NOTES FROM MATH 8230: SYMPLECTIC TOPOLOGY, UGA, SPRING 2019, BY MIKE USHER

CONTENTS

1. GEOMETRY AND HAMILTONIAN FLOWS ON $\mathbb{R}^{2n}$

1.1. **Hamilton's equations.** Initial interest in symplectic geometry came from its role in Hamilton's formulation of classical mechanics. I will begin by explaining enough about Hamiltonian mechanics to indicate how symplectic geometry naturally arises. I will not go very deeply into the physics (in particular I won't say anything at all about Lagrangian mechanics, for example; [MS, Chapter 1] discusses this a little bit, and [Ar] much more so.)

Hamilton described the state of an n-dimensional physical system using a total of $2n$ coordinates $p_1, \dots, p_n, q_1, \dots, q_n$; in some standard examples (*e.g.* in Example 1.1) q_i denotes the ith position coordinate and p_i denotes the ith momentum coordinate. The way in which the system evolves is dictated by a "Hamiltonian function" $H \colon \mathbb{R}^{2n} \to \mathbb{R}$. If the state of the system at time t is $(p_1(t), \dots, p_n(t), q_1(t), \dots, q_n(t))$, **Hamilton's equations** say that, for $i = 1, \dots, n$,

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i}. \end{aligned} \tag{1}$$

Here the dots over p_i and q_i refer to differentiation with respect to t.

Example 1.1. *The most famous Hamiltonian functions H look like*

$$H(\vec{p}, \vec{q}) = \frac{\|\vec{p}\|^2}{2m} + U(\vec{q})$$

where $m > 0$ denotes mass, $U \colon \mathbb{R}^n \to \mathbb{R}$ is a smooth[1] *function, and we abbreviate $\vec{p} = (p_1, \dots, p_n)$ and $\vec{q} = (q_1, \dots, q_n)$. Then Hamilton's equations read*

$$\begin{aligned} \dot{p}_i &= -\frac{\partial U}{\partial q_i} \\ \dot{q}_i &= \frac{p_i}{m}, \end{aligned}$$

or more concisely

$$\begin{aligned} \dot{\vec{p}} &= -\nabla U \\ \dot{\vec{q}} &= \frac{1}{m}\vec{p}. \end{aligned}$$

The second equation says that $\vec{p} = m\dot{\vec{q}}$ is indeed the momentum (according to the usual high school physics definition) of the system, and then the first equation says that

$$m\ddot{\vec{q}} = -\nabla U.$$

This latter equation—a second-order equation taking place in n-dimensional space, as opposed to Hamilton's equations which give a first-order equation taking place in $2n$-dimensional space—is Newton's Second Law of Motion for a system being acted on by a force $\vec{F} = -\nabla U$. Then U has the physical interpretation of potential energy, and the whole Hamiltonian $H(\vec{p}, \vec{q}) = \frac{\|\vec{p}\|^2}{2m} + U(\vec{q}) = \frac{m\|\dot{\vec{q}}\|^2}{2} + U(\vec{q})$ is the sum of the kinetic energy and the potential energy.

If you know a little about differential equations, you have probably seen before the idea that one can reduce a second-order system (like Newton's Second Law) to a first-order system (like Hamilton's equations) by doubling the number of variables.

Exercise 1.2. *Take $n = 3$, let $\vec{A} \colon \mathbb{R}^3 \to \mathbb{R}^3$ be smooth, and let e, m be constants with $m > 0$. Consider the Hamiltonian*

$$H(\vec{p}, \vec{q}) = \frac{1}{2m}\|\vec{p} - e\vec{A}(\vec{q})\|^2.$$

Show that Hamilton's equations give rise to the second-order equation

$$m\ddot{\vec{q}} = e\dot{\vec{q}} \times (\nabla \times \vec{A}) \tag{2}$$

for $\vec{q}(t)$, where $\times$ denotes cross product.

[1] I always intend "smooth" to mean "infinitely differentiable."

Remark 1.3. The equation (2) is the "Lorentz Force Law" for a particle of mass m and electric charge e moving in a magnetic field $\vec{B} = \nabla \times \vec{A}$. In particular this example comes from a simple and natural physical setting. It should become clear early in the course of solving Exercise 1.2 that in this case it is not true that $\vec{p}$ is equal to the classical momentum $m\dot{\vec{q}}$. More generally, from the perspective of symplectic geometry, the coordinates p_i and q_i can take on many different meanings. This is already apparent from the basic form of Hamilton's equations: replacing p_i and q_i by, respectively, q_i and $-p_i$ leaves those equations unchanged, so there is a symmetry that effectively interchanges "position" and "momentum;" this is not something that one would likely anticipate based on older (Newtonian or Lagrangian) formulations of classical mechanics.

1.2. **Divergence-free vector fields.** I would now like to make the case that considering Hamilton's equations leads one to interesting geometry. I'll use x as my generic name for an element of $\mathbb{R}^{2n}$ (so we can write $x = (\vec{p}, \vec{q})$). A more concise expression of Hamilton's equations is

$$\dot{x}(t) = X_H(x(t))$$

where the **Hamiltonian vector field** $X_H \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is defined by

$$X_H = \left(-\frac{\partial H}{\partial q_1}, \ldots, -\frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n} \right),$$

or, in notation that I hope is self-explanatory,

$$X_H(\vec{p}, \vec{q}) = \left(-\nabla^{(q)} H, \nabla^{(p)} H \right).$$

A first indication that X_H is a special kind of vector field is:

Proposition 1.4. *For any smooth function $H \colon \mathbb{R}^{2n} \to \mathbb{R}$, the corresponding Hamiltonian vector field X_H is divergence-free: $\nabla \cdot X_H = 0$.*

(Of course, the divergence of a vector field $V = (V_1, \ldots, V_k)$ on $\mathbb{R}^k$ is defined to be $\nabla \cdot V = \sum_{i=1}^k \frac{\partial V_i}{\partial x_i}$.)

Proof. We find

$$\nabla \cdot X_H = \sum_{i=1}^n \frac{\partial}{\partial p_i} \left(-\frac{\partial H}{\partial q_i} \right) + \sum_{i=1}^n \frac{\partial}{\partial q_i} \left(\frac{\partial H}{\partial p_i} \right) = \sum_{i=1}^n \left(-\frac{\partial^2 H}{\partial p_i \partial q_i} + \frac{\partial^2 H}{\partial q_i \partial p_i} \right) = 0 \tag{3}$$

by equality of mixed partials (which applies since H is assumed smooth—C^2 would indeed be enough). □

Let us consider the geometric significance of a vector field being divergence-free. Recall that the divergence theorem asserts that, if $E \subset \mathbb{R}^k$ is a compact region with piecewise smooth boundary ∂E, and if V is a vector field on $\mathbb{R}^k$, then the flux of V through ∂E is given by

$$\int_{\partial E} V \cdot d\vec{S} = \int_E (\nabla \cdot V) d^k x.$$

So if $\nabla \cdot V = 0$, as is the case when $V = X_H$, then the flux of V through the boundary ∂E of any compact region E (with piecewise smooth boundary) is 0.

Remark 1.5. Expressing all of this in the language of differential forms—and perhaps also giving some reassurance that the case(s) of the divergence theorem that you learned about in multivariable calculus still work in higher dimension—let $\mathbf{vol}_k = dx_1 \wedge \cdots \wedge dx_k$ be the standard volume form on

$\mathbb{R}^k$. So the (k-dimensional) volume of a region E is by definition $\mathrm{vol}(E) = \int_E \mathbf{vol}_k$. Corresponding to the vector field $V = (V_1, \ldots, V_k)$ is the $(k-1)$-form[2]

$$\iota_V \mathbf{vol}_k = \sum_{i=1}^{k} (-1)^{i-1} V_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k.$$

One can verify that the flux of V through an oriented hypersurface S is then $\int_S (\iota_V \mathbf{vol}_k)$. (Or, if you prefer, you can simply define the flux by this formula.) It's easy to check that the exterior derivative of $\iota_V \mathbf{vol}_k$ is given by

$$d(\iota_V \mathbf{vol}_k) = \left(\sum_{i=1}^{k} \frac{\partial V_i}{\partial x_i} \right) dx_1 \wedge \cdots dx_k = (\nabla \cdot V)\mathbf{vol}_k.$$

So (the differential forms version of) Stokes' theorem shows that the flux of V through the piecewise smooth boundary of a compact region E is

$$\int_{\partial E} \iota_V \mathbf{vol}_k = \int_E (\nabla \cdot V)\mathbf{vol}_k,$$

consistently with the divergence theorem as stated above.

