In a later section we shall have more to say about the Laplace operator in relation to the Penrose transform. Returning for the present to the interpretation of other sheaf cohomology groups, we get a very similar picture for the groups $H^1(\tilde{U}, \mathcal{O}(-m))$ $m > 2$. Any element Φ of this cohomology group gives rise to an element $\varphi_x \in H^1(P_x, \mathcal{O}(-m))$ for all $x \in U$. As shown in section 1 this vector space (in which φ_x takes its values) has dimension $m-1$. Thus, for $U \subset R^4$, φ is an $(m-1)$-component function and it satisfies a first-order differential equation. For $m = 3$ this is the (mass-less) Dirac equation and for $m = 4$ it is the (self-dual) Maxwell equation. All these can be exhibited in conformally invariant form.

For $m < 2$ we have already met the case $m = 0$ in a different context. For other values one has to proceed slightly differently. Restricting Φ to any P_x now gives zero since $H^1(P_x, \mathcal{O}(-m)) = 0$ for $m < 1$. However the isomorphism (2.1) still holds with 2 replaced by m. In particular taking $n = 2 - m$ we see that

$$\mathcal{S}^{2-m}(S^-) \cong H^1(P_3 - P_1, \mathcal{O}(-m))_{2-m} . \tag{2.3}$$

Based on this isomorphism we can now assign to any $\Phi \in H^1(P_3 - P_1, \mathcal{O}(-m))$ a k-component function φ on R^4, where $k = 3 - m$. This is done as follows. Given $x \in R^4$ we take this as origin for scalar magnification and then pick out the component Φ_{2-m} of Φ. By (2.3) we get a vector in the k-dimensional space $\mathcal{S}^{2-m}(S^-)$. An alternative description is to consider the ν-th formal neighbourhoods $P_x^{(\nu)}$ of the line P_x: this means that we include not only values on P_x but also normal derivatives up to order ν. Taking $\nu = 2 - m$ we restrict Φ to give an element

$$\varphi_x \in H^1(P_x^{(\nu)}, \mathcal{O}(-m)) .$$

It is not hard to show that this is equivalent to the previous defi

Again one shows that the functions φ arising this way are pr solutions of the appropriate first-order differential equation. I we again get the (mass-less) Dirac equation but this time for th type of spinor from the case $m = 3$. In general the equations for m are essentially the same, except that they are of opposite type when we reverse orientation.

3. – Linear equations in a Yang-Mills background.

As we saw in Chapter IV a vector bundle E on S^4 with an connection lifts to give a bundle $\tilde{E}$ on $P_3(C)$ which has a natural h structure. As in the preceding section we can then form the s mology groups $H^1(P_3, \tilde{E}(-m))$ or their counterparts over an It is natural to look for an interpretation of these vector space of the original bundle E on S^4. When E is a trivial bundle the section 2 tell us that we get spaces of solutions of certain stanc differential equations. These differential equations have obviou parts for a non-trivial bundle in which the derivatives are repla propriate covariant derivatives built from the connection. It i tremely plausible that our sheaf cohomology groups will corr these covariant differential equations, and in this section we shall this is indeed the case.

As mentioned in Chapter IV the vector bundle $\tilde{E}$ is trivial line P_x, for $x \in S^4$, and furthermore is trivial along the first for bourhood $P_x^{(1)}$. We shall first give the proof of this statement. I is any holomorphic vector bundle on P_3, defined in a neighbou fixed projective line P_1, triviality of V restricted to P_1 implies triv restricted to $P_1^{(1)}$. To see this we consider first the exact sequence

$$0 \to J^2/J \to \mathcal{O}/J^2 \to \mathcal{O}/J \to 0 \tag{3.1}$$

where $\mathcal{O}$ is the sheaf of holomorphic functions and J is the ideal s (consisting of functions vanishing on P_1). The three sheaves in then be equivalently described as follows:

$\mathcal{O}/J = \mathcal{O}(P_1)$, holomorphic functions on P_1

$\mathcal{O}/J^2 = \mathcal{O}(P_1^{(1)})$, holomorphic « functions » on $P_1^{(1)}$

$J^2/J =$ sheaf of sections of the co-normal bundle N^* of P_1 in

If $f \in \mathcal{O}/J^2$ then it has a value on P_1, namely its image in $\mathcal{O}/J$, and extra components corresponding to the first-order terms in the Taylor expansion normal to P_1. These first-order terms lie in J^2/J.

Since the normal bundle $N \cong L^{-1} \oplus L^{-1}$ we have $N^* \cong L \oplus L$ and so in particular

$$H^0(P_1, N^*) = H^1(P_1, N^*) = 0$$

(see § 1). From the cohomology exact sequence of (3.1) this implies that the global sections of $\mathcal{O}/J^2$ map isomorphically onto those of $\mathcal{O}/J$: in other words a global function on $P_1^{(1)}$ is constant, as we should expect.

Suppose now we introduce the vector bundle V, assumed trivial on P_1, and tensor (3.1) with V. We get an exact sequence:

$$0 \to V^{(0)} \otimes N^* \to V^{(1)} \to V^{(0)} \to 0 \qquad (3.2)$$

where $V^{(0)}$, $V^{(1)}$ denote the restriction of V to P_1 and $P_1^{(1)}$ respectively. Since $V^{(0)}$ is trivial we again have

$$H^0(P_1, V^{(0)} \otimes N^*) = H^1(P_1, V^{(0)} \otimes N^*) = 0$$

and (3.2) then implies as before that global sections of $V^{(1)}$ map isomorphically onto global sections of $V^{(0)}$. This means that a basis of (constant) sections of the trivial bundle $V^{(0)}$ has a unique extension to $V^{(1)}$, showing in particular that $V^{(1)}$ is trivial as claimed.

When $V = \tilde{E}$ arises from a bundle E on S^4 as in Chapter IV the fibre E_x can be identified with the space of global sections of $V_x^{(0)}$ (i.e. the restriction of V to P_x). Global sections of $V_x^{(1)}$ give vectors in $E_x^{(1)}$, the first jet space of sections of E at x (i.e. including first derivatives). The fact that every global section of $V_x^{(0)}$ extends uniquely to a global section of $V_x^{(1)}$ means that we have a unique way of extending vectors from E_x to $E_x^{(1)}$, i.e. we have a connection on E. This is essentially Ward's definition of the connection as explained in Chapter IV. Note that the property of quadric cones used in Chapter IV, § 3 for Ward's construction is equivalent to the vanishing of $H^1(P_3, \mathcal{O}(-1))$ as one sees by an exact sequence argument; a more elementary argument can however be used based on the fact that a quadric surface factorizes as $P_1 \times P_1$.

