Part II

Algebraic Geometry

Paper 1, Section II
25I Algebraic Geometry

Let k be an algebraically closed field and let $V \subset \mathbb{A}_k^n$ be a non-empty affine variety. Show that V is a finite union of irreducible subvarieties.

Let V_1 and V_2 be subvarieties of $\mathbb{A}_k^n$ given by the vanishing loci of ideals I_1 and I_2 respectively. Prove the following assertions.

(i) The variety $V_1 \cap V_2$ is equal to the vanishing locus of the ideal $I_1 + I_2$.

(ii) The variety $V_1 \cup V_2$ is equal to the vanishing locus of the ideal $I_1 \cap I_2$.

Decompose the vanishing locus

$$\mathbb{V}(X^2 + Y^2 - 1,\, X^2 - Z^2 - 1) \subset \mathbb{A}_{\mathbb{C}}^3.$$

into irreducible components.

Let $V \subset \mathbb{A}_k^3$ be the union of the three coordinate axes. Let W be the union of three distinct lines through the point $(0,0)$ in $\mathbb{A}_k^2$. Prove that W is not isomorphic to V.

Paper 2, Section II
25I Algebraic Geometry

Let k be an algebraically closed field and $n \geqslant 1$. Exhibit $GL(n,k)$ as an open subset of affine space $\mathbb{A}_k^{n^2}$. Deduce that $GL(n,k)$ is smooth. Prove that it is also irreducible.

Prove that $GL(n,k)$ is isomorphic to a closed subvariety in an affine space.

Show that the matrix multiplication map

$$GL(n,k) \times GL(n,k) \to GL(n,k)$$

that sends a pair of matrices to their product is a morphism.

Prove that any morphism from $\mathbb{A}_k^n$ to $\mathbb{A}_k^1 \setminus \{0\}$ is constant.

Prove that for $n \geqslant 2$ any morphism from $\mathbb{P}_k^n$ to $\mathbb{P}_k^1$ is constant.

Paper 3, Section II

24I Algebraic Geometry

In this question, all varieties are over an algebraically closed field k of characteristic zero.

What does it mean for a projective variety to be *smooth*? Give an example of a smooth affine variety $X \subset \mathbb{A}^n_k$ whose projective closure $\overline{X} \subset \mathbb{P}^n_k$ is not smooth.

What is the *genus* of a smooth projective curve? Let $X \subset \mathbb{P}^4_k$ be the hypersurface $V(X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_4^3)$. Prove that X contains a smooth curve of genus 1.

Let $C \subset \mathbb{P}^2_k$ be an irreducible curve of degree 2. Prove that C is isomorphic to $\mathbb{P}^1_k$.

We define a *generalized conic* in $\mathbb{P}^2_k$ to be the vanishing locus of a non-zero homogeneous quadratic polynomial in 3 variables. Show that there is a bijection between the set of generalized conics in $\mathbb{P}^2_k$ and the projective space $\mathbb{P}^5_k$, which maps the conic $V(f)$ to the point whose coordinates are the coefficients of f.

(i) Let $R^\circ \subset \mathbb{P}^5_k$ be the subset of conics that consist of unions of two distinct lines. Prove that R° is not Zariski closed, and calculate its dimension.

(ii) Let I be the homogeneous ideal of polynomials vanishing on R°. Determine generators for the ideal I.

Paper 4, Section II

24I Algebraic Geometry

Let C be a smooth irreducible projective algebraic curve over an algebraically closed field.

Let D be an effective divisor on C. Prove that the vector space $L(D)$ of rational functions with poles bounded by D is finite dimensional.

Let D and E be linearly equivalent divisors on C. Exhibit an isomorphism between the vector spaces $L(D)$ and $L(E)$.

What is a *canonical divisor* on C? State the Riemann–Roch theorem and use it to calculate the degree of a canonical divisor in terms of the genus of C.

Prove that the canonical divisor on a smooth cubic plane curve is linearly equivalent to the zero divisor.

Paper 1, Section II

25F Algebraic Geometry

Let k be an algebraically closed field of characteristic zero. Prove that an affine variety $V \subset \mathbb{A}_k^n$ is irreducible if and only if the associated ideal $I(V)$ of polynomials that vanish on V is prime.

Prove that the variety $\mathbb{V}(y^2 - x^3) \subset \mathbb{A}_k^2$ is irreducible.

State what it means for an affine variety over k to be *smooth* and determine whether or not $\mathbb{V}(y^2 - x^3)$ is smooth.

Paper 2, Section II

24F Algebraic Geometry

Let k be an algebraically closed field of characteristic not equal to 2 and let $V \subset \mathbb{P}_k^3$ be a nonsingular quadric surface.

(a) Prove that V is birational to $\mathbb{P}_k^2$.

(b) Prove that there exists a pair of disjoint lines on V.

(c) Prove that the affine variety $W = \mathbb{V}(xyz - 1) \subset \mathbb{A}_k^3$ does not contain any lines.

Paper 3, Section II

24F Algebraic Geometry

(i) Suppose $f(x, y) = 0$ is an affine equation whose projective completion is a smooth projective curve. Give a basis for the vector space of holomorphic differential forms on this curve. [You are not required to prove your assertion.]

Let $C \subset \mathbb{P}^2$ be the plane curve given by the vanishing of the polynomial

$$X_0^4 - X_1^4 - X_2^4 = 0$$

over the complex numbers.

(ii) Prove that C is nonsingular.