One virtue of this formulation of the divergence theorem is that it immediately extends to the general case of a smooth k-dimensional manifold M endowed with a nowhere-vanishing k-form[3] $\mathbf{vol}_M$: given a vector field V on M one can *define* the divergence $\nabla \cdot V$ of V to be the smooth function on M that obeys $d(\iota_V \mathbf{vol}_M) = (\nabla \cdot V)\mathbf{vol}_M$, and then Stokes' theorem immediately yields a version of the divergence theorem for codimension-zero submanifolds-with-boundary of M.

If one thinks in terms of the usual interpretation of a vector field as describing the velocity distribution of a fluid, and of the flux through a hypersurface as representing the net flow of the fluid through a membrane, the divergence theorem shows that if the vector field is divergence-free and the membrane is the boundary of a compact region then there will be as much fluid entering the region as leaving it. Or one could imagine that the membrane is elastic (but impermeable) and the fluid is exerting pressure on it; then the membrane will tend to expand outward at some points (where $V \cdot d\vec{S}$ is positive) and contract inward at others, but the statement that the flux is zero indicates that these balance each other out in such a way that, although the shape of the membrane changes, the total volume enclosed remains the same.

To make the statements in the previous paragraph into proper mathematics we should speak precisely about flows of vector fields, a notion that plays a central role in symplectic and contact topology. If V is a (smooth) vector field on $\mathbb{R}^k$ and $x_0 \in \mathbb{R}^k$, the existence and uniqueness theorem for ODE's ([L, Theorems 17.17 and 17.18]) shows that there is an open interval $I_{x_0} \subset \mathbb{R}$ around 0 and a unique map $\gamma^V_{x_0} \colon I_{x_0} \to \mathbb{R}^k$ having the property that $\gamma^V_{x_0}(0) = x_0$ and $\dot{\gamma}^V_{x_0}(t) = V(\gamma^V_{x_0}(t))$ for all $t \in I_{x_0}$. The vector field V is said to be "complete" if we can take $I_{x_0} = (-\infty, \infty)$ for every x_0; we will typically assume implicitly that the vector fields that we work with are complete. Every Lipschitz vector field is complete, so it suffices for V to have uniform bounds on its partial derivatives. If V is complete, then we get a map

$$\Psi^V \colon \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k \quad \text{defined by} \quad \Psi^V(t, x) = \gamma^V_x(t).$$

[2]More generally, recall that if V is a vector field and θ is a differential k-form then one has an "interior product" $\iota_V \theta$, which is a $(k-1)$-form defined by, for all vectors $w_1, \ldots, w_{k-1}$, $(\iota_V \theta)(w_1, \ldots, w_{k-1}) = \theta(V, w_1, \ldots, w_{k-1})$.

[3]Recall that this determines an orientation of M, and under the assumption that M is oriented specifying $\mathbf{vol}_M$ is basically the same as giving a measure on M that satisfies a smoothness hypothesis.

So if the vector field V describes the velocity of a fluid and a molecule of the fluid is located at the position x at time 0, then at time t that molecule will be located at the position $\Psi^V(t,x)$. By "smooth dependence on initial conditions" ([L, Theorem 17.19]), the smoothness of V implies that Ψ^V is also smooth. Evidently $\Psi^V(0,x) = x$ for all x, and a moment's thought (using the "uniqueness" part of existence and uniqueness) shows that, for $s,t \in \mathbb{R}$ and $x \in \mathbb{R}^k$,

$$\Psi^V(s+t,x) = \Psi^V(s,\Psi^V(t,x)). \tag{4}$$

Definition 1.6. If V is a complete vector field on $\mathbb{R}^k$ and $t \in \mathbb{R}$, the *time-t flow* of V is the map $\psi^{V,t} \colon \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$\psi^{V,t}(x) = \Psi^V(t,x) = \gamma_x^V(t).$$

Proposition 1.7. *For any complete vector field V we have:*

(i) *$\psi^{V,0}$ is the identity, and for $s,t \in \mathbb{R}$ we have $\psi^{V,s+t} = \psi^{V,s} \circ \psi^{V,t}$.*
(ii) *For $x \in \mathbb{R}$, $\frac{d}{dt}\psi^{V,t}(x) = V(\psi^{V,t}(x))$.*
(iii) *$\psi^{V,t}$ is a diffeomorphism, with inverse $\psi^{V,-t}$.*

Proof. (i) follows directly from (4) and the definitions. (ii) just follows from the definition of γ_x^V: we have

$$\frac{d}{dt}\psi^{V,t}(x) = \frac{d}{dt}\left(\gamma_x^V(t)\right) = V(\gamma_x^V(t)) = V(\psi^{V,t}(x)).$$

Finally, since the map Ψ^V is smooth, so are all of the maps $\psi^{V,t}$, and moreover (i) shows that $\psi^{V,t}$ and $\psi^{V,-t}$ are each other's inverses. □

Here then is a very geometric interpretation of the divergence-free condition, which should seem consistent with our prior considerations involving the divergence theorem.

Theorem 1.8. *A smooth, complete vector field V on $\mathbb{R}^k$ satisfies $\nabla \cdot V = 0$ if and only if, for all $t \in \mathbb{R}$ and all measurable subsets $E \subset \mathbb{R}^k$, we have $\mathrm{vol}(\psi^{V,t}(E)) = \mathrm{vol}(E)$.*

Partial proof. I will use the following fact, which you may or may not yet know how to prove (it uses Cartan's formula):

Fact 1.9. We have $\nabla \cdot V = 0$ (*i.e.*, $d\iota_V\mathbf{vol}_k = 0$) if and only if, for all t, $\psi^{V,t*}\mathbf{vol}_k = \mathbf{vol}_k$.

Assuming this, if $\nabla \cdot V = 0$ and if E is a compact region with piecewise smooth boundary, then the change-of-variables formula yields

$$\mathrm{vol}(\psi^{V,t}(E)) = \int_{\psi^{V,t}(E)} \mathbf{vol}_k = \int_E \psi^{V,t*}\mathbf{vol}_k = \int_E \mathbf{vol}_k = \mathrm{vol}(E).$$

The case that E is an arbitray measurable subset follows from the special case that E has piecewise smooth boundary (or even just that E is a product of intervals) by standard approximation arguments.

Conversely, if $\mathrm{vol}(\psi^{V,t}(E)) = \mathrm{vol}(E)$ for every measurable E (or even just every ball E), then the change-of-variables formula as above yields, for every ball E,

$$\int_E (\psi^{V,t*}\mathbf{vol}_k - \mathbf{vol}_k) = 0.$$

If there were some point at which the (smooth) differential form $\psi^{V,t*}\mathbf{vol}_k - \mathbf{vol}_k$ were nonzero, then taking E to be a small ball around that point would then yield a contradiction. (More precisely, E should be so small that the function f defined by $\psi^{V,t*}\mathbf{vol}_k - \mathbf{vol}_k = f\mathbf{vol}_k$ has the same sign throughout E.) So $\psi^{V,t*}\mathbf{vol}_k = \mathbf{vol}_k$ everywhere, whence $\nabla \cdot V = 0$ by Fact 1.9.