We turn now to the question of interpreting the sheaf cohomology groups $H^1(\tilde{U}, \tilde{E}(-m))$ where $\tilde{U} \subset P_3(C)$ corresponds to an open set $U \subset S^4$. We begin by considering the case $m > 2$. Then for any $x \in U$, an element

$\Phi \in H^1(\tilde{U}, \tilde{E}(-m))$ defines

$$\varphi_x \in H^1(P_x, \tilde{E}_x^{(0)}(-m)) \cong E_x \otimes H^1(P_x, \mathcal{O}(-m))$$

(since $\tilde{E}$ is trivial on each P_x). In other words φ is a section
where W_m is the vector-bundle (with fibre C^{m-1}) which we i
previous section. For $m > 2$ there is a first-order differential e
sections of W_m whose solutions correspond to elements of $H^1(\tilde{U}$
This differential equation may be identified by computing in the
neighbourhood $P_x^{(1)}$. Repeating this computation with $\tilde{E}(-m)$,
triviality of $\tilde{E}_x^{(1)}$, and the way in which this triviality defines the
on E, it follows that φ satisfies the obvious differential equation i
of $E \otimes W_m$. For example if $m = 3$, W_m is the 2-component sp
the equation is the (mass-less) Dirac equation and the equation
is the Dirac equation extended to E using its connection. In mo
terms this is the Dirac equation in a Yang-Mills background fi

For $m < 2$, the equations are again first-order, and a similar l
more complicated argument leads to the same conclusion.

Finally we come to the case $m = 2$ which is both the most
and technically the most complicated since it involves a second-
ator, the Laplacian. General considerations, and computations of
type, lead to the conclusion that elements $\Phi \in H^1(\tilde{U}, \tilde{E}(-2))$
to sections of $E \otimes W_2$ over U (where W_2 is the line-bundle of f
of densities as in Section 2) which satisfy a second-order differentia
Using the triviality of $\tilde{E}$ on $P_x^{(1)}$ identifies the first-order and se
parts of this equation but leaves unidentified the zero-order pa
calculation is therefore necessary to determine this zero-order
most elegant and informative way of performing this calculation is
in detail in [29]. The idea is to exploit the inter-relation betwe
ferent cohomology groups $H^1(\tilde{U}, \tilde{E}(-m))$ for various m, arising
fact that multiplication by each of the linear coordinates $z_1, .$
$\tilde{E}(-m)$ to $\tilde{E}(-m+1)$. Formally we note that $z_1, \ldots, z_4$ are
the holomorphic sections over $P_3(C)$ of the bundle L^{-1}. Conv
statement into one on S^4 provides a link between the differential
for various integers m. Knowledge of the equation for $m \neq 2$ ca
used to help identify the equation for $m = 2$. In more concrete
second-order operator has to annihilate elements of the form z_α
satisfies the appropriate Dirac equation (corresponding to $m = 3$).
in [29] this information is enough to identify our equation as

$$\left(\nabla^*\nabla + \frac{R}{6}\right)\varphi = 0 \qquad (3.3)$$

where ∇ is the covariant derivative of φ, ∇^* its adjoint relative to the standard metric of S^4, and φ is now considered as a pure scalar function (densities being identified with scalars by using the volume form of S^4).

An important consequence of (3.3) is that it has no global solutions on P_3 except $\varphi = 0$. This is because the operator $\nabla^*\nabla + R/6$ is positive just as in Section 2. We conclude that

$$H^1(P_3(C), \tilde{E}(-2)) = 0 . \tag{3.4}$$

Unlike the case when $\tilde{E}$ is trivial (3.4) is not a general result of algebraic geometry. There are many vector bundles V which do not satisfy (3.4). All that general theory tells us is that

$$H^1(P_3, V(-m)) = 0 \quad \text{if } |m| > m_0(V)$$

where m_0 is a sufficiently large integer. The proof of (3.4) depends crucially on the fact that $\tilde{E}$ comes from a unitary bundle on S^4.

For $m > 2$ by composing the first-order operator with its adjoint one gets a second-order operator which can also be shown to be positive. This leads to the further result that:

$$H^1(P_3(C), \tilde{E}(-m)) = 0 \qquad m > 2 . \tag{3.5}$$

However, as we shall see in Chapter VII, (3.5) follows easily from (3.6). The reason is that multiplication by a monomial of degree $m-2$ in $z_1, \ldots, z_4$ shifts us from $H^1(P_3(C), \tilde{E}(-m))$ to $H^1(P_3(C), \tilde{E}(-2))$.

So far we have explained that sheaf cohomology groups correspond to solutions of various linear equations in the background field of Yang-Mills instantons. For quantum calculations one needs, not only the solutions of the homogeneous linear equation, but the full Green's function or « propagator ». Using the explicit description of instantons given in Chapter II explicit and fairly simple formulae have been derived for some of these Green's functions [13] [15].

4. – The 't Hooft Ansatz.

This section is a digression in which we shall explain how solutions of the anti-self-duality equations can be constructed from solutions of linear equations. This point of view was explained in [4] and it leads to the 't Hooft Ansatz and natural generalizations of it.

We shall consider the group $SL(2, C)$, ignoring for the pres constraints. Equivalently we work with 2-dimensional vect with a skew form. According to the results of Chapter IV an $SL(2, C)$ connections on $U \subset S^4$ correspond to holomorphic vect on $\tilde{U} \subset P_3(C)$ which are trivial along all fibres P_x for $x \in U$. O construct such vector bundles is to produce extensions of line-l other words to use only the triangular matrices in $SL(2, C)$ for g formations. As explained in Section 1 such extensions are classif ments of a sheaf cohomology group. Thus we can construct $SL(2,$ by first picking a line-bundle N on $\tilde{U}$ and then choosing an ele $H^1(\tilde{U}, N^2)$. This gives a vector bundle V with N as sub-bund as quotient.

In particular we can take N to be the standard line-bundle L its powers L^m. Then to every $\Phi \in H^1(\tilde{U}, L^{2m}) = H^1(\tilde{U}, \mathcal{O}(-2m))$ holomorphic vector bundle over $\tilde{U}$. To descend to S^4 we need bundle to be trivial on each P_x with $x \in \tilde{U}$. To investigate this q consider a fixed projective line P_1 and ask which elements of $H^1(P_1,$ give the trivial bundle. Note for example that the zero element direct sum $L^m \oplus L^{-m}$ which is never trivial (for $m \neq 0$). As we a general element always gives the trivial bundle but special ele rise to direct sums $L^r \oplus L^{-r}$ for $r = 1, 2, \ldots, m$.