(iii) Let ℓ be a line in $\mathbb{P}^2$ and define D to be the divisor $\ell \cap C$. Prove that D is a canonical divisor on C.

(iv) Calculate the minimum degree d such that there exists a non-constant map

$$C \to \mathbb{P}^1$$

of degree d.

[*You may use any results from the lectures provided that they are stated clearly.*]

Paper 4, Section II

24F Algebraic Geometry

Let $P_0, \ldots, P_n$ be a basis for the homogeneous polynomials of degree n in variables Z_0 and Z_1. Then the image of the map $\mathbb{P}^1 \to \mathbb{P}^n$ given by

$$[Z_0, Z_1] \mapsto [P_0(Z_0, Z_1), \ldots, P_n(Z_0, Z_1)]$$

is called a rational normal curve.

Let $p_1, \ldots, p_{n+3}$ be a collection of points in general linear position in $\mathbb{P}^n$. Prove that there exists a unique rational normal curve in $\mathbb{P}^n$ passing through these points.

Choose a basis of homogeneous polynomials of degree 3 as above, and give generators for the homogeneous ideal of the corresponding rational normal curve.

Paper 4, Section II

24F Algebraic Geometry

(a) Let $X \subseteq \mathbb{P}^2$ be a smooth projective plane curve, defined by a homogeneous polynomial $F(x, y, z)$ of degree d over the complex numbers $\mathbb{C}$.

(i) Define the divisor $[X \cap H]$, where H is a hyperplane in $\mathbb{P}^2$ not contained in X, and prove that it has degree d.

(ii) Give (without proof) an expression for the degree of $\mathcal{K}_X$ in terms of d.

(iii) Show that X does not have genus 2.

(b) Let X be a smooth projective curve of genus g over the complex numbers $\mathbb{C}$. For $p \in X$ let

$$G(p) = \{n \in \mathbb{N} \mid \text{ there is } no\ f \in k(X) \text{ with } v_p(f) = n, \text{ and } v_q(f) \leqslant 0 \text{ for all } q \neq p\}.$$

(i) Define $\ell(D)$, for a divisor D.

(ii) Show that for all $p \in X$,

$$\ell(np) = \begin{cases} \ell((n-1)p) & \text{for } n \in G(p) \\ \ell((n-1)p) + 1 & \text{otherwise.} \end{cases}$$

(iii) Show that $G(p)$ has exactly g elements. [*Hint: What happens for large n?*]

(iv) Now suppose that X has genus 2. Show that $G(p) = \{1, 2\}$ or $G(p) = \{1, 3\}$.

[*In this question $\mathbb{N}$ denotes the set of positive integers.*]

Paper 3, Section II

24F Algebraic Geometry

Let $W \subseteq \mathbb{A}^2$ be the curve defined by the equation $y^3 = x^4 + 1$ over the complex numbers $\mathbb{C}$, and let $X \subseteq \mathbb{P}^2$ be its closure.

(a) Show X is smooth.

(b) Determine the ramification points of the map $X \to \mathbb{P}^1$ defined by

$$(x : y : z) \mapsto (x : z).$$

Using this, determine the Euler characteristic and genus of X, stating clearly any theorems that you are using.

(c) Let $\omega = \frac{dx}{y^2} \in \mathcal{K}_X$. Compute $\nu_p(\omega)$ for all $p \in X$, and determine a basis for $\mathcal{L}(\mathcal{K}_X)$.

Paper 2, Section II

24F Algebraic Geometry

(a) Let A be a commutative algebra over a field k, and $p : A \to k$ a k-linear homomorphism. Define $Der(A, p)$, the derivations of A centered in p, and define the *tangent space* T_pA in terms of this.

Show directly from your definition that if $f \in A$ is not a zero divisor and $p(f) \neq 0$, then the natural map $T_pA[\frac{1}{f}] \to T_pA$ is an isomorphism.

(b) Suppose k is an algebraically closed field and $\lambda_i \in k$ for $1 \leqslant i \leqslant r$. Let

$$X = \{(x, y) \in \mathbb{A}^2 \mid x \neq 0, y \neq 0, y^2 = (x - \lambda_1) \cdots (x - \lambda_r)\}.$$

Find a surjective map $X \to \mathbb{A}^1$. Justify your answer.

Paper 1, Section II

25F Algebraic Geometry

(a) Let k be an algebraically closed field of characteristic 0. Consider the algebraic variety $V \subset \mathbb{A}^3$ defined over k by the polynomials

$$xy, \quad y^2 - z^3 + xz, \quad \text{and} \quad x(x + y + 2z + 1).$$

Determine

(i) the irreducible components of V,

(ii) the tangent space at each point of V,

(iii) for each irreducible component, the smooth points of that component, and

(iv) the dimensions of the irreducible components.

(b) Let $L \supseteq K$ be a finite extension of fields, and $\dim_K L = n$. Identify L with $\mathbb{A}^n$ over K and show that

$$U = \{\alpha \in L \mid K[\alpha] = L\}$$

is the complement in $\mathbb{A}^n$ of the vanishing set of some polynomial. [You need not show that U is non-empty. You may assume that $K[\alpha] = L$ if and only if $1, \alpha, \ldots, \alpha^{n-1}$ form a basis of L over K.]

Paper 4, Section II

24I Algebraic Geometry

State a theorem which describes the canonical divisor of a smooth plane curve C in terms of the divisor of a hyperplane section. Express the degree of the canonical divisor K_C and the genus of C in terms of the degree of C. [You need not prove these statements.]