This shows that Fact 1.9 implies Theorem 1.8. I am calling this a "partial proof" because I am not providing here a proof of Fact 1.9. □

1.3. **Hamiltonian flows and the standard symplectic form.** Theorem 1.8 and Proposition 1.4 combine to show that the flows $\psi^{X_H,t}$ of Hamiltonian vector fields—*i.e.*, the maps $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ that send each possible initial state of a physical system to the state to which it evolves after t units of time— are *volume-preserving* diffeomorphisms. This fact, known as Liouville's theorem, was an early indication that Hamiltonian mechanics has interesting geometric features. Using the language of differential forms, which were not known to Liouville and his contemporaries, one can refine this statement to a statement about 2-dimensional areas "enclosed" by curves instead of $2n$-dimensional volumes enlosed by hypersurfaces. Roughly this refinement corresponds to the fact that each of the n individual terms $-\frac{\partial^2 H}{\partial p_i \partial q_i} + \frac{\partial^2 H}{\partial q_i \partial p_i}$ in (3) vanishes, whereas Proposition 1.4 just relies on the fact that their sum vanishes.

To discuss 2-dimensional (signed) areas "enclosed" by curves we introduce the **standard symplectic form** on $\mathbb{R}^{2n}$:

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

This is a 2-form on $\mathbb{R}^{2n}$, which then determines the signed area of any oriented compact surface in $\mathbb{R}^{2n}$ by integration.

Definition 1.10. Let $\gamma\colon S^1 \to \mathbb{R}^{2n}$ be a smooth map. The area "enclosed" by γ is given by

$$A(\gamma) = \int_{D^2} u^*\omega_0,$$

for any choice of smooth map $u\colon D^2 \to \mathbb{R}^{2n}$ such that $u|_{S^1} = \gamma$.

Here D^2 is the closed unit disk in $\mathbb{C}$, and we view S^1 as its boundary. One can think of u as consisting of an n-tuple of maps $(u_1, \ldots, u_n)$ where $u_i\colon D^2 \to \mathbb{R}^2$ is given by projecting u to the $p_i q_i$-plane, and then $A(\gamma)$ is the sum of the signed areas of the u_i. Here by "signed area" we mean the regions of D^2 on which u_i is orientation-preserving count positively while those on which it is orientation-reversing count negatively.

This definition may appear problematic because it defines $A(\gamma)$ in terms of u instead of γ. Of course a suitable map u exists because $\mathbb{R}^{2n}$ is simply connected (and because of standard results allowing one to approximate continuous maps by smooth ones), but it is far from unique. However, observe that ω_0 is an *exact* 2-form; for instance one has

$$\omega_0 = d\left(\sum_{i=1}^{n} p_i dq_i\right),$$

and so by Stokes' theorem

$$A(\gamma) = \int_{S^1} \gamma^*\left(\sum_{i=1}^{n} p_i dq_i\right)$$

which does not depend on u.

We will show soon that flows of Hamiltonian vector fields have the key property that they preserve areas enclosed by curves; if $n > 1$ this is a strictly stronger statement than the fact that they are volume-preserving. To begin we observe that Hamiltonian vector fields have a nice relationship to ω_0:

Proposition 1.11. *If $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ is smooth the corresponding Hamiltonian vector field $X_H = (-\nabla^{(q)}H, \nabla^{(p)}H)$ obeys*

$$\iota_{X_H}\omega_0 = -dH.$$

Moreover the only vector field V obeying the identity $\iota_V\omega_0 = -dH$ is $V = X_H$.

Proof. We can simply evaluate both sides of the equation on all of the coordinate vector fields ∂_{p_i} and ∂_{q_i}, as these collectively form a basis for the tangent space at each point of $\mathbb{R}^{2n}$. Of course $dH(\partial_{p_i}) = \frac{\partial H}{\partial p_i}$ and $dH(\partial_{q_i}) = \frac{\partial H}{\partial q_i}$. Since $\omega_0 = \sum_i dp_i \wedge dq_i$, a general vector field $V = (V_1, \ldots, V_{2n})$ has $\iota_V \omega_0(\partial_{q_i}) = \omega_0(V, \partial_{q_i}) = dp_i(V) = V_i$ while similarly $\iota_V \omega_0(\partial_{p_i}) = -dq_i(V) = -V_{n+i}$. So $\iota_V \omega_0 = -dH$ if and only if, for each i, both $V_i = -\frac{\partial H}{\partial q_i}$ and $V_{n+i} = \frac{\partial H}{\partial p_i}$. In other words, $\iota_V \omega_0 = -dH$ if and only if $V = X_H$. □

Remark 1.12. Later we will define a symplectic manifold to be a pair (M, ω) consisting of a smooth manifold M together with a 2-form ω locally modeled on ω_0. In view of Proposition 1.11, one can extend Hamilton's equations to a general symplectic manifold (M, ω): given a smooth function $H\colon M \to \mathbb{R}$ one gets a Hamiltonian vector field X_H by requiring $\iota_{X_H}\omega = -dH$, and then Hamilton's equations for a path $x\colon I \to M$ are $\dot{x}(t) = X_H(x(t))$.

Remark 1.13. Different sign conventions can be found in the literature; in particular in [MS] the Hamiltonian vector field is defined by $\iota_{X_H}\omega = dH$. Since one presumably wants Hamilton's equations to work out in the physically-correct manner, if one adopts their convention then one should also take the standard symplectic form on $\mathbb{R}^{2n}$ to be $\sum_i dq_i \wedge dp_i$ instead of $\sum_i dp_i \wedge dq_i$.

Here is a precise formulation of the statement that Hamiltonian flows preserve 2-dimensional area.

Proposition 1.14. *Let $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function whose Hamiltonian vector field X_H is complete, and let $\gamma\colon S^1 \to \mathbb{R}^{2n}$ be smooth. Then for all $T \in \mathbb{R}$ we have*

$$A(\psi^{X_H,T} \circ \gamma) = A(\gamma).$$

Proof. For notational convenience assume $T \geq 0$; there's no loss of generality in this since $\psi^{X_H,-T} = \psi^{X_{-H},T}$ so if necessary we could replace H by $-H$. Let $u\colon D^2 \to \mathbb{R}^{2n}$ be smooth with $u|_{S^1} = \gamma$. Thus by definition $A(\gamma) = \int_{D^2} u^*\omega_0$.

Define $\Gamma\colon [0, T] \times S^1 \to \mathbb{R}^{2n}$ by

$$\Gamma(t, \theta) = \psi^{X_H,t}(\gamma(\theta)) \tag{5}$$

for $\theta \in S^1$. Since $\psi^{X_H,0}$ is the identity, we have $\Gamma(0, \theta) = \gamma(\theta)$. Thus we can join $u\colon D^2 \to \mathbb{R}^{2n}$ with $\Gamma\colon [0, T] \times S^1 \to \mathbb{R}^{2n}$ by gluing ∂D^2 to $S^1 \times \{0\}$, yielding a continuous and piecewise smooth map from a topological disk, which restricts to the boundary $S^1 \times \{T\}$ of the disk to $\psi^{X_H,T} \circ \gamma$. By Stokes' theorem this map can be used to compute $A(\psi^{X_H,T} \circ \gamma)$, and so we have

$$A(\psi^{X_H,T} \circ \gamma) = \int_{D^2} u^*\omega_0 + \int_{[0,1]\times S^1} \Gamma^*\omega_0 = A(\gamma) + \int_{[0,T]\times S^1} \Gamma^*\omega_0.$$