For simplicity consider first the case $m = 1$, then $H^1(P_1,$ one-dimensional and every non-zero Φ gives the trivial bundle [1] next the case $m = 2$, then $H^1(P_1, \mathcal{O}(-4))$ is a 3-dimensional Section 1 we saw that a basis is given by the transition functio cycles) z, z^2, z^3. A more invariant description of the space $H^1(P$ is to say that it is the dual of the space of quadratic forms in the geneous coordinates (z_1, z_2) of P_1. This arises because of the mul

$$H^0(P_1, \mathcal{O}(2)) \otimes H^1(P_1, \mathcal{O}(-4)) \to H^1(P_1, \mathcal{O}(-2)) \cong C .$$

Using our basis it is easy to verify that this multiplication gives th duality. This is in fact a very special case of the much more gene theorem of Serre [39]. Thus if $P_1 = P_1(V)$, i.e. V is the copy of lines are represented by points of P_1, we have a natural isomor

$$H^1(P_1, \mathcal{O}(-4)) \cong S^2(V)^* \cong S^2(V^*) . \tag{4.1}$$

Hence elements $\Phi \in H^1(P_1, \mathcal{O}(-4))$ represent quadratic form Then Φ defines the trivial bundle over P_1 if and only if the corr quadratic form on V^* is non-singular (i.e. not a perfect squa $\Phi \neq 0$ but corresponds to a perfect square the bundle is isomorphic t

The generalization of this to arbitrary $m > 0$ is now fairly straightforward. We have a natural isomorphism

$$H^1(P_1, \mathcal{O}(-2m)) \cong \mathcal{S}^{2m-2}(V^*) . \tag{4.2}$$

If $\Phi \in H^1(P_1, \mathcal{O}(-2m))$, let f be the corresponding homogeneous polynomial of degree $2m-2$ on V^*. Any such polynomial can be written as a linear combination of perfect powers:

$$f = \sum_{i=1}^{m-2} \lambda^i u_i^{2m-2}$$

where $\lambda_i \in C$ and u_i is a linear form on V^*. The general result proved in [1] is that Φ defines the trivial bundle unless f can be expressed as a combination of less than $m-2$ perfect powers (or is a limit of such). More generally if f is a combination of $m-2-r$ perfect powers (or a limit of such), with r being maximal, then Φ defines the bundle $L^r \oplus L^{-r}$.

We know therefore the precise conditions to be imposed on an element $\Phi \in H^1(\tilde{U}, \mathcal{O}(-2m))$ in order to get a trivial bundle on all P_x with $x \in U$ and so to descend to a bundle on U with anti-self-dual $SL(2, C)$-connection. By the results described in Section 2, Φ corresponds to a section of the vector bundle $\mathcal{S}^{2m-2}(S^+)$ satisfying the appropriate linear differential equation. Here S^+ denotes the half-spin bundle corresponding to instantons, whereas S^- corresponds to anti-instantons. We have also used the metric on S^4 or R^4 to identify S^+ with its dual $(S^+)^*$, although to exhibit full conformal invariance this should not be done.

To sum up we have given an implicit construction for anti-self-dual $SL(2, C)$ connections starting from a section φ of $\mathcal{S}^{2m-2}(S^+)$ satisfying the relevant differential equation. To get a solution this section must be everywhere « general » i.e. not a combination of less than $(m-2)$ perfect powers.

For $m = 1$, φ is a scalar field satisfying the Laplace equation and nowhere zero. For $m = 2$, φ is a self-dual solution of Maxwell's equations which is nowhere the square of a spinor.

The case $m = 1$ is the Ansatz employed by 't Hooft. The explicit formulae for $m = 2$ are given in [4]. It is important to note that the solution of the non-linear Yang-Mills equations obtained in such a way may have a larger domain of regularity than the linear field φ from which we start. For example it is well-known, for the 't Hooft Ansatz, that a $1/r^2$ singularity for φ disappears when we construct the Yang-Mills field. Geometrically this corresponds to the fact that the reduction from $SL(2, C)$ to the triangular matrices may only be valid in a smaller open set. The situation is quite clear if we work on the whole of $P_3(C)$ as we shall now explain.

Let W be a 2-dimensional holomorphic vector bundle o
Although W may have no globally defined holomorphic section,
zero, general theorems tell us that $W \otimes L^{-m} = W(m)$ has a nor
tion if m is sufficiently large. Let s be such a section, then $s =$
closed algebraic subariety Σ of $P_3(C)$ and $s \neq 0$ on the comp
open set. On this open set s generates a trivial sub-line-bundl
and so exhibits $W(m)$ as an extension. Tensoring back by L^m t
that, over this open set W has L^m as a sub-line-bundle. If W is a
bundle then the quotient is necessarily L^{-m} and we are in the stan
tion described above. Thus the subvariety Σ given by $s = 0$
as a singularity of the extension element Φ, or the corresponding lin
but is not a singularity of the vector bundle W.

If m is the least integer for which $W(m)$ has global sections th
show that Σ necessarily has complex dimension 1, i.e. it is an algeb
Conversely given any curve Σ one can describe (a) when it ari
bundle W as above and (b) how to reconstruct W from Σ when
tions of (a) are satisfied [23]. In particular when Σ is connected i
determines W. If Σ has several connected components Σ_i then
on non-zero constants c_i (up to a common factor).

When $m = 1$, Σ has to be a collection of $k+1$ disjoint li
$k = c_2(W)$ is the anti-instanton number). If we take these lines
i.e. fibres of $P_3(C) \to S^4$, we recover the 't Hooft Ansatz dep
$(k+1)$ points of S^4 and corresponding weights $c_1, \ldots, c_{k+1}$. Wh
Σ has to be a collection of disjoint elliptic curves, whereas
Σ involves rather special curves of high genus.

For the bundles given by the Horrocks construction of Cha
can show that one need only take $m > \sqrt{3k+1} - 1$. Hartshorn
shown that this bound is best possible in the sense that smaller
not give all $-k$-instantons.

We conclude with a few further comments on the Ansatz for
In order to give an explicit formula for the potential one must c
finite gauge and it is desirable to pick the gauge as simply a
Geometrically this means that we have to pick a basis of $H^0(P_x, W$
ing smoothly on $x \in R^4$. Recall that in the appropriate open set
as an extension or exact sequence of bundles

$$0 \to L^m \to W \to L^{-m} \to 0 .$$

Restricting to P_x and taking cohomology we get an exact s
vector spaces:

$$0 \to H^0(P_x, W_x) \to H^0(P_x, \mathcal{O}(m)) \xrightarrow{\delta} H^1(P_x, \mathcal{O}(-m)) \to 0$$

Now $H^0(P_x, \mathcal{O}(m)) \cong \mathcal{S}^m(S_x^+)$ and

$$H^1(P_x, \mathcal{O}(-m)) \cong \mathcal{S}^{m-2}(S_x^+)^*$$

and the map δ between them is given by φ_x. Thus if $W = \tilde{E}$, the fibre $E_x = H^0(P_x, W_x)$ appears as the kernel of the homomorphism

$$\varphi_x \colon \mathcal{S}^m(S_x^+) \to \mathcal{S}^{m-2}(S_x^+)^* .$$

Now over R^4 the bundle S^+ has a natural trivialization and so therefore do $\mathcal{S}^m(S^+)$ and $\mathcal{S}^{m-2}(S^+)$. However the kernel of φ_x clearly varies with x and so the explicit description of the potential is best given by using the natural basis of the larger space $\mathcal{S}^m(S^+)$. Note that this only applies for $m \geqslant 2$ because for $m = 1$ the second space $\mathcal{S}^{m-2} = 0$. These remarks explain the apparent singularity in the formula for $m = 2$ given in [4]: this occurs because a basis for $\operatorname{Ker} \varphi_x$ has been taken by projecting a sub-set of basis vectors of S^+, and different subsets are needed for different positions of x in R^4.

5. – Relation with Radon transform.

In this section we make yet another digression to explain how the Penrose transform is related to other well-known transforms.