From now on, we work over $\mathbb{C}$. Consider the curve in $\mathbf{A}^2$ defined by the equation

$$y + x^3 + xy^3 = 0.$$

Let C be its projective completion. Show that C is smooth.

Compute the genus of C by applying the Riemann–Hurwitz theorem to the morphism $C \to \mathbf{P}^1$ induced from the rational map $(x, y) \mapsto y$. [You may assume that the discriminant of $x^3 + ax + b$ is $-4a^3 - 27b^2$.]

Paper 3, Section II

24I Algebraic Geometry

(a) State the Riemann–Roch theorem.

(b) Let E be a smooth projective curve of genus 1 over an algebraically closed field k, with $\operatorname{char} k \neq 2, 3$. Show that there exists an isomorphism from E to the plane cubic in $\mathbf{P}^2$ defined by the equation

$$y^2 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3),$$

for some distinct $\lambda_1, \lambda_2, \lambda_3 \in k$.

(c) Let Q be the point at infinity on E. Show that the map $E \to Cl^0(E)$, $P \mapsto [P - Q]$ is an isomorphism.

Describe how this defines a group structure on E. Denote addition by $\boxplus$. Determine all the points $P \in E$ with $P \boxplus P = Q$ in terms of the equation of the plane curve in part (b).

Paper 2, Section II

24I Algebraic Geometry

(a) Let $X \subseteq \mathbf{A}^n$ be an affine algebraic variety defined over the field k.

Define the *tangent space* T_pX for $p \in X$, and the *dimension* of X in terms of T_pX.

Suppose that k is an algebraically closed field with $\operatorname{char} k > 0$. Show directly from your definition that if $X = Z(f)$, where $f \in k[x_1, \ldots, x_n]$ is irreducible, then $\dim X = n-1$.

[Any form of the Nullstellensatz may be used if you state it clearly.]

(b) Suppose that $\operatorname{char} k = 0$, and let W be the vector space of homogeneous polynomials of degree d in 3 variables over k. Show that

$$U = \{(f, p) \in W \times k^3 \mid Z(f-1) \text{ is a smooth surface at } p\}$$

is a non-empty Zariski open subset of $W \times k^3$.

Paper 1, Section II

25I Algebraic Geometry

(a) Let k be an uncountable field, $\mathcal{M} \subseteq k[x_1, \ldots, x_n]$ a maximal ideal and $A = k[x_1, \ldots, x_n]/\mathcal{M}$.

Show that every element of A is algebraic over k.

(b) Now assume that k is algebraically closed. Suppose that $J \subset k[x_1, \ldots, x_n]$ is an ideal, and that $f \in k[x_1, \ldots, x_n]$ vanishes on $Z(J)$. Using the result of part (a) or otherwise, show that $f^N \in J$ for some $N \geqslant 1$.

(c) Let $f : X \to Y$ be a morphism of affine algebraic varieties. Show $\overline{f(X)} = Y$ if and only if the map $f^* : k[Y] \to k[X]$ is injective.

Suppose now that $\overline{f(X)} = Y$, and that X and Y are irreducible. Define the *dimension* of X, $\dim X$, and show $\dim X \geqslant \dim Y$. [You may use whichever definition of $\dim X$ you find most convenient.]

Paper 2, Section II

22I Algebraic Geometry

Let k be an algebraically closed field of any characteristic.

(a) Define what it means for a variety X to be *non-singular* at a point $P \in X$.

(b) Let $X \subseteq \mathbb{P}^n$ be a hypersurface $Z(f)$ for $f \in k[x_0, \ldots, x_n]$ an irreducible homogeneous polynomial. Show that the set of singular points of X is $Z(I)$, where $I \subseteq k[x_0, \ldots, x_n]$ is the ideal generated by $\partial f/\partial x_0, \ldots, \partial f/\partial x_n$.

(c) Consider the projective plane curve corresponding to the affine curve in $\mathbb{A}^2$ given by the equation

$$x^4 + x^2y^2 + y^2 + 1 = 0.$$

Find the singular points of this projective curve if char $k \neq 2$. What goes wrong if char $k = 2$?

Paper 3, Section II

22I Algebraic Geometry

(a) Define what it means to give a *rational map* between algebraic varieties. Define a *birational map*.

(b) Let

$$X = Z(y^2 - x^2(x-1)) \subseteq \mathbb{A}^2.$$

Define a birational map from X to $\mathbb{A}^1$. [*Hint: Consider lines through the origin.*]

(c) Let $Y \subseteq \mathbb{A}^3$ be the surface given by the equation

$$x_1^2x_2 + x_2^2x_3 + x_3^2x_1 = 0.$$

Consider the blow-up $X \subseteq \mathbb{A}^3 \times \mathbb{P}^2$ of $\mathbb{A}^3$ at the origin, i.e. the subvariety of $\mathbb{A}^3 \times \mathbb{P}^2$ defined by the equations $x_iy_j = x_jy_i$ for $1 \leqslant i < j \leqslant 3$, with y_1, y_2, y_3 coordinates on $\mathbb{P}^2$. Let $\varphi : X \to \mathbb{A}^3$ be the projection and $E = \varphi^{-1}(0)$. Recall that the proper transform $\widetilde{Y}$ of Y is the closure of $\varphi^{-1}(Y) \setminus E$ in X. Give equations for $\widetilde{Y}$, and describe the fibres of the morphism $\varphi|_{\widetilde{Y}} : \widetilde{Y} \to Y$.