So it suffices to show that $\int_{[0,T]\times S^1} \Gamma^*\omega_0 = 0$, where Γ is defined by (5). To do this, note that

$$\int_{[0,T]\times S^1} \Gamma^*\omega_0 = \int_0^T \int_0^{2\pi} (\omega_0)_{\Gamma(t,\theta)} (\Gamma_*\partial_t, \Gamma_*\partial_\theta)\, d\theta\, dt = \int_0^T \left(\int_0^{2\pi} (\omega_0)_{\psi^{X_H,t}\circ\gamma(\theta)} \left(X_H, (\psi^{X_H,t} \circ \gamma)_*\partial_\theta\right) d\theta \right) dt$$

where we have used Proposition 1.7 (ii). But we shall see that the inner integral is zero: indeed by Proposition 1.11

$$\begin{aligned}\int_0^{2\pi} (\omega_0)_{\psi^{X_H,t}\circ\gamma(\theta)} \left(X_H, (\psi^{X_H,t} \circ \gamma)_*\partial_\theta\right) d\theta &= -\int_0^{2\pi} (dH)_{\psi^{X_H,t}\circ\gamma(\theta)}((\psi^{X_H,t} \circ \gamma)_*\partial_\theta) d\theta \\ &= -\int_0^{2\pi} \frac{d}{d\theta}\left(H(\psi^{X_H,t} \circ \gamma(\theta))\right) d\theta = 0\end{aligned}$$

by the Fundamental Theorem of Calculus, since the integral is over the whole circle on which 0 and 2π are identified. □

Corollary 1.15. *Let $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function whose Hamiltonian vector field X_H is complete. Then for all $T \in \mathbb{R}$ we have $\psi^{X_H,T*}\omega_0 = \omega_0$.*

Proof. It follows immediately from Proposition 1.14 and the definitions that if $u\colon D^2 \to \mathbb{R}^{2n}$ is smooth then $\int_{D^2}(\psi^{X_H,T} \circ u)^*\omega_0 = \int_{D^2} u^*\omega_0$, *i.e.* that

$$\int_{D^2} u^*\left(\psi^{X_H,T*}\omega_0 - \omega_0\right) = 0. \tag{6}$$

In general if a two-form $\eta \in \Omega^2(\mathbb{R}^{2n})$ is not identically zero, so that there are $x \in \mathbb{R}^{2n}$ and $v, w \in T_x\mathbb{R}^{2n}$ with $\eta_x(v,w) \neq 0$, then one can construct[4] a map $u\colon D^2 \to \mathbb{R}^{2n}$ to a small neighborhood of x (say with derivative at 0 having image spanned by v and w) such that $u^*\eta$ is a positive multiple of the area form on D^2 everywhere, and so $\int_{D^2} u^*\eta > 0$. This reasoning, applied with $\eta = \psi^{X_H,T*}\omega_0 - \omega_0$, shows that (6) forces $\psi^{X_H,T*}\omega_0$ to equal ω_0 everywhere. □

Remark 1.16. Note that the differential form $\omega_0^{\wedge n}$ (*i.e.*, the result of taking the wedge product of ω_0 with itself n times) is equal to $n!dp_1 \wedge dq_1 \wedge \cdots \wedge dp_n \wedge dq_n$, *i.e.* to $(-1)^{n(n-1)/2}n!\mathbf{vol}_{2n}$. So a map $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ that satisfies $\phi^*\omega_0 = \omega_0$ also satisfies (by naturality of the wedge product with respect to pullback, implying $\phi^*(\omega_0^{\wedge n}) = (\phi^*\omega_0)^{\wedge n}$)

$$\phi^*\mathbf{vol}_{2n} = \mathbf{vol}_{2n}.$$

Thus Corollary 1.15 recovers the fact that Hamiltonian flows are volume-preserving. This also follows from Theorem 1.8, but the argument just given does not make use of Fact 1.9.

1.4. **Symmetries of Hamilton's equations.** At the end of Remark 1.3 we observed that Hamilton's equations are unchanged if we make a coordinate change replacing the coordinates p_i, q_i by, respectively, $q_i, -p_i$. Let us be a bit more precise (and maybe a little pedantic) about what we mean for Hamilton's equations to "unchanged" under a coordinate transformation. Such a transformation is encoded by a diffeomorphism $\psi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ (or more generally one could just take ψ to be a diffeomorphism between open subsets of $\mathbb{R}^{2n}$). So for example perhaps $\psi(\vec{p},\vec{q}) = (\vec{q}, -\vec{p})$. Breaking this down into coordinates, we get new functions $p_1', \ldots, p_n', q_1', \ldots, q_n'$ on $\mathbb{R}^{2n}$ (in the example $p_i' = q_i$ and $q_i' = -p_i$), and the bijectivity of ψ means that every point $(\vec{p},\vec{q})$ in $\mathbb{R}^{2n}$ is determined uniquely by the values of the $2n$ functions $p_1', \ldots, q_n'$ at that point.

So if we have a function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$, sending $(\vec{p},\vec{q})$ to $H(\vec{p},\vec{q})$, this function can instead be viewed as a function of the $2n$ values $p_1', \ldots, q_n'$, since specifying those values gives a unique point of $\mathbb{R}^{2n}$ which is sent by H to a particular real number. This is a rule that assigns to a $(2n)$-tuple of real numbers $(p_1', \ldots, q_n')$ a new real number, *i.e.* it is a function $\mathbb{R}^{2n} \to \mathbb{R}$, but it is not the function H—rather it is $H \circ \psi^{-1}$. Indeed ψ^{-1} sends $(p_1', \ldots, q_n')$ to the appropriate $(p_1, \ldots, q_n)$, which can then be plugged into H. This maybe becomes clearer if we introduce two different names for $\mathbb{R}^{2n}$: say E is "$\mathbb{R}^{2n}$ with coordinates $(p_1, \ldots, q_n)$" and E' is "$\mathbb{R}^{2n}$ with coordinates $(p_1', \ldots, q_n')$" and then regard E and E' as simply different spaces. So we initially have a function $H\colon E \to \mathbb{R}$ and a diffeomorphism $\psi\colon E \to E'$, and the only natural way of getting a function $E' \to \mathbb{R}$ out of this setup is to use $H \circ \psi^{-1}$.

With this said we make the following definition:

Definition 1.17. We say that a diffeomorphism $\psi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *is a symmetry of Hamilton's equations* provided that, for every smooth function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ and every $x\colon I \to \mathbb{R}^{2n}$ where I is an interval

[4] To be a little more explicit about how to do this, by interchanging v and w if necessary assume that $\eta_x(v,w) > 0$. Then if $\epsilon > 0$ is small enough, by continuity considerations we will continue to have $\eta_{x+\epsilon s v + \epsilon t w}(v,w) > 0$ whenever $(s,t) \in D^2$. For such a small ϵ, define $u\colon D^2 \to \mathbb{R}^{2n}$ by $u(s,t) = x + \epsilon s v + \epsilon t w$. Then $(u^*\eta)_{(s,t)} = \eta_{x+\epsilon s v+\epsilon t w}(v,w)ds \wedge dt$ is a positive multiple of $ds \wedge dt$ at each $(s,t) \in D^2$.

in $\mathbb{R}$, it holds that

$$\frac{dx}{dt} = X_H(x(t)) \text{ for all t} \quad \text{if and only if} \quad \frac{d(\psi\circ x)}{dt} = X_{H\circ\psi^{-1}}(\psi\circ x(t)) \text{ for all t.}$$

So ψ should take solutions of Hamilton's equations for any given H to solutions of Hamilton's equations for the appropriately-transformed version $H\circ\psi^{-1}$ of H. Here is an easy rephrasing:

Proposition 1.18. *A diffeomorphism $\psi\colon \mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is a symmetry of Hamilton's equations if and only if, for every smooth function $H\colon \mathbb{R}^{2n}\to\mathbb{R}$ and every $x\in\mathbb{R}^{2n}$, we have $\psi_*(X_H(x)) = X_{H\circ\psi^{-1}}(\psi(x))$.*

Proof. This follows directly from the fact that, by the chain rule,

$$\frac{d(\psi\circ x)}{dt} = \psi_*\frac{dx}{dt}.$$

(I suppose I am also using the existence theorem for ODE's for the forward implication; think about why.) □