We recall first the classical Radon transform which associates to a function f on R^3 its integral over affine planes. Thus to each f we obtain a transformed function φ defined on the space of planes π by:

$$\varphi(\pi) = \int_\pi f . \tag{5.1}$$

The space of all planes in R^3 is a 3-dimensional manifold which can be identified with $P_3(R)$ with a point removed. There is an inversion formula which defines f in terms of an integral of φ. The Radon transform is closely related to the Fourier transform and is therefore useful for solving constant coefficient linear differential equations [20].

An obvious generalization of the Radon transform is to associate to f its integral over affine lines l in R^3. Thus we define φ by

$$\varphi(l) = \int_l f . \tag{5.2}$$

It is a function on the space of all lines in R^3. This space is a su the space of all lines in $P_3(R)$, namely the real Klein quadric ter III). This quadric has signature $(3, 3)$ and contains two s real projective planes. Our subspace is obtained by omitting on one system.

In (5.1) and (5.2) the measure on π or l, with respect to whicl gration is performed, is the usual Euclidean area or length re Consequently both transforms are compatible with the group of motions, this group acting naturally on the space of planes or li possible however to extend both transforms to the whole projecti $P_3(R)$, compatibly with the action of $SL(4, R)$, provided we replace by appropriate types of density. In general a k-th root of a de be called a k^{-1}-density. Then in (5.1) φ, f must be taken as $\frac{1}{4}$ and respectively, while in (5.2) φ, f must be $\frac{1}{2}$ and $\frac{1}{2}$ densities. More if we integrate over $(p-1)$-planes in R^{p+q-1} the relevant de $1/(p+q)$ and $p/(p+q)$.

In this extended form (5.2) therefore transforms the $\frac{1}{2}$-density to the $\frac{1}{2}$-density φ on the real Klein quadric $Q_4(R)$. Since $\dim Q_4 =$ it is unreasonable to expect a universal inversion formula, as Instead one expects φ to satisfy some condition in order for it transform of an f. In fact this condition is that φ be a solution c formally invariant second order equation $\Delta\varphi = 0$, where Δ is the analogue of the conformal Laplacian [21], [45]. Note that $Q_4(R)$ i formal compactification of the space R^4 with the $(2, 2)$ signature is an « ultra-hyperbolic » operator, i.e. one which in flat space is

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2} .$$

The group $SL(4, R)$ acts on $Q_4(R)$ by conformal transformations serves the operator Δ.

In these transforms we have so far ignored regularity questio The precise correspondence holds either for C^∞ functions or for re functions. Note that, unlike the positive definite case when Δ the equation $\Delta\varphi = 0$ does not imply analyticity of φ.

If we deal with analytic functions then the transform (5.2) can at in the complexification. Thus, if we extend f as a complex ana tion to some neighbourhood of $P_3(R)$ in $P_3(C)$, then (5.2) becomes integral. This leads naturally to the sheaf cohomology occurring i rose transform as we shall now explain.

To begin with let us recall that for the complex projective l we have

(5.3) $$H^1(P_1(C), \mathcal{O}(-2)) \cong C\,.$$

In section 1 this isomorphism was given in terms of explicit representative cocycles, and this can be made more intrinsic by recalling that $\mathcal{O}(-2) \cong \Omega^1$ the sheaf of holomorphic differentials on $P_1(C)$ (note that the tangent bundle of $P_1(C)$ is L^{-2}). If $\omega = f(z)\,dz$ is a holomorphic differential defined near the circle $|z| = 1$, we can take $\omega = \omega_{0\infty}$ to define a 1-cocycle relative to the standard covering of $P_1(C)$ by the open sets U_0, U_∞ given by $|z| < 1 + \varepsilon$, $|z| > 1 - \varepsilon$. The isomorphism (5.3) is then given by the contour integral

(5.4) $$\omega \to \frac{1}{2\pi i}\int \omega$$

taken around $|z| = 1$ (oriented in the conventional manner as the boundary of $|z| < 1$).

We now return to the transform (5.2) in which f, as explained earlier, should be taken as a $\frac{1}{2}$-density on $P_3(R)$. If we fix the orientation of $P_3(R)$ a density on $P_3(R)$ can be identified with an exterior differential 3-form and hence with a section of the real line-bundle L^4. Thus f can be viewed as a section of L^2 and complexified accordingly.

Next we fix a line l_0 in $P_3(R)$ and consider all nearby lines l, parametrized by a small open set $V \subset Q_4(R)$. The complexifications of the lines l fill out an open set U in $P_3(C)$ as one can verify. If we orient l_0, and extend this orientation by continuity to all nearby l, we can define open subsets U_0, U_∞ of U swept out by the (ε-enlarged) upper and lower hemispheres of each $P_1(C)$ in our family. If ε is small enough $U_0 \cap U_\infty$ will be close to $P_3(R)$ and hence within the domain of definition of f. Hence taking $f_{0\infty} = f$ we get an element

(5.5) $$(f) \in H^1(U, \mathcal{O}(-2))\,.$$

By the Penrose transform, as explained in Section 2, (f) defines a $\frac{1}{2}$-density φ, on the appropriate open set of $Q_4(C)$, satisfying $\Delta\varphi = 0$. The value of φ at the point parametrizing a given $P_1(C)$ is given by restricting (f) to this $P_1(C)$ and using (5.3). Comparing (5.2) and (5.4) we see that they coincide, up to the factor $2\pi i$. Thus the Penrose description of solutions of $\Delta\varphi = 0$ via sheaf cohomology classes can be considered as the natural complex-analytic description of the integral transform (5.2).

Although the transform (5.2), taken globally on the whole of $P_3(R)$, is invertible (i.e. $\varphi \equiv 0 \Rightarrow f \equiv 0$), the same is not true locally. Thus if f, defined originally in U, extends holomorphically to U_0 or U_∞, or more

generally is the difference of two such extensions, then $(f) = 0$
in V. This clarifies the role of the sheaf cohomology group whic
absorbs the local non-invertibility, in other words $\varphi \equiv 0$ in $V \Rightarrow (f$

It is perhaps worth emphasizing that the construction of the
mology class (f) from the $\frac{1}{2}$-density f depended on orienting the
that U_0 and U_∞ could be unambiguously defined. Now for topologi
we cannot coherently and continuously orient all real lines l in
space $Q_4(R)$ is not simply-connected. Thus a global f does *not* de
ment of the global cohomology group $H^1(P_3(C), \mathcal{O}(-2))$. This
contradiction that would otherwise arise from the vanishing of
mology group and the global invertibility of (5.3).

CHAPTER VII

Theorems on Algebraic Bundles

1. – Cohomology of the Horrocks construction.

In Chapter V we gave the construction of Horrocks for producing symplectic vector bundles on $P_3(C)$. In this chapter we shall prove a theorem due to Barth [8] which shows how the Horrocks bundles may be characterized in cohomological terms. Combined with the results of the preceding chapter on the vanishing of appropriate sheaf cohomology groups for instanton bundles this will finally prove that the Horrocks construction gives all instanton bundles.

We begin in this section by examining the Horrocks construction in greater detail and deducing its cohomological properties. At this stage, and also in the next section, we work purely over the complex numbers. Reality questions do not enter until later.