Paper 4, Section II
23I Algebraic Geometry

(a) Let X and Y be non-singular projective curves over a field k and let $\varphi : X \to Y$ be a non-constant morphism. Define the *ramification degree* e_P of φ at a point $P \in X$.

(b) Suppose $\operatorname{char} k \neq 2$. Let $X = Z(f)$ be the plane cubic with $f = x_0x_2^2 - x_1^3 + x_0^2x_1$, and let $Y = \mathbb{P}^1$. Explain how the projection

$$(x_0 : x_1 : x_2) \mapsto (x_0 : x_1)$$

defines a morphism $\varphi : X \to Y$. Determine the degree of φ and the ramification degrees e_P for all $P \in X$.

(c) Let X be a non-singular projective curve and let $P \in X$. Show that there is a non-constant rational function on X which is regular on $X \setminus \{P\}$.

Paper 1, Section II
24I Algebraic Geometry
Let k be an algebraically closed field.

(a) Let X and Y be varieties defined over k. Given a function $f : X \to Y$, define what it means for f to be a *morphism of varieties.*

(b) If X is an affine variety, show that the coordinate ring $A(X)$ coincides with the ring of regular functions on X. [*Hint: You may assume a form of the Hilbert Nullstellensatz.*]

(c) Now suppose X and Y are affine varieties. Show that if X and Y are isomorphic, then there is an isomorphism of k-algebras $A(X) \cong A(Y)$.

(d) Show that $Z(x^2 - y^3) \subseteq \mathbb{A}^2$ is not isomorphic to $\mathbb{A}^1$.

Paper 3, Section II

21H Algebraic Geometry

(a) Let X be an affine variety. Define the *tangent space* of X at a point P. Say what it means for the variety to be *singular* at P.

Define the *dimension* of X in terms of (i) the tangent spaces of X, and (ii) Krull dimension.

(b) Consider the ideal I generated by the set $\{y, y^2 - x^3 + xy^3\} \subseteq k[x, y]$. What is $Z(I) \subseteq \mathbb{A}^2$?

Using the generators of the ideal, calculate the tangent space of a point in $Z(I)$. What has gone wrong? [A complete argument is not necessary.]

(c) Calculate the dimension of the tangent space at each point $p \in X$ for $X = Z(x - y^2, x - zw) \subseteq \mathbb{A}^4$, and determine the location of the singularities of X.

Paper 2, Section II

22H Algebraic Geometry

In this question we work over an algebraically closed field of characteristic zero. Let $X^o = Z(x^6 + xy^5 + y^6 - 1) \subset \mathbb{A}^2$ and let $X \subset \mathbb{P}^2$ be the closure of X^o in $\mathbb{P}^2$.

(a) Show that X is a non-singular curve.

(b) Show that $\omega = dx/(5xy^4 + 6y^5)$ is a regular differential on X.

(c) Compute the divisor of ω. What is the genus of X?

Paper 4, Section II

22H Algebraic Geometry

(a) Let C be a smooth projective curve, and let D be an effective divisor on C. Explain how D defines a morphism ϕ_D from C to some projective space.

State a necessary and sufficient condition on D so that the pull-back of a hyperplane via ϕ_D is an element of the linear system $|D|$.

State necessary and sufficient conditions for ϕ_D to be an isomorphism onto its image.

(b) Let C now have genus 2, and let K be an effective canonical divisor. Show that the morphism ϕ_K is a morphism of degree 2 from C to $\mathbb{P}^1$.

Consider the divisor $K + P_1 + P_2$ for points P_i with $P_1 + P_2 \not\sim K$. Show that the linear system associated to this divisor induces a morphism ϕ from C to a quartic curve in $\mathbb{P}^2$. Show furthermore that $\phi(P) = \phi(Q)$, with $P \neq Q$, if and only if $\{P, Q\} = \{P_1, P_2\}$.

[*You may assume the Riemann–Roch theorem.*]

Paper 1, Section II

23H Algebraic Geometry

Let k be an algebraically closed field.

(a) Let X and Y be affine varieties defined over k. Given a map $f : X \to Y$, define what it means for f to be a *morphism of affine varieties*.

(b) Let $f : \mathbb{A}^1 \to \mathbb{A}^3$ be the map given by

$$f(t) = (t, t^2, t^3).$$

Show that f is a morphism. Show that the image of f is a closed subvariety of $\mathbb{A}^3$ and determine its ideal.

(c) Let $g : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^7$ be the map given by

$$g\big((s_1, t_1), (s_2, t_2), (s_3, t_3)\big) = (s_1s_2s_3,\ s_1s_2t_3,\ s_1t_2s_3,\ s_1t_2t_3,\ t_1s_2s_3,\ t_1s_2t_3,\ t_1t_2s_3,\ t_1t_2t_3).$$

Show that the image of g is a closed subvariety of $\mathbb{P}^7$.

Paper 4, Section II
20F Algebraic Geometry

(i) Explain how a linear system on a curve C may induce a morphism from C to projective space. What condition on the linear system is necessary to yield a morphism $f : C \to \mathbb{P}^n$ such that the pull-back of a hyperplane section is an element of the linear system? What condition is necessary to imply the morphism is an embedding?

(ii) State the Riemann–Roch theorem for curves.