Example 1.19. *Take $n=1$, let $H(p,q)=p^2+q^3$, and for our diffeomorphism use $\psi(p,q)=(q,-p)$ (so we are changing coordinates to $p'=q$ and $q'=-p$). Then $\psi^{-1}(p',q')=(-q',p')$, so $H\circ\psi^{-1}(p',q') = (q')^2+(p')^3$. We see that*

$$X_H(p,q) = (-\nabla^{(q)}H, \nabla^{(p)}H) = (-3q^2, 2p)$$

and likewise

$$X_{H\circ\psi^{-1}}(p',q') = (-2q', 3(p')^2).$$

So for $(p,q)\in\mathbb{R}^2$ we have

$$X_{H\circ\psi^{-1}}(\psi(p,q)) = X_{H\circ\psi^{-1}}(q,-p) = (2p, 3q^2)$$

which is indeed the image of $X_H(p,q)=(-3q^2,2p)$ under ψ_* (since ψ, being a linear map, is its own derivative). Said differently, Hamilton's equations in the original (p,q) coordinates (using $H=p^2+q^3$) say

$$\begin{aligned}\dot{p} &= -3q^2\\ \dot{q} &= 2p\end{aligned}$$

while Hamilton's equations in the (p',q') coordinates (using $H\circ\psi = (q')^2+(p')^3$) say

$$\begin{aligned}\dot{p}' &= -2q'\\ \dot{q}' &= 3(p')^2.\end{aligned}$$

Given that $p'=q$ and that $q'=-p$ these systems of equations are indeed equivalent.

We now show that being a symmetry of Hamilton's equations is the same as being a symmetry of the standard symplectic form.

Proposition 1.20. *A diffeomorphism $\psi\colon\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is a symmetry of Hamilton's equations if and only if $\psi^*\omega_0=\omega_0$.*

Proof. We have $\psi^*\omega_0=\omega_0$ if and only if, for all $x\in\mathbb{R}^{2n}$ and all $v,w\in T_x\mathbb{R}^{2n}$, it holds that $\omega_x(v,w)=\omega_{\psi(x)}(\psi_*v,\psi_*w)$. By Exercise 1.21 below, this is equivalent to the statement that, for every $x\in\mathbb{R}^{2n}$, every smooth $H\colon\mathbb{R}^{2n}\to\mathbb{R}$, and every $w\in T_x\mathbb{R}^{2n}$, we have

$$\omega_x(X_H,w) = \omega_{\psi(x)}(\psi_*X_H, \psi_*w).$$

Now observe that, regardless of whether $\psi^*\omega_0 = \omega_0$, for all x, H, w as above we have,

$$\begin{aligned}\omega_{\psi(x)}(X_{H\circ\psi^{-1}}, \psi_* w) &= -d(H\circ\psi^{-1})_{\psi(x)}(\psi_* w) = -(\psi^{-1*}dH)_{\psi(x)}(\psi_* w) \\ &= -dH_x(w) = \omega_x(X_H, w)\end{aligned}$$

where we have used Proposition 1.11 twice (once for $X_{H\circ\psi^{-1}}$ and once for X_H). Consequently the condition that $\psi^*\omega_0 = \omega_0$ is equivalent to the condition that, for all $x \in \mathbb{R}^{2n}$, all smooth $H\colon \mathbb{R}^{2n} \to \mathbb{R}$, and all $w \in T_x\mathbb{R}^{2n}$ we have

$$(\iota_{X_{H\circ\psi^{-1}}}\omega)_{\psi(x)}(\psi_* w) = (\iota_{\psi_* X_H}\omega)_{\psi(x)}(\psi_* w). \tag{7}$$

But since ψ is a diffeomorphism (so both ψ and ψ_* are bijective), Proposition 1.11 shows that $X_{H\circ\psi^{-1}}$ is the *only* vector field V having the property that $(\iota_V\omega)_{\psi(x)}(\psi_* w) = -d(H\circ\psi^{-1})_{\psi(x)}(\psi_* w)$ for all x and w, so (7) is equivalent to the statement that $\psi_* X_H = X_{H\circ\psi^{-1}}$ everywhere, as desired. □

Exercise 1.21. *Prove that if $x \in \mathbb{R}^{2n}$ and $v \in T_x\mathbb{R}^{2n}$, then there is a smooth function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ such that $X_H(x) = v$.*

We have thus shown that the the standard symplectic form ω_0 on $\mathbb{R}^{2n}$ is related to Hamilton's equations in two different ways: Hamiltonian flows preserve ω_0 and, separately from this, the coordinate transformations that are symmetries of Hamilton's equations are just the same ones as those that preserve ω_0. In particular the flow of one Hamiltonian H (and of course there are many possible H, as H can be any smooth map with X_H Lipschitz) gives a symmetry of the versions of Hamilton's equations for all other possible Hamiltonians. Motivated by the apparent importance of the symplectic form ω_0, our next task will be to understand some of its linear algebraic properties.

2. Linear symplectic geometry

2.1. **Alternating bilinear forms.**

Let us point out some simple abstract properties of the standard symplectic form

$$\omega_0 = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + \cdots + dp_n \wedge dq_n$$

on $\mathbb{R}^{2n}$. The previous section treats this as a *differential* 2-form on $\mathbb{R}^{2n}$, *i.e.* a smoothly varying choice of alternating bilinear form $(\omega_0)_x$ on each of the various tangent spaces $T_x\mathbb{R}^{2n}$. But ω_0 is the simplest kind of differential 2-form on $\mathbb{R}^{2n}$, namely one with constant coefficients when expressed in the standard coordinate basis, so that if $x, y \in \mathbb{R}^{2n}$ the bilinear forms $(\omega_0)_x\colon T_x\mathbb{R}^{2n} \times T_x\mathbb{R}^{2n} \to \mathbb{R}$ and $(\omega_0)_y\colon T_y\mathbb{R}^{2n} \times T_y\mathbb{R}^{2n} \to \mathbb{R}$ are the same where we use the standard identification of $T_x\mathbb{R}^{2n}$ and $T_y\mathbb{R}^{2n}$ with $\mathbb{R}^{2n}$ which is induced by the vector space structure of $\mathbb{R}^{2n}$. (Said differently, if τ is any translation of $\mathbb{R}^{2n}$ then $\tau^*\omega_0 = \omega_0$.)

More generally, if V is a vector space over $\mathbb{R}$ and $k \in \mathbb{N}$, any alternating, k-linear map $\eta\colon V^k \to \mathbb{R}$ induces a differential form on V, whose value at $x \in V$ is the alternating, k-linear function $\eta_x\colon (T_xV)^k \to \mathbb{R}$ that is obtained from η by using the canonical identification of V with T_xV (if we think of elements of T_xV as equivalence classes of arcs through x, this identification sends $v \in V$ to the equivalence class the arc $t \mapsto x + tv$). We will simultaneously use the notation η to refer to the original map $V^k \to \mathbb{R}$ and to the corresponding differential k-form, and refer to such objects η as "linear k-forms" (to distinguish them from more general differential k-forms). So the first abstract statement to make about ω_0 is that it is a linear 2-form on $\mathbb{R}^{2n}$. For the rest of the section we will view ω_0 as a map $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$, rather than the differential form that this map canonically determines.

If V is a finite-dimensional vector space over $\mathbb{R}$, let us consider some properties of linear 2-forms $\omega\colon V \times V \to \mathbb{R}$. So by definition ω is an alternating (*i.e.*, $\omega(v,w) = -\omega(w,v)$) bilinear map. In the following we'll use B as our notation for a general bilinear map; we'll only name it as ω if we additionally know that it is alternating. Giving a bilinear map $B\colon V \times V \to \mathbb{R}$ is the same thing as giving a linear map $\theta_B\colon V \to V^*$. Namely, given B, for any $v \in V^*$ one defines $\theta_B(v) \in V^*$ by $(\theta_B(v))(w) = B(v,w)$; the linearity of B in its second argument shows that $\theta_B(v)$ is linear, so indeed belongs to V^*, and the linearity of B in its first argument shows that $v \mapsto \theta_B(v)$ is indeed a linear map. Conversely if $\theta\colon V \to V^*$ is linear then $(v,w) \mapsto (\theta(v))(w)$ is obviously bilinear.