Throughout this chapter we shall be making extensive use of the machinery and standard results of sheaf cohomology which have only been lightly touched on in previous chapters. Inevitably therefore this chapter will be more technical, but we shall try to recall basic facts as and when they are needed. In any case, it is the arguments of this chapter which exhibit the full power of cohomology theory and the reader may find it instructive to try, where possible, to reinterpret the methods in real 4-space terms. We shall at the appropriate stage make some comments in this direction.

We recall that the Horrocks construction for $Sp(n, C)$-bundles, as explained in Chapter V, Section 2, starts from a linear map

(1.1) $$A(z): W \to V$$

with appropriate properties. Here $\dim W = k$, $\dim V = 2k + 2n$, $z = (z_1, \ldots, z_4)$ and V has a non-degenerate skew-form. For each $z \neq 0$ the

image $U_z = A(z)W \subset V$ is assumed k-dimensional and isotropic, contained in its polar space U_z^0. We then put $E_z = U_z^0/U_z$ whic
vector space of dimension $2n$ with an inherited non-degenerate s
This is the fibre over $(z) \in P_3(C)$ of the required bundle E.

Since $A(z)$ is linear in z we can view (1.1) as a homomorphism
bundles over $P_3(C)$:

(1.2) $$A: W(-1) \to V$$

where $W(-1) = W \otimes L$. In other words the bundle $U \subset V$ is i
to $W(-1)$. By duality V/U^0 is then isomorphic to $W^*(1)$. We c
all bundles relevant to the construction in the following display
sequences:

(1.3) $$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & W(-1) & \to & Q^* & \to & E & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & W(-1) & \xrightarrow{A} & V & \to & Q & \to & 0 \\ & & & & \downarrow{\scriptstyle A^*} & & \downarrow & & \\ & & & & W^*(1) & = & W^*(1) & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where $Q = V/U$, $Q^* = U^0$. This diagram is self-dual, and the
skew in an appropriate sense, being induced by that of V.

In dealing with sheaf cohomology groups on $P_3(C)$ we shall,
confusion can arise, omit the symbol $P_3(C)$ and simply write $H^q(S$
cohomology of a sheaf S. The cohomology of the bundles E, Q i
then given by

PROPOSITION (1.4).

$$H^0(Q) \cong V \qquad H^0(E(-n)) = 0 \quad \text{for } n \geq 1$$
$$H^0(Q(-n)) = 0 \quad \text{for } n \geq 1 \qquad H^1(E(-n)) = 0 \quad \text{for } n \geq 2$$
$$H^1(Q(n)) = 0 \quad \text{for all } n \qquad H^1(E(-1)) \cong W^* .$$

PROOF. The middle row of (1.3), together with the cohomology p
of $W(-1)$ and V give at once the results on Q [we use here the v
of $H^q(\mathcal{O}(n))$ for $q = 1, 2$ and all n, for $q = 0$ and $n < 0$]. Using this
tion on Q in the last column we then deduce the results on E.

As noted in Chapter V the second Chern class $c_2(E)$ can be read off from (1.3) and we get $c_2(E) = k$. More formally if x is the standard generator of $H^4(P_3, Z)$ the total Chern polynomials are given by:

$$c(W(-1)) = (1-x)^k, \quad c(W^*(1)) = (1+x)^k, \quad c(V) = 1$$

hence

$$c(Q) = (1-x)^{-k}$$

and

$$c(E) = (1-x)^{-k}(1+x)^{-k} = (1-x^2)^{-k} = 1 + kx^2$$

showing that $c_2(E) = k$.

Returning now to (1.3) let us use the last column to investigate $H^1(E(n))$ for $n > -1$. We see that

$$H^0(W^*(n+1)) \to H^1(E(n))$$

is surjective. Using the identification of W^* given by (1.4) this can be replaced by the surjectivity of

$$H^1(E(-1)) \otimes H^0(\mathcal{O}(n+1)) \to H^1(E(n)) .$$

Thus, if we introduce the graded module

$$M = \oplus M_n , \quad M_n = H^1(E(n))$$

over the ring of polynomials in $z_1, \ldots, z_4$, we see that M has the properties:

$$(1.5) \qquad \begin{cases} M_n = 0 \text{ for } n < -1 \text{ and } n \text{ sufficiently large} \\ \dim M_{-1} = k \\ M_{-1} \text{ generates } M . \end{cases}$$

The properties (1.5) of the module M reflect the simple nature of the Horrocks construction. There are in fact vector bundles E given by more complicated constructions, in which A is not assumed linear in $z_1, \ldots, z_4$, for which the associated module M is more complicated. This is part of the general theory of Horrocks but fortunately for our purposes the simple case leading to (1.5) is sufficient.

2. – Theorem of Barth.

We shall now give Barth's theorem [8] which gives sufficient c a symplectic bundle E in order that it should arise from the Horr struction. We consider first the following assumption on E:

(2.1) *For some line l in P_3, the restriction of E to l is trivial.*

This assumption is related to the notion of semi-stability: for when E is 2-dimensional (2.1) is equivalent to semi-stability ar vanishing of $H^0(E(-1))$ [7]. Note that (2.1) implies automaticall is trivial on the « general » line of P_3 (see Chapter IV). For bundle from S^4, via the basic construction of Chapter IV, (2.1) is certainly since E is trivial on all real lines, i.e. the fibres of $P_3(C) \to S^4$.

Our second assumption is that E satisfies the vanishing theore

$$(2.2) \qquad H^1(E(-2)) = 0$$

which is satisfied by bundles coming from the Horrocks con (see (1.4)), and also for bundles corresponding to anti-self-dua connections on S^4 by the results of Chapter VI.

Barth's theorem asserts that (2.1) and (2.2) are sufficient, na

(2.3) Theorem. *Let E be a symplectic vector bundle on $P_3(C)$ (2.1) and (2.2). Then E arises by the Horrocks construction from a li*

$$A(z): W \to V$$

unique up to isomorphism.

The uniqueness in (2.3) means that, if (A, W, V) and $(A', W',$ isomorphic symplectic bundles, there are isomorphisms $W \to W'$, (preserving skew-forms) taking A into A'.

The idea of the proof of Theorem (2.3) is to show that the disp can be canonically reconstructed from E alone. As a preliminary first show that the module $M = \bigoplus_n H^1(E(n))$ satisfies the conditio a necessary consequence of (1.3).