(iii) Show that any divisor of degree 5 on a curve C of genus 2 induces an embedding.

Paper 3, Section II
20F Algebraic Geometry

(i) Let X be an affine variety. Define the *tangent space* of X at a point P. Say what it means for the variety to be singular at P.

(ii) Find the singularities of the surface in $\mathbb{P}^3$ given by the equation

$$xyz + yzw + zwx + wxy = 0.$$

(iii) Consider $C = Z(x^2 - y^3) \subseteq \mathbb{A}^2$. Let $X \to \mathbb{A}^2$ be the blowup of the origin. Compute the proper transform of C in X, and show it is non-singular.

Paper 2, Section II
21F Algebraic Geometry

(i) Define the radical of an ideal.

(ii) Assume the following statement: If k is an algebraically closed field and $I \subseteq k[x_1, \ldots, x_n]$ is an ideal, then either $I = (1)$ or $Z(I) \neq \emptyset$. Prove the Hilbert Nullstellensatz, namely that if $I \subseteq k[x_1, \ldots, x_n]$ with k algebraically closed, then

$$I(Z(I)) = \sqrt{I}.$$

(iii) Show that if A is a commutative ring and $I, J \subseteq A$ are ideals, then

$$\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$

(iv) Is

$$\sqrt{I + J} = \sqrt{I} + \sqrt{J}\,?$$

Give a proof or a counterexample.

Paper 1, Section II
21F Algebraic Geometry

Let k be an algebraically closed field.

(i) Let X and Y be affine varieties defined over k. Given a map $f : X \to Y$, define what it means for f to be a morphism of affine varieties.

(ii) With X, Y still affine varieties over k, show that there is a one-to-one correspondence between $\mathrm{Hom}(X, Y)$, the set of morphisms between X and Y, and $\mathrm{Hom}(A(Y), A(X))$, the set of k-algebra homomorphisms between $A(Y)$ and $A(X)$.

(iii) Let $f : \mathbb{A}^2 \to \mathbb{A}^4$ be given by $f(t, u) = (u, t, t^2, tu)$. Show that the image of f is an affine variety X, and find a set of generators for $I(X)$.

Paper 4, Section II

23H Algebraic Geometry

Let X be a smooth projective curve of genus $g > 0$ over an algebraically closed field of characteristic $\neq 2$, and suppose there is a degree 2 morphism $\pi : X \to \mathbf{P}^1$. How many ramification points of π are there?

Suppose Q and R are distinct ramification points of π. Show that $Q \not\sim R$, but $2Q \sim 2R$.

Now suppose $g = 2$. Show that every divisor of degree 2 on X is linearly equivalent to $P + P'$ for some $P, P' \in X$, and deduce that every divisor of degree 0 is linearly equivalent to $P_1 - P_2$ for some $P_1, P_2 \in X$.

Show that the subgroup $\{[D] \in Cl^0(X) \mid 2[D] = 0\}$ of the divisor class group of X has order 16.

Paper 3, Section II

23H Algebraic Geometry

Let $f \in k[x]$ be a polynomial with distinct roots, $\deg f = d > 2$, $\operatorname{char} k = 0$, and let $C \subseteq \mathbf{P}^2$ be the projective closure of the affine curve

$$y^{d-1} = f(x).$$

Show that C is smooth, with a single point at ∞.

Pick an appropriate $\omega \in \Omega^1_{k(C)/k}$ and compute the valuation $v_q(\omega)$ for all $q \in C$.

Hence determine $\deg \mathcal{K}_C$.

Paper 2, Section II

24H Algebraic Geometry

(i) Let k be an algebraically closed field, $n \geqslant 1$, and S a subset of k^n.

Let $I(S) = \{f \in k[x_1, \ldots, x_n] \mid f(p) = 0 \text{ when } p \in S\}$. Show that $I(S)$ is an ideal, and that $k[x_1, \ldots, x_n]/I(S)$ does not have any non-zero nilpotent elements.

Let $X \subseteq \mathbf{A}^n$, $Y \subseteq \mathbf{A}^m$ be affine varieties, and $\Phi : k[Y] \to k[X]$ be a k-algebra homomorphism. Show that Φ determines a map of sets from X to Y.

(ii) Let X be an irreducible affine variety. Define the *dimension* of X, $\dim X$ (in terms of the tangent spaces of X) and the *transcendence dimension* of X, $\operatorname{tr.dim} X$.

State the Noether normalization theorem. Using this, or otherwise, prove that the transcendence dimension of X equals the dimension of X.

Paper 1, Section II

24H Algebraic Geometry

Let k be an algebraically closed field and $n \geqslant 1$. We say that $f \in k[x_1, \ldots, x_n]$ is *singular* at $p \in \mathbf{A}^n$ if either p is a singularity of the hypersurface $\{f = 0\}$ or f has an irreducible factor h of multiplicity strictly greater than one with $h(p) = 0$. Given $d \geqslant 1$, let $X = \{f \in k[x_1, \ldots, x_n] \mid \deg f \leqslant d\}$ and let

$$Y = \{(f, p) \in X \times \mathbf{A}^n \mid f \text{ is singular at } p\}.$$

(i) Show that $X \simeq \mathbf{A}^N$ for some N (you need not determine N) and that Y is a Zariski closed subvariety of $X \times \mathbf{A}^n$.

(ii) Show that the fibres of the projection map $Y \to \mathbf{A}^n$ are linear subspaces of dimension $N - (n+1)$. Conclude that $\dim Y < \dim X$.