Because of the assumption that V is finite-dimensional, we have a canonical identification of V with the double-dual V^{**}, sending v to the linear functional on V^* defined by $\alpha \mapsto \alpha(v)$ (finite-dimensionality of V is equivalent to this being an isomorphism). In view of this, if $\theta\colon V \to V^*$ is linear we can regard the adjoint $\theta^*\colon V^{**} \to V^*$ (defined by $(\theta^*\beta)(v) = \beta(\theta v)$ for $v \in V$ and $\beta \in V^{**}$) as being a map from V to V^*, just like θ is.

Exercise 2.1. *Let V be a finite-dimensional vector space over $\mathbb{R}$, let $B\colon V \times V \to \mathbb{R}$ be bilinear, and as above define $\theta_B\colon V \to V^*$ by $(\theta_B(v))(w) = B(v,w)$.*

(i) *Show that, where we use the canonical identification of V with V^{**} mentioned in the previous paragraph, the adjoint $\theta_B^*\colon V \to V^*$ is given by $(\theta_B(v))(w) = B(w,v)$. In particular B is alternating iff $\theta_B^* = -\theta_B$.*

(ii) *Choose a basis $\mathcal{E} = \{e_1,\ldots,e_k\}$ for V, and let M be the $k \times k$ matrix that represents the map $\theta_B\colon V \to V^*$ by a $k \times k$ matrix with respect to the basis $\mathcal{E}$ for V and the dual basis $\mathcal{E}^* = \{e^1,\ldots,e^k\}$ for V^*. Show that B is alternating iff M is a skew-symmetric matrix (*i.e. *$M = -M^T$).*

Definition 2.2. The **rank** of a bilinear form $B\colon V \times V \to \mathbb{R}$ is the rank (*i.e.*, the dimension of the image) of the corresponding map $\theta_B\colon V \to V^*$.

Equivalently, the rank of B is the same as the rank of the corresponding matrix M from Exercise 2.1 (ii).

Proposition 2.3. *Let $\omega\colon V \times V \to \mathbb{R}$ be a linear 2-form on a k-dimensional vector space V over $\mathbb{R}$. Then the rank of ω is an* even *number $2n$, and there is a basis $\{e_1,\ldots,e_n,f_1,\ldots,f_n,g_1,\ldots,g_{k-2n}\}$ for V such that*

$$\omega(e_i,f_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}, \quad \omega(e_i,e_j) = \omega(f_i,f_j) = \omega(e_i,g_j) = \omega(f_i,g_j) = \omega(g_i,g_j) = 0. \tag{8}$$

Remark 2.4. If there is a basis $\{e_1,\ldots,e_n,f_1,\ldots f_n,g_1,\ldots,g_{k-2n}\}$ as in the statement of the theorem, then θ_ω evidently sends e_i to the dual basis element f^i, sends f_i to the dual basis element $-e^i$, and sends g_i to zero. So the statement about the rank follows from the existence of the indicated basis, and the matrix M from Exercise 2.1 (ii) takes the block form

$$\begin{pmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where I denotes the $n \times n$ identity.

Proof. The proof is by induction on k. If ω is identically zero (as it must be if V is zero- or one-dimensional, thus taking care of the base case of the induction) then the result is trivial: we will have $n = 0$ and for our basis we can just take any basis $\{g_1,\ldots,g_k\}$ for V. So assume that ω is not identically zero and that the result holds for all alternating bilinear forms ω' on all vector spaces V' of dimension strictly smaller than k. Since ω is not identically zero we can find $e,f \in V$ such that $\omega(e,f) = 1$. Write $W = \mathrm{span}\{e,f\}$, and consider the ω-orthogonal complement

$$W^\omega = \{v \in V | \omega(e,v) = \omega(f,v) = 0\},$$

which is evidently a subspace of V. The key point is now:

Claim 2.5. We have $V = W \oplus W^{\omega}$.

To prove the claim, first note that if $v = ae + bf \in W \cap W^{\omega}$ then $a = \omega(v, f) = 0$ and $b = \omega(e, v) = 0$ so $v = 0$, confirming that W and W^{ω} have trivial intersection. More interestingly, we can use an analogue of the Gram-Schmidt procedure to express a general $v \in V$ as a sum of an element of W and an element of W^{ω}. Indeed, if $v \in V$, then take

$$v' = v + \omega(v, e)f - \omega(v, f)e.$$

Then

$$\omega(v', e) = \omega(v, e) + \omega(v, e)\omega(f, e) = 0 \quad \text{and} \quad \omega(v', f) = \omega(v, f) - \omega(v, f)\omega(e, f) = 0$$

so $v' \in W^{\omega}$, in view of which $v = v' - \omega(v, e)f + \omega(v, f)e \in W + W^{\omega}$.

Having established the claim, observe that we can restrict ω to (pairs of) vectors in the subspace W^{ω} and apply the inductive hypothesis to this restricted linear 2-form, giving a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n, g_1, \ldots, g_{k-2-2n}\}$ for W^{ω} as in the statement of the theorem. But then

$$\{e, e_1, \ldots, e_n, f, f_1, \ldots, f_n, g_1, \ldots, g_{k-2-2n}\}$$

is a basis for $V = W \oplus W^{\omega}$ that satisfies the desired properties. □

Note that the fact that $V = W \oplus W^{\omega}$, which was important on the proof, depended on the specific subspace W that we used. For example, specializing to the case $n = 2, k = 4$, $V = \text{span}\{e_1, e_2, f_1, f_2\}$, if we take $W = \text{span}\{e_1, e_2\}$ we have $W^{\omega} = W$, quite differently from the situation in Claim 2.5.

Corollary 2.6. *Let V and V' be vector spaces of the same finite dimension k, and let $\omega\colon V \times V \to \mathbb{R}$ and $\omega'\colon V' \times V' \to \mathbb{R}$ be linear 2-forms having the same rank. Then there is a linear isomorphism $A\colon V \to V'$ such that $A^*\omega' = \omega$.*

Proof. Use Proposition 2.3 to construct a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n, g_1, \ldots, g_{k-2n}\}$ for V that obeys (8), and a basis $\{e'_1, \ldots, e'_n, f'_1, \ldots, f'_n, g'_1, \ldots, g'_{k-2n}\}$ for V' that obeys the version of (8) with ω replaced by ω'. The unique linear map $A\colon V \to V'$ obeying $Ae_i = e'_i$, $Af_i = f'_i$, and $Ag_i = g'_i$ then satisfies $A^*\omega' = \omega$ □

Definition 2.7.
- (i) A *linear symplectic form* on a vector space V is a linear 2-form ω on V such that the associated map $\theta_{\omega}\colon V \to V^*$ is injective.[5]
- (ii) A *symplectic vector space* is a pair (V, ω) consisting of a finite-dimensional vector space V and a linear symplectic form ω on V.

Corollary 2.8. *If (V, ω) is a symplectic vector space then $\dim V$ is even.*

Proof. Indeed, if $k = \dim V$ the symplectic condition amounts to the rank of ω being k, but the rank is even by Proposition 2.3. □

Corollary 2.9. *If (V, ω) is a symplectic vector space then there is a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ for V such that, denoting the dual basis for V by $\{e^1, \ldots, e^n, f^1, \ldots, f^n\}$, we have*

$$\omega = \sum_{i=1}^{n} e^i \wedge f^i. \tag{9}$$

Proof. Again this follows directly from Proposition 2.3, as in the symplectic case we have $k = 2n$, and both sides of (9) evaluate in the same way on basis elements. □

[5] If V is finite-dimensional, as it will be in all our examples, the rank plus nullity theorem of course shows that θ_{ω} is injective iff it is bijective. The condition that θ_{ω} is injective is often called **non-degeneracy**; rephrasing slightly, it says that if $v \in V$ and $v \neq 0$ then there is $w \in V$ such that $\omega(v, w) \neq 0$.