Take any plane P_2 containing the line l, then E is trivial on the line in this P_2 and so, for $n < 0$, $E(n)$ can have no non-zero sect this P_2. Now consider the standard exact sequence

$$0 \to E(-1) \to E \to E|P_2 \to 0 .$$

Tensoring with $\mathcal{O}(n)$ and taking cohomology gives the long exact sequence

$$\to H^0(P_2, E(n)) \to H^1(P_3, E(n-1)) \to H^1(P_3, E(n)) \to$$

Taking $n = -2, -3, \ldots$ in turn and using the vanishing of $H^0(P_2, E(n))$ we deduce inductively that $H^1(P_3, E(n)) = 0$ for all $n < -2$, which is the first part of (1.5). To prove the last part we take coordinates in P_3 so that l is given by $z_1 = z_2 = 0$. Then we have the exact sequence, resolving $\mathcal{O}_l$

$$0 \to \mathcal{O}(-2) \xrightarrow{\alpha} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \xrightarrow{\beta} \mathcal{O} \to \mathcal{O}_l \to 0 \tag{2.4}$$

where $\alpha = (z_2, -z_1)$ and $\beta = (z_1, z_2)$. The image of β is the ideal sheaf J of the line l. Tensoring (2.4) with $E(n)$, and taking cohomology we deduce the exact sequences

$$\begin{aligned} &\to H^1(E(n-1)) \oplus H^1(E(n-1)) \to H^1(J(n)) \to H^2(E(n-2)) \to \\ &\qquad\qquad\qquad\qquad\qquad \| \\ &\qquad \to H^1(J(n)) \to H^1(E(n)) \to H^1(l, E(n)) \to 0\,. \end{aligned} \tag{2.5}$$

Since E is trivial on l, $H^1(l, E(n)) = 0$ for $n \geq -1$, and by Serre duality $H^2(E(n-2))$ is dual to $H^1(E(-n-2))$ and so vanishes for $n \geq 0$ (as proved above): note that we have used $E \cong E^*$ here. Hence (2.5) asserts that the map

$$M_{n-1} \oplus M_{n-1} \xrightarrow{(z_1, z_2)} M_n$$

is surjective for all $n \geq 0$. This implies that M_{-1} generates M as a module (and is a somewhat stronger statement since only two of the variables are required). Finally to check the dimension of $M_{-1} = H^1(E(-1))$ we note that all the other cohomology groups $H^q(E(-1))$, $q \neq 1$, vanish. For $q = 2$ this follows from Serre duality and the vanishing of M_{-3}, for $q = 0$ it follows from the triviality of E on general lines and for $q = 3$ we apply Serre duality and reduce to the same argument. Thus $\dim M_{-1}$ can be computed from the Riemann-Roch theorem which evaluates

$$\sum_{q=0}^{3} (-1)^q \dim H^q(E(-1))$$

in terms of the Chern classes. Since $c_1(E) = 0$, $c_2(E) = k$ we get $\dim M_{-1} = a + bk$, where a, b are constants independent of E. These can either be obtained from the detailed Riemann-Roch formula or more simply we take

explicit examples arising from $-k$-instantons. Either way $\dim M_{-1} = k$ completing the verification of (1.5).

We turn now to the construction of the display (1.3) and we the last column. Defining $W^* = H^1(E(-1))$ and using the inte of H^1 in terms of extensions (cf. Chapter VI) we construct the

$$0 \to E \to Q \to W^*(1) \to 0 \tag{2.6}$$

corresponding to the identity element of

$$H^1(W \otimes E(-1)) = W \otimes W^* = \mathrm{End}(W)\,.$$

The effect of this is that, in the cohomology sequence of (2.6 with $\mathcal{O}(-1)$

$$\to H^0(W^*) \xrightarrow{\delta} H^1(E(-1)) \to H^1(Q(-1)) \to H^1(W^*)$$

the coboundary δ is the identity. Since $H^1(W^*) = 0$ it fol $H^1(Q(-1)) = 0$. If we tensor (2.6) in general with $\mathcal{O}(n)$ then fo we deduce

$$H^1(Q(-n)) \cong H^1(E(-n)) = 0$$

while for $n \geq 0$ we get

$$W^* \otimes H^0(\mathcal{O}(n+1)) \xrightarrow{\delta} H^1(E(n)) \to H^1(Q(n)) \to 0\,. \tag{2.7}$$

In (2.7) δ can be identified with the module multiplication from M and the surjectivity just proved (i.e. (1.5)) tells us that $H^1(Q(n)) =$ we have proved

$$H^1(Q(n)) = 0 \text{ for all } n\,. \tag{2.8}$$

We have now constructed the last column of (1.3) and establ key property (2.8) of the bundle Q. Dualizing we get the first rov The cohomology sequence of this first row tells us that

$$H^1(Q^*(n)) \cong H^1(E(n)) \quad \text{for all } n\,.$$

Interpreted in terms of extensions (with $n = -1$) this shows tha an extension of Q^* by $W^*(1)$ compatible with each such extensi In particular this gives us the middle column of (1.3), for a suitat bundle V.

From the middle row of (1.3), using (2.8), we deduce

$$H^1(V(n)) = 0 \quad \text{for all } n . \tag{2.9}$$

By Serre duality (2.8) implies that $H^2(Q^*(n)) = 0$ for all n, and hence from the middle column of (1.3) we get

$$H^2(V(n)) = 0 \quad \text{for all } n . \tag{2.10}$$

From (2.9) and (2.10), we can now deduce that V is a trivial bundle. We require the following special case of the general theory of Horrocks [28].

PROPOSITION (2.11). *Let V be a vector bundle over P_3 such that $H^1(V(n)) = H^2(V(n)) = 0$ for all n. Then V is isomorphic to a direct sum of line-bundles.*

PROOF. Fix $P_1 \subset P_2 \subset P_3$ and let

$$V|_{P_1} \cong K|_{P_1} \tag{2.12}$$

where $K \cong L^{n_1} \oplus \ldots \oplus L^{n_m}$ is a sum of line-bundles. It will be enough to shows that the isomorphism φ of (2.12) extends to P_3 as a homomorphism, because the points where φ is not an isomorphism form an algebraic surface (local equation $\det \varphi = 0$) not meeting P_1 and so necessarily vacuous. We show that φ extends first to P_2 then to P_3 by using the exact sequences

$$0 \to V(n-1)|_{P_2} \to V(n)|_{P_2} \to V(n)|_{P_1} \to 0$$
$$0 \to V(n-1) \to V(n) \to V(n)|_{P_2} \to 0 .$$

The second sequence and the vanishing of $H^1(V(n))$, $H^2(V(n))$ shows that $H^1(P_2, V(n)) = 0$ and that every section of $V(n)$ over P_2 extends to P_3. The vanishing of $H^1(P_2, V(n)) = 0$ applied to the first sequence shows similarly that every section of $V(n)$ over P_1 extends to P_2. Since φ (or rather φ^{-1}) is a direct sum of sections $V(n_i)$, the proposition follows.

We now apply (2.11) to the bundle V constructed above. To prove finally that the integers n_i are all zero, i.e. that V is trivial, it is enough to restrict V to one line l. But when (1.3) is restricted to l, since $E|l$ is trivial, all the extensions split and it follows that $V|l$ is trivial.

We have now reconstructed the display (1.3) from E. It remains only to show that it is skew-symmetric and in particular that V has a canonical skew-form. But dualizing (1.3) and using the skew duality $E \cong E^*$ we get an isomorphic diagram, showing that $V \cong V^*$. With a little more care

(cf. [8]) one checks that this isomorphism is skew, completing of Barth's theorem. The uniqueness of the display (1.3) follow canonical nature of our reconstruction, in which no arbitra were involved.