(iii) Hence show that $\{f \in X \mid \deg f = d,\ Z(f) \text{ smooth}\}$ is dense in X.

[You may use standard results from lectures if they are accurately quoted.]

Paper 3, Section II

23H Algebraic Geometry

Let $C \subset \mathbb{P}^2$ be the plane curve given by the polynomial

$$X_0^n - X_1^n - X_2^n$$

over the field of complex numbers, where $n \geqslant 3$.

(i) Show that C is nonsingular.

(ii) Compute the divisors of the rational functions

$$x = \frac{X_1}{X_0}, \quad y = \frac{X_2}{X_0}$$

on C.

(iii) Consider the morphism $\phi = (X_0 : X_1) \colon C \to \mathbb{P}^1$. Compute its ramification points and degree.

(iv) Show that a basis for the space of regular differentials on C is

$$\left\{ x^i y^j \omega_0 \;\middle|\; i, j \geqslant 0,\ i + j \leqslant n - 3 \right\}$$

where $\omega_0 = dx/y^{n-1}$.

Paper 4, Section II

23H Algebraic Geometry

Let C be a nonsingular projective curve, and D a divisor on C of degree d.

(i) State the Riemann–Roch theorem for D, giving a brief explanation of each term. Deduce that if $d > 2g - 2$ then $\ell(D) = 1 - g + d$.

(ii) Show that, for every $P \in C$,

$$\ell(D - P) \geqslant \ell(D) - 1.$$

Deduce that $\ell(D) \leqslant 1 + d$. Show also that if $\ell(D) > 1$, then $\ell(D - P) = \ell(D) - 1$ for all but finitely many $P \in C$.

(iii) Deduce that for every $d \geqslant g - 1$ there exists a divisor D of degree d with $\ell(D) = 1 - g + d$.

Paper 2, Section II

24H Algebraic Geometry

Let $V \subset \mathbb{P}^3$ be an irreducible quadric surface.

(i) Show that if V is singular, then every nonsingular point lies in exactly one line in V, and that all the lines meet in the singular point, which is unique.

(ii) Show that if V is nonsingular then each point of V lies on exactly two lines of V.

Let V be nonsingular, P_0 a point of V, and $\Pi \subset \mathbb{P}^3$ a plane not containing P_0. Show that the projection from P_0 to Π is a birational map $f \colon V \dashrightarrow \Pi$. At what points does f fail to be regular? At what points does f^{-1} fail to be regular? Justify your answers.

Paper 1, Section II

24H Algebraic Geometry

Let $V \subset \mathbb{A}^n$ be an affine variety over an algebraically closed field k. What does it mean to say that V is *irreducible*? Show that any non-empty affine variety $V \subset \mathbb{A}^n$ is the union of a finite number of irreducible affine varieties $V_j \subset \mathbb{A}^n$.

Define the *ideal* $I(V)$ of V. Show that $I(V)$ is a prime ideal if and only if V is irreducible.

Assume that the base field k has characteristic zero. Determine the irreducible components of

$$V\big(X_1X_2,\, X_1X_3 + X_2^2 - 1,\, X_1^2(X_1 - X_3)\big) \subset \mathbb{A}^3\,.$$

Paper 4, Section II

23I Algebraic Geometry

Let X be a smooth projective curve of genus 2, defined over the complex numbers. Show that there is a morphism $f : X \to \mathbf{P}^1$ which is a double cover, ramified at six points.

Explain briefly why X cannot be embedded into $\mathbf{P}^2$.

For any positive integer n, show that there is a smooth affine plane curve which is a double cover of $\mathbf{A}^1$ ramified at n points.

[State clearly any theorems that you use.]

Paper 3, Section II

23I Algebraic Geometry

Let $X \subset \mathbf{P}^2(\mathbf{C})$ be the projective closure of the affine curve $y^3 = x^4 + 1$. Let ω denote the differential dx/y^2. Show that X is smooth, and compute $v_p(\omega)$ for all $p \in X$.

Calculate the genus of X.

Paper 2, Section II

24I Algebraic Geometry

Let k be a field, J an ideal of $k[x_1, \ldots, x_n]$, and let $R = k[x_1, \ldots, x_n]/J$. Define the radical $\sqrt{J}$ of J and show that it is also an ideal.

The Nullstellensatz says that if J is a maximal ideal, then the inclusion $k \subseteq R$ is an *algebraic* extension of fields. Suppose from now on that k is algebraically closed. Assuming the above statement of the Nullstellensatz, prove the following.

(i) If J is a maximal ideal, then $J = (x_1 - a_1, \ldots, x_n - a_n)$, for some $(a_1, \ldots, a_n) \in k^n$.

(ii) If $J \neq k[x_1, \ldots, x_n]$, then $Z(J) \neq \emptyset$, where

$$Z(J) = \{a \in k^n \mid f(a) = 0 \text{ for all } f \in J\}.$$

(iii) For V an affine subvariety of k^n, we set

$$I(V) = \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0 \text{ for all } a \in V\}.$$

Prove that $J = I(V)$ for some affine subvariety $V \subseteq k^n$, if and only if $J = \sqrt{J}$.

[*Hint. Given $f \in J$, you may wish to consider the ideal in $k[x_1, \ldots, x_n, y]$ generated by J and $yf - 1$.*]

(iv) If A is a finitely generated algebra over k, and A does not contain nilpotent elements, then there is an affine variety $V \subseteq k^n$, for some n, with $A = k[x_1, \ldots, x_n]/I(V)$.