Corollary 2.9 should make the definition $\omega_0 = \sum_{i=1}^n dp_i \wedge dq_i$ of the standard symplectic form on $\mathbb{R}^{2n}$ seem somewhat less arbitrary: ω_0 has been chosen to be a linear symplectic form on $\mathbb{R}^{2n}$ (a completely coordinate-free condition), and subject to this requirement it is automatic that there will be linear coordinates $(p_1, \ldots, q_n)$ on $\mathbb{R}^{2n}$ in terms of which ω_0 can be expressed by the indicated formula. Another expression of the same point is that Corollary 2.6 proves that if ω is any linear symplectic form on $\mathbb{R}^{2n}$ then there is a linear isomorphism $A\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $A^*\omega = \omega_0$.

Exercise 2.10. *Let ω be a linear 2-form on a finite-dimensional vector space V, and define*

$$V^\omega = \{v \in V | (\forall w \in V)(\omega(v,w) = 0)\}.$$

(In other words, $V^\omega = \ker \theta_\omega$.) Prove that there is a linear symplectic *form $\underline{\omega}$ on the quotient vector space V/V^ω uniquely characterized by the property that $\pi^*\underline{\omega} = \omega$ where $\pi\colon V \to V/V^\omega$ is the quotient projection.*

2.2. **Subspaces of symplectic vector spaces.** We will now focus on symplectic vector spaces (V, ω) and on their subspaces. If $W \le V$ is a subspace then, as already seen in special cases above, we can consider the *symplectic orthogonal complement*

$$W^\omega = \{v \in V | (\forall w \in W)(\omega(v,w) = 0)\}.$$

Hopefully the motivation for the name is clear: if instead of being an alternating bilinear form ω were a symmetric, positive definite bilinear form (*i.e.*, an inner product) then W^ω would be the orthogonal complement in the usual sense; in particular we would have $V = W \oplus W^\omega$. The example mentioned just before Corollary 2.6 shows that the latter relation does not typically hold in the symplectic case, but we do have the following parallel:

Proposition 2.11. *Let (V, ω) be a symplectic vector space and $W \le V$. Then there is a short exact sequence*

$$0 \longrightarrow W^\omega \xrightarrow{\iota} V \xrightarrow{\theta_\omega^W} W^* \longrightarrow 0 \tag{10}$$

where $\iota\colon W^\omega \to V$ is the inclusion and θ_ω^W is defined by $(\theta_\omega^W(v))(w) = \omega(v,w)$. Thus there is a natural[6] *isomorphism $\frac{V}{W^\omega} \cong W^*$, and hence*

$$\dim V = \dim W + \dim W^\omega.$$

Proof. If $v \in V$, we have

$$v \in \ker \theta_\omega^W \Leftrightarrow (\forall w \in W)(\omega(v,w) = 0) \Leftrightarrow v \in \mathrm{Im}(\iota),$$

so the indicated sequence is exact at V. Of course ι is injective, so to show exactness of the whole sequence we just need to check that θ_ω^W is surjective. Now if $j_W\colon W \to V$ is the inclusion, inducing (by taking adjoints) the restriction map $j_W^*\colon V^* \to W^*$, observe that $\theta_\omega^W = j_W^* \circ \theta_\omega$. But $\theta_\omega\colon V \to V^*$ is an isomorphism by definition of a symplectic vector space, and j_W^* is surjective since j_W is injective (or, more simply, because any linear functional on W extends to V), so θ_ω^W is indeed surjective and so (10) is a short exact sequence.

[6] Loosely, "natural" just means "canonical" or "independent of choices" (in particular, independent of any choice of basis). If you like you can phrase this in terms of the category-theoretic notion of natural transformations: consider the category whose objects are triples (V, W, ω) where (V, ω) is a symplectic vector space and $W \le V$, with morphisms given by linear isomorphisms that respect both the linear symplectic form and the subspace, and then one obtains two functors to the category of vector spaces using, respectively, $(V, W, \omega) \mapsto \frac{V}{W^\omega}$ or $(V, W, \omega) \mapsto W^*$, and we're giving a natural isomorphism between these functors.

As is always the case for short exact sequences, it then follows that θ_ω^W descends to the quotient $\frac{V}{\iota(W^\omega)} = \frac{V}{W^\omega}$ as an isomorphism to W^*; naturality is straightforward to check. Since $\dim W^* = \dim W$ and $\dim \frac{V}{W^\omega} = \dim V - \dim W^\omega$ the last statement of the proposition is immediate. □

Corollary 2.12. *If (V,ω) is a symplectic vector space and W is any subspace then $(W^\omega)^\omega = W$.*

Proof. That $W \leq (W^\omega)^\omega$ is immediate from the definitions. But by Proposition 2.11 (twice) we have

$$\dim(W^\omega)^\omega = \dim V - \dim W^\omega = \dim W$$

so the inclusion $W \leq (W^\omega)^\omega$ must be an equality. □

A symplectic vector space has various kinds of subspaces, distinguished from one another by their relations to their symplectic orthogonal complements. Here are the types that are considered most often:

Definition 2.13. Let (V,ω) be a symplectic vector space and let $W \leq V$. We say that W is:

- a **symplectic** subspace if $W \cap W^\omega = \{0\}$;
- an **isotropic** subspace if $W \subset W^\omega$;
- a **coisotropic** subspace if $W \supset W^\omega$;
- a **Lagrangian** subspace if $W = W^\omega$.

So a Lagrangian subspace is a subspace that is both isotropic and coisotropic. By Proposition 2.11, a Lagrangian subspace necessarily has dimension $\frac{\dim V}{2}$, and an isotropic subspace (or for that matter a coisotropic subspace) is Lagrangian iff its dimension is $\frac{\dim V}{2}$. By Corollary 2.12, W is symplectic iff W^ω is symplectic, while W is isotropic iff W^ω is *co*isotropic. Corollary 2.12 also implies that the dimension of an isotropic subspace can never be larger than $\frac{\dim V}{2}$.

Note that a simpler way of saying that W is isotropic is just that, for all $v, w \in W$, we have $\omega(v,w) = 0$. For example, in $\mathbb{R}^{2n}$ with its standard symplectic structure, the subspace obtained by varying some of the coordinates p_i while holding all $q_i = 0$ (or vice versa) will be isotropic.

The definition of a "symplectic" subspace is justified by the fact that the condition $W \cap W^\omega = \{0\}$ is equivalent to the condition that the restriction ω_W of ω to $W \times W$ makes (W, ω_W) a symplectic vector space; indeed the map $\theta_{\omega_W} \colon W \to W^*$ has kernel equal to $W \cap W^\omega$.

A prototypical example of a symplectic subspace of $(\mathbb{R}^{2n}, \omega_0)$ is given by choosing $k \leq n$ and taking

$$\{(p_1, \ldots, p_k, 0, \ldots, 0, q_1, \ldots, q_k, 0, \ldots, 0)\};$$

perhaps more intuitively, if we regard each $p_i q_i$-plane as a copy of the complex plane $\mathbb{C}$, and hence $\mathbb{R}^{2n}$ as $\mathbb{C}^n$, this example is the subspace $\mathbb{C}^k$ spanned by the first k complex coordinates.