It is clear from our proof that exactly the same argument w orthogonal bundle, i.e. a vector bundle E with a non-degenerat form. The duality of the corresponding diagram to (1.3) is now

In conclusion we should point out that a more general theorem is given in [17] (and has been further generalized in [9]) which g sary and sufficient conditions for a bundle to arise from the Ho struction. Condition (2.2) is still kept but (2.1) is relaxed. In t module M, conditions (1.5) are still fulfilled but it is not assu of the 4 variables $(z_1, \ldots, z_4)$ are enough to generate M from $M_{}$ stanton bundles condition (2.1) is satisfied and therefore the extra of [17] is not needed.

3. – Reality constraints.

We come now to the question of imposing the necessary reality on the Horrocks construction to produce instanton bundles. In § 2 we imposed a Galois action σ on the triple (A, V, W), with on V and $\sigma^2 = 1$ on W, which enabled us to obtain an $Sp(n)$-bun with anti-self-dual connection. We want now to show the co that every such connection arises in this way.

According to Chapter IV, § 2 every such connection corresp holomorphic vector bundle E over $P_3(C)$ with

(i) a holomorphic symplectic structure

(ii) an anti-linear σ, compatible with (i), with σ on $P_3(C)$, $\sigma^2 = -1$, and inducing a positive hermitian form o on $\tilde{E}$ along all real lines of $P_3(C)$.

Since $\tilde{E}$ is trivial on all real lines of $P_3(C)$ and satisfies the vani dition $H^1(\tilde{E}(-2)) = 0$ (Chapter VI, § 3) we can apply Barth's t the previous section and deduce that $\tilde{E}$ is constructed canonical triple (A, V, W). The anti-linear map σ then induces an antilinear This follows from the uniqueness part of Barth's theorem: we c and the complex conjugate of $\sigma^*(\tilde{E})$. The uniqueness also shows t $\sigma^2 = -1$ and is compatible with the symplectic form.

To derive all the conditions imposed in Chapter V, § 2 we ne to show that the hermitian form induced by σ on V is positive. N that, for any $(z) \in P_3(C)$ we have an orthogonal decomposition:

$$V = U_z \oplus E_z \oplus U_{\sigma z}$$

where $U_{\sigma z}^{\perp} = U_z^{\circ} = U_z \oplus E_z$. The hermitian form restricted to U_z is therefore definite. Applying σ shows that the sign of this definite form is the same in $U_{\sigma z}$ as in U_z. The form is positive on E_z because of its original definition. Hence the form is either positive on all V or else it has signature $(2n, 2k)$. But in the latter case the fibres E_z would lie in the positive cone and the bundle would be topologically trivial (deformable to a fixed positive subspace), contradicting the fact that $c_2(\tilde{E}) = k \neq 0$.

This completes the proof that the construction of Chapter V, made explicit in Chapter II, gives all anti-instantons for $Sp(n)$. Exactly analogous arguments work for the orthogonal group. Finally for the unitary group $U(n)$ we can embed this in $SO(2n)$ and consider an additional anti-linear J. Again using the uniqueness part of Barth's theorem we deduce that we get an operator J on V with the appropriate properties.

4. – The Drinfeld-Manin description.

In the proof of Barth's theorem in § 2 we reconstructed the display (1.3) from the symplectic vector bundle E over P_3. The vector space W^* was identified with the cohomology group $H^1(E(-1))$. The vector space V can also be identified as a cohomology group in such a way that the linear map A (or its dual A^*) acquires a simple cohomological interpretation. This enables one to reformulate Barth's theorem rather more elegantly by directly exhibiting the triple (A, V, W) in terms of the cohomology of E. This reformulation has the advantage of making the additional reality conditions more transparent.

We shall explain how to interpret V cohomologically. For fuller details and a more systematic account see [18].

From the middle column of (1.3) we get the cohomology sequence:

$$0 \to H^0(Q^*) \to H^0(V) \to H^0(W^*(1)) \to H^1(Q^*) \to H^1(V) . \tag{4.1}$$

Since V is a trivial bundle $H^0(V) \cong V$, $H^1(V) = 0$.

From the top row we have

$$H^q(Q^*) \cong H^q(E) , \quad q = 0, 1$$

and finally we recall that

$$W^* \cong H^1(E(-1)) .$$

Substituting these into (4.1) we get

$$0 \to H^0(E) \to V \to H^1(E(-1)) \otimes H^0(\mathcal{O}(1)) \xrightarrow{\mu} H^1(E) \to 0 \tag{4.2}$$

where μ can be identified with the natural module multiplicati
module M.

On the other hand we have a natural exact sequence on P_3:

$$0 \to \mathcal{O}(-1) \to C^4 \to T(-1) \to 0$$

where T is the tangent bundle. Dualizing and tensoring with $E($

$$0 \to E \otimes \Omega^1 \to (C^4)^* \otimes E(-1) \to E \to 0 .$$

The cohomology sequence of this is

$$0 \to H^0(E) \to H^1(E \otimes \Omega^1) \to H^1(E(-1)) \otimes (C^4)^* \xrightarrow{\mu} H^1(E) \tag{4.3}$$

where μ is the natural multiplication (recall that $H^0(\mathcal{O}(1))$ is th linear forms $(C^4)^*$).

Comparing (4.2) with (4.3) suggests that there should be isomorphism

$$V \cong H^*(E \otimes \Omega^1) . \tag{4.4}$$

This is clear if $H^0(E) = 0$ which is the case for instanton bundle taining a trivial summand (e.g. for $Sp(1)$-bundles and $k \neq 0$). I eral case V splits as a direct sum

$$V \cong H^0(E) \oplus V' \tag{4.5}$$

using the skew-form on V. Now $H^1(E \otimes \Omega^1)$ has a natural skew-f by the cup-product (and the skew-form on E):

$$H^1(E \otimes \Omega^1) \otimes H^1(E \otimes \Omega^1) \to H^2(\Omega^2) \cong C \tag{4.6}$$

and so it decomposes analogously to (4.5). This gives us the isomorp and one must then check directly that this isomorphism is compa the skew-forms.

Thus the linear map A^* associated to E can be identified with the map

$$H^1(E\otimes\Omega^1)\to H^1(E(-1))\otimes H^0(\mathcal{O}(1))$$

occurring in (4.3).

Any additional structure on E such as σ or J then passes naturally to these cohomology groups. Note that in the orthogonal case the multiplication (4.6) is symmetric.