Assuming $\text{char}(k) \neq 2$, find $\sqrt{J}$ when J is the ideal $(x(x-y)^2, y(x+y)^2)$ in $k[x, y]$.

Paper 1, Section II

24I Algebraic Geometry

(a) Let X be an affine variety, $k[X]$ its ring of functions, and let $p \in X$. Assume k is algebraically closed. Define the *tangent space* T_pX at p. Prove the following assertions.

(i) A morphism of affine varieties $f : X \to Y$ induces a linear map

$$df : T_pX \to T_{f(p)}Y.$$

(ii) If $g \in k[X]$ and $U := \{x \in X \mid g(x) \neq 0\}$, then U has the natural structure of an affine variety, and the natural morphism of U into X induces an isomorphism $T_pU \to T_pX$ for all $p \in U$.

(iii) For all $s \geqslant 0$, the subset $\{x \in X \mid \dim T_xX \geqslant s\}$ is a Zariski-closed subvariety of X.

(b) Show that the set of nilpotent 2×2 matrices

$$X = \{x \in \mathrm{Mat}_2(k) \mid x^2 = 0\}$$

may be realised as an affine surface in $\mathbf{A}^3$, and determine its tangent space at all points $x \in X$.

Define what it means for two varieties Y_1 and Y_2 to be *birationally equivalent*, and show that the variety X of nilpotent 2×2 matrices is birationally equivalent to $\mathbf{A}^2$.

Paper 1, Section II
24H Algebraic Geometry

(i) Let X be an affine variety over an algebraically closed field. Define what it means for X to be *irreducible*, and show that if U is a non-empty open subset of an irreducible X, then U is dense in X.

(ii) Show that $n \times n$ matrices with distinct eigenvalues form an affine variety, and are a Zariski open subvariety of affine space $\mathbb{A}^{n^2}$ over an algebraically closed field.

(iii) Let $\mathrm{char}_A(x) = \det(xI - A)$ be the characteristic polynomial of A. Show that the $n \times n$ matrices A such that $\mathrm{char}_A(A) = 0$ form a Zariski closed subvariety of $\mathbb{A}^{n^2}$.

Hence conclude that this subvariety is all of $\mathbb{A}^{n^2}$.

Paper 2, Section II
24H Algebraic Geometry

(i) Let k be an algebraically closed field, and let I be an ideal in $k[x_0, \ldots, x_n]$. Define what it means for I to be homogeneous.

Now let $Z \subseteq \mathbb{A}^{n+1}$ be a Zariski closed subvariety invariant under $k^* = k - \{0\}$; that is, if $z \in Z$ and $\lambda \in k^*$, then $\lambda z \in Z$. Show that $I(Z)$ is a homogeneous ideal.

(ii) Let $f \in k[x_1, \ldots, x_{n-1}]$, and let $\Gamma = \{(x, f(x)) \mid x \in \mathbb{A}^{n-1}\} \subseteq \mathbb{A}^n$ be the graph of f. Let $\overline{\Gamma}$ be the closure of Γ in $\mathbb{P}^n$.

Write, in terms of f, the homogeneous equations defining $\overline{\Gamma}$.

Assume that k is an algebraically closed field of characteristic zero. Now suppose $n = 3$ and $f(x, y) = y^3 - x^2 \in k[x, y]$. Find the singular points of the projective surface $\overline{\Gamma}$.

Paper 3, Section II

23H Algebraic Geometry

Let X be a smooth projective curve over an algebraically closed field k of characteristic 0.

(i) Let D be a divisor on X.

Define $\mathcal{L}(D)$, and show $\dim \mathcal{L}(D) \leqslant \deg D + 1$.

(ii) Define the space of *rational differentials* $\Omega^1_{k(X)/k}$.

If p is a point on X, and t a local parameter at p, show that $\Omega^1_{k(X)/k} = k(X)dt$.

Use that equality to give a definition of $v_p(\omega) \in \mathbb{Z}$, for $\omega \in \Omega^1_{k(X)/k}$, $p \in X$. [You need not show that your definition is independent of the choice of local parameter.]

Paper 4, Section II

23H Algebraic Geometry

Let X be a smooth projective curve over an algebraically closed field k.

State the Riemann–Roch theorem, briefly defining all the terms that appear.

Now suppose X has genus 1, and let $P_\infty \in X$.

Compute $\mathcal{L}(nP_\infty)$ for $n \leqslant 6$. Show that ϕ_{3P_∞} defines an isomorphism of X with a smooth plane curve in $\mathbb{P}^2$ which is defined by a polynomial of degree 3.

Paper 1, Section II
24G Algebraic Geometry

(i) Let $X = \{(x, y) \in \mathbb{C}^2 \mid x^2 = y^3\}$. Show that X is birational to $\mathbf{A}^1$, but not isomorphic to it.

(ii) Let X be an affine variety. Define the *dimension* of X in terms of the tangent spaces of X.

(iii) Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial, where k is an algebraically closed field of arbitrary characteristic. Show that $\dim Z(f) = n - 1$.

[You may assume the Nullstellensatz.]

Paper 2, Section II
24G Algebraic Geometry

Let $X = X_{n,m,r}$ be the set of $n \times m$ matrices of rank at most r over a field k. Show that $X_{n,m,r}$ is naturally an affine subvariety of $\mathbf{A}^{nm}$ and that $X_{n,m,r}$ is a Zariski closed subvariety of $X_{n,m,r+1}$.

Show that if $r < \min(n, m)$, then 0 is a singular point of X.

Determine the dimension of $X_{5,2,1}$.

Paper 3, Section II
23G Algebraic Geometry

(i) Let X be a curve, and $p \in X$ be a smooth point on X. Define what a *local parameter* at p is.

Now let $f : X \dashrightarrow Y$ be a rational map to a quasi-projective variety Y. Show that if Y is projective, f extends to a morphism defined at p.

Give an example where this fails if Y is not projective, and an example of a morphism $f : \mathbb{C}^2 \setminus \{0\} \to \mathbf{P}^1$ which does not extend to 0.

(ii) Let $V = Z(X_0^8 + X_1^8 + X_2^8)$ and $W = Z(X_0^4 + X_1^4 + X_2^4)$ be curves in $\mathbf{P}^2$ over a field of characteristic not equal to 2. Let $\phi : V \to W$ be the map $[X_0 : X_1 : X_2] \mapsto [X_0^2 : X_1^2 : X_2^2]$. Determine the degree of ϕ, and the ramification e_p for all $p \in V$.

Paper 4, Section II

23G Algebraic Geometry

Let $E \subseteq \mathbf{P}^2$ be the projective curve obtained from the affine curve $y^2 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$, where the λ_i are distinct and $\lambda_1 \lambda_2 \lambda_3 \neq 0$.

(i) Show there is a unique point at infinity, P_∞.

(ii) Compute $\operatorname{div}(x)$, $\operatorname{div}(y)$.

(iii) Show $\mathcal{L}(P_\infty) = k$.

(iv) Compute $l(nP_\infty)$ for all n.

[You may *not* use the Riemann–Roch theorem.]

Paper 1, Section II

24G Algebraic Geometry

Define what is meant by a *rational map* from a projective variety $V \subset \mathbb{P}^n$ to $\mathbb{P}^m$. What is a *regular point* of a rational map?

Consider the rational map $\phi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by

$$(X_0 : X_1 : X_2) \mapsto (X_1X_2 : X_0X_2 : X_0X_1).$$

Show that ϕ is not regular at the points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ and that it is regular elsewhere, and that it is a birational map from $\mathbb{P}^2$ to itself.

Let $V \subset \mathbb{P}^2$ be the plane curve given by the vanishing of the polynomial $X_0^2X_1^3 + X_1^2X_2^3 + X_2^2X_0^3$ over a field of characteristic zero. Show that V is irreducible, and that ϕ determines a birational equivalence between V and a nonsingular plane quartic.

Paper 2, Section II

24G Algebraic Geometry

Let V be an irreducible variety over an algebraically closed field k. Define the *tangent space* of V at a point P. Show that for any integer $r \geqslant 0$, the set $\{P \in V \mid \dim T_{V,P} \geqslant r\}$ is a closed subvariety of V.

Assume that k has characteristic different from 2. Let $V = V(I) \subset \mathbb{P}^4$ be the variety given by the ideal $I = (F, G) \subset k[X_0, \ldots, X_4]$, where

$$F = X_1X_2 + X_3X_4, \qquad G = X_0X_1 + X_3^2 + X_4^2.$$

Determine the singular subvariety of V, and compute $\dim T_{V,P}$ at each singular point P. [You may assume that V is irreducible.]

Paper 3, Section II

23G Algebraic Geometry

Let V be a smooth projective curve, and let D be an effective divisor on V. Explain how D defines a morphism ϕ_D from V to some projective space. State the necessary and sufficient conditions for ϕ_D to be finite. State the necessary and sufficient conditions for ϕ_D to be an isomorphism onto its image.

Let V have genus 2, and let K be an effective canonical divisor. Show that the morphism ϕ_K is a morphism of degree 2 from V to $\mathbb{P}^1$.

By considering the divisor $K + P_1 + P_2$ for points P_i with $P_1 + P_2 \not\sim K$, show that there exists a birational morphism from V to a singular plane quartic.

[You may assume the Riemann–Roch Theorem.]

Paper 4, Section II

23G Algebraic Geometry

State the Riemann–Roch theorem for a smooth projective curve V, and use it to outline a proof of the Riemann–Hurwitz formula for a non-constant morphism between projective nonsingular curves in characteristic zero.

Let $V \subset \mathbb{P}^2$ be a smooth projective plane cubic over an algebraically closed field k of characteristic zero, written in normal form $X_0X_2^2 = F(X_0, X_1)$ for a homogeneous cubic polynomial F, and let $P_0 = (0 : 0 : 1)$ be the point at infinity. Taking the group law on V for which P_0 is the identity element, let $P \in V$ be a point of order 3. Show that there exists a linear form $H \in k[X_0, X_1, X_2]$ such that $V \cap V(H) = \{P\}$.

Let $H_1,\ H_2 \in k[X_0, X_1, X_2]$ be nonzero linear forms. Suppose the lines $\{H_i = 0\}$ are distinct, do not meet at a point of V, and are nowhere tangent to V. Let $W \subset \mathbb{P}^3$ be given by the vanishing of the polynomials

$$X_0X_2^2 - F(X_0, X_1), \quad X_3^2 - H_1(X_0, X_1, X_2)H_2(X_0, X_1, X_2).$$

Show that W has genus 4. [You may assume without proof that W is an irreducible smooth curve.]