Since $H^2(P_3, \Omega^2)$ maps isomorphically onto $H^2(P_2, \Omega^2)$ it follows that (4.6) factors through the restriction to P_2 where it coincides with Serre duality. The non-degeneracy of (4.6) therefore implies the injectivity of

$$H^1(P_3, E\otimes\Omega^1(P_3))\to H^1(P_2, E\otimes\Omega^1(P_2)) .$$

Concerning restriction from P_3 to P_2 we draw attention to the following. Let P_2 and $\sigma(P_2)$ be considered as a degenerate quadric with real structure. In general for any real quadric $Q\subset P_3$ two real bundles E, F on P_3, corresponding to anti-instanton $Sp(n)$-bundles on S^4, are isomorphic if and only if their restrictions to Q are isomorphic. To see this we have to show that any isomorphism $\varphi\colon E\to F$ over Q extends as a homomorphism over P_3 (it is then necessarily an isomorphism by an argument used earlier). Now $G = \mathrm{Hom}(E, F)$ again corresponds to an anti-instanton bundle and so satisfies the vanishing condition $H^1(P_3, G(-2)) = 0$. Applied to the exact sequence

$$0\to G(-2)\to G\to G|_Q\to 0$$

this shows precisely that every section of G over Q extends to a section over P_3. Taking $Q = P_2\cup\sigma(P_2)$ we deduce that $E\simeq F$ over P_3 (as real bundles i.e. commuting with σ) if and only if $E\simeq F$ over P_2 and the isomorphism respects σ over the real line $P_1 = P_2\cap\sigma(P_2)$. Since E, F are trivial over real lines the real structure over P_1 corresponds to a real structure or hermitian metric on C^{2n}. Thus we deduce that the map from the space of moduli of real $Sp(n)$ anti-instantons to the space of moduli of complex symplectic bundles over $P_2(C)$ has fibre the symmetric space $Sp(n, C)/Sp(n)$. For example when $n = 1$ this fibre is the hyperbolic 3-space. The $(8k-3)$-manifold of $SU(2)$ $-k$-instantons therefore fibres over an $8k-6 = 2(4k-3)$-manifold of bundles on $P_2(C)$. But a dimension count (cf. [3]) shows that $(4k-3)$ is the complex dimension of the relevant space of (stable) bundles on $P_2(C)$. Hence, after factoring out by the hyperbolic 3-space, we get a complex structure on the real moduli space.

This result can be interpreted roughly as follows. Starting in I
coordinates to identify it with C^2. Then an anti-self-dual connec
a holomorphic bundle on C^2. If we require the bundle to extend
holomorphic bundle extends to $P_2(C)$ and acquires a real structı
The surprising result is that this data uniquely determines th
solution: in other words it is enough to fix one set of complex c
if we work globally, whereas the main local result of Chapter IV
we needed all complex coordinates to interpret the anti-self-duality
It would be very interesting to have a direct differential-geome
of this result. For this it would be necessary to prove the exis
unique hermitian metric for the holomorphic bundle which sat
propriate conditions.

CHAPTER VIII

Further Problems

1. – Euclidean approach to instantons.

As we have seen, the proof that the Horrocks construction yields all instantons uses the full machinery of algebraic geometry. On the other hand the construction itself has a simple description in the Euclidean 4-space without resorting to the twistor picture. It is natural therefore to ask if we can produce an alternative proof of the completeness working only in the Euclidean 4-space. As a first step it seems necessary to describe the canonical data used in the Horrocks construction in Euclidean terms. In particular the vector spaces V and W should have such a description. We recall that, in sheaf cohomology terms, we have

$$W^* = H^1(E(-1))$$
$$V \;= H^1(E\otimes \Omega^1)\,.$$

According to the results of Chapter VI we already know that $H^1(E(-1))$ can be identified with the space of $E\otimes S^-$ satisfying the coupled Dirac equation (recall that with our sign conventions we produce anti-instantons): here S^- is the negative spin bundle over S^4. We shall now show how the other cohomology group can be similarly interpreted.

Using the standard metric of S^4 the basic anti-instanton bundle is precisely the bundle S^- and, when lifted to $P_3(C)$, it gives the holomorphic bundle orthogonal to the fibres as explained in Chapter V, § 1. Thus on $P_3(C)$ we have the exact sequence

$$0\to \tilde{S}^-\otimes L^*\to T\to L^{-2}\to 0$$

or dualizing,

$$0\to L^2\to \Omega^1\to \tilde{S}^-\otimes L\to 0\,.$$

Tensoring with E gives

$$0\to E(-2)\to E\otimes\Omega^1\to E\otimes\tilde{S}^-(-1)\to 0\,.$$

Taking cohomology and using the vanishing of $H^q(E(-2))$ fo (these are duals of each other) we see that

$$H^1(E\otimes\Omega^1)\cong H^1(E\otimes\tilde{S}^-(-1))\,.$$

Thus $H^1(E\otimes\Omega^1)$ can be identified with the space of solutions of equation on $(E\otimes S^-)\otimes S^-$ (we have for simplicity used the same for the holomorphic bundle on $P_3(C)$ and the corresponding anti bundle on S^4). Now sections of $S^-\otimes S^-$ can be naturally ident pairs (f,ω) where f is a scalar function and ω is an anti-self-du this is a matter of examining the representations of Spin (4). the Dirac operator coupled to S^- can be identified with the Hodg $d+d^*$ on differential forms. The corresponding results hold i further couple to E provided d is replaced by its covariant an Thus we see that $H^1(E\otimes\Omega^1)$ can be identified with the spac (f,ω) satisfying the equation

$$Df=-D^*\omega \tag{1.1}$$

where f is a section of E and ω a section of $E\otimes\Omega^2_-$.

When E is irreducible it has no covariant constant sections e and so ω, in (1.1), uniquely determines f.

The geometrical description of the bundle E, arising from the construction, gives it as a sub-bundle of the trivial bundle $S^4\times V$. gonal projection V then becomes identified with a certain space of E. Conversely to find the embedding of E into $S^4\times V$, it is give this space of sections. It seems reasonable to conjecture that of sections is related to the pairs (f,ω) by interior and exterio with the curvature F.

From this point of view one would have to show the approp degeneracy of this space of sections (so as to get an embedding $S^4\times V$). In addition one would have to verify that the original on E coincided with the connection induced from the embedd last part appears to be the hardest.

A direct interpretation of the self-dual Yang-Mills equations of analytic function theory of a quaternionic variable has been by F. Gürsey. It would be very interesting to investigate in detai tion between this point of view and our complex methods. This might help to produce a proof of the completeness of the instanton construction in the Euclidean framework.

2. – General solutions of the Yang-Mills equations.

In these lectures I have concentrated on the instanton problems which correspond to absolute minima of the Yang-Mills functional on S^4. I shall now review what is known about general solutions of the Yang-Mills equations, corresponding to critical points which are not absolute minima.

First of all there is a recent result of Bourguignon, Lawson and Simons [11] which shows that there are no other local minima. It is at present unknown if other critical points exist, but [11] asserts that any such points must be unstable. The proof consists in showing that the second variation is indefinite.

The twistor interpretation of instantons does not immediately apply to other Yang-Mills solutions, but Witten [44] and Green et al. [24] have shown how to generalize Ward's ideas to the general case. However, the twistor interpretation is now more complicated and its potential has not yet been exploited.

Topological aspects of Yang-Mills theory related to ideas of Morse theory have been studied in [5]. In this and other respects there are close analogies with the non-linear σ model in 2 dimensions and these analogies suggest that no other critical points exist.

Finally, K. Uhlenbeck has recently shown [41] that square integrability on R^4 for Yang-Mills solutions automatically ensures the extension to S^4. More precisely there is a purely local result asserting that a smooth connection defined in a neighbourhood of 0 (but not at 0), which is locally square integrable and satisfies the Yang-Mills equations, automatically extends to a smooth connection defined at 0. In other words, point singularities are « removable ». Applied to the point at ∞, after a conformal transformation, this yields the extension from R^4 to S^4.

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