Exercise 2.14. *(a) Prove that in the symplectic vector space $(\mathbb{R}^4, \omega_0)$ every subspace is symplectic, isotropic, or coisotropic. (Hint: If V is the subspace, consider the possible values of the rank of $\omega_0|_{V\times V}$.)*

(b) Give an example of a subspace of $\mathbb{R}^6$ that is neither symplectic, nor isotropic, nor coisotropic (with respect to ω_0).

For any vector space V and subspace W of V, let us define an *algebraic complement* of W in V to be any subspace $U \leq V$ such that $W \oplus U = V$. Of course such a U always exists: if we take a basis for W and then extend it to a basis for V, the span of the elements in the larger basis that do not belong to W will be an algebraic complement to W in V. However U is very far from unique, and in the absence of additional structure there is no canonical choice of U. If U is an algebraic complement of W in V, the restriction to U of the quotient projection $V \to V/W$ gives an isomorphism $U \cong V/W$. (If you are not familiar with this fact you should convince yourself of it now.)

If (V,ω) is a symplectic vector space and W is a *symplectic* subspace, there is a preferred algebraic complement to W, namely the symplectic orthogonal complement W^{ω}, which is itself a symplectic subspace. Less canonically but still usefully, we have:

Proposition 2.15. *If (V,ω) is a symplectic vector space and $C\leq V$ is a coisotropic subspace, then there is an isotropic subspace $I\leq V$ which is an algebraic complement to C in V. In particular, if C is a Lagrangian subspace, then we can write $V=C\oplus I$ where I is also Lagrangian.*

(Of course C^{ω} is isotropic, but it isn't an algebraic complement to C, so we'll have to do something else.)

Proof. [7] We prove this by induction on $\dim V-\dim C$. As the base case we can use the case $\dim V-\dim C=0$, in which case we just take $I=\{0\}$. Assuming then that $\dim V-\dim C\geq 1$, let $x\in V\setminus C$ and let $D=C\oplus\mathrm{span}\{x\}$. Then $D^{\omega}\subset C^{\omega}\subset C\subset D$, so D is coisotropic. Also $\dim V-\dim D<\dim V-\dim C$, so we can apply the inductive hypothesis to get an isotropic subspace J with $D\oplus J=V$, *i.e.* $C\oplus\mathrm{span}\{x\}\oplus J=V$.

We might then want to set $I=\mathrm{span}\{x\}\oplus J$, but this is probably not isotropic because we likely have $\omega(x,y)\neq 0$ for some $y\in J$. To fix this, note that since $D\oplus J=V$ we have $D^{\omega}\cap J^{\omega}=\{0\}$, and so by Proposition 2.11 we see that the restriction of $v\mapsto(\iota_v\omega)|_J$ to D^{ω} defines an isomorphism $D^{\omega}\to J^*$. In particular there is $w\in D^{\omega}$ such that $\omega(w,y)=\omega(x,y)$ for all $y\in J$. So if we set $I=\mathrm{span}\{x-w\}\oplus J$ then I will be isotropic. Moreover, since $D^{\omega}\subset C^{\omega}\subset C$ and $D\in C^{\omega}$, we have $D=C\oplus\mathrm{span}\{x\}=C\oplus\mathrm{span}\{x-w\}$. So the fact that $C\oplus\mathrm{span}\{x\}\oplus J=V$ implies that $C\oplus I=C\oplus\mathrm{span}\{x-w\}\oplus J=V$.

We have established the first sentence of the proposition, and this immediately implies the second since if C is Lagrangian then I will be an isotropic subspace of dimension $\frac{\dim V}{2}$ which is thus also Lagrangian. □

Corollary 2.6 shows that if (V,ω) and (V',ω') are two symplectic vector spaces of the same dimension then there is a *linear symplectomorphism* from $V\to V'$, *i.e.* a linear map $A\colon V\to V'$ such that $A^*\omega'=\omega$. If one additionally has subspaces $W\leq V$ and $W'\leq V'$ having the same dimension, one can ask whether this linear symplectomorphism can be arranged to additionally map W to W'. If ω,ω' were inner products instead of linear symplectic forms, with A required to intertwine the inner products, this would indeed be possible as one can see by making appropriate use of orthonormal bases. But in the symplectic context it is typically not possible, as one can see using symplectic orthogonal complements: if $A^*\omega'=\omega$ then it's easy to see that $A(W^{\omega})=(AW)^{\omega'}$, so we will not be able to map W to W' by a linear symplectomorphism if, for instance, W is a (proper) symplectic subspace while W' is coisotropic.

However if one sticks to subspaces each belonging to a specific one of the classes of Definition 2.13 then one does have results along these lines. We'll just show this in the symplectic and Lagrangian cases since these are the most important. The symplectic case is easier thanks to what we've already done:

Proposition 2.16. *Let (V,ω) and (V',ω') be symplectic vector spaces and let $W\leq V$ and $W'\leq V'$ be symplectic subspaces. Assume that $\dim V=\dim V'$, and that $\dim W=\dim W'$. Then there is a linear symplectomorphism $A\colon V\to V'$ such that $A(W)=W'$.*

Proof. The vector spaces W,W^{ω},W', and $(W')^{\omega'}$, when endowed with the alternating bilinear forms ω or ω' as appropriate, are symplectic vector spaces in their own right. So by Corollary 2.6 there are linear symplectomorphisms $A_1\colon W\to W'$ and $A_2\colon W^{\omega}\to(W')^{\omega'}$. But $V=W\oplus W^{\omega}$ and

[7] If/when you know about compatible complex structures you might try to find an easier proof of this; I've chosen to phrase the proof so as to invoke as little extra structure as possible, at the cost of being perhaps a little trickier.

$V' = W' \oplus (W')^{\omega'}$, so we can just define $A\colon V \to V'$ by, for $x \in W$ and $y \in W^{\omega}$, $A(x+y) = A_1 x + A_2 y$. It's easy to see from the definitions that this is a linear symplectomorphism, and it obviously sends W to W'. □

We now turn to Lagrangian subspaces. The following is a convenient model for how a Lagrangian subspace can sit inside a larger symplectic vector space; it could be considered a version of the symplectic structure on the cotangent bundle to a smooth manifold, in the special case that the manifold is just a vector space.

Definition 2.17. Let L be a finite-dimensional vector space, with dual space L^*. The *canonical symplectic form* ω^L on the direct sum $L^* \oplus L$ is defined by

$$\omega^L((\alpha, x), (\beta, y)) = \alpha(y) - \beta(x) \quad \text{for } x, y \in L,\ \alpha, \beta \in L^*$$

It is easy to check that ω^L is a linear symplectic form, and that both L and L^* are Lagrangian subspaces of the symplectic vector space $(L^* \oplus L, \omega^L)$.

Exercise 2.18. *(a) Show that if $N \leq L^* \oplus L$ is has the property that $N \oplus L = L^* \oplus L$, then there is a unique linear map $A_N\colon L^* \to L$ such that $N = \{(\alpha, A_N\alpha) | \alpha \in L^*\}$.*

(b) Prove that, with N and A_N as in part (a), the subspace N is Lagrangian if and only if the map $A_N\colon L^* \to L$ is symmetric (in the sense that the adjoint map $A_N^*\colon L^* \to L^{**}$ coincides with A_N under the standard identification of L^{**} with L that is recalled above Exercise 2.1).

Proposition 2.19. *Let L_0 and L_1 be two finite-dimensional vector spaces and let $A\colon L_0 \to L_1$ be a linear isomorphism. There is then a unique linear symplectomorphism $\hat{A}\colon L_0^* \oplus L_0 \to L_1^* \oplus L_1$ having the following properties:*

(i) *$\hat{A}(L_0) = L_1$, and $\hat{A}|_{L_0} = A$.*

(ii) *$\hat{A}(L_0^*) = L_1^*$.*

Specifically, $\hat{A}$ is given by

$$\hat{A}(\alpha, x) = ((A^{-1})^*\alpha, Ax) \quad \text{for } \alpha \in L_0^*, x \in L_0 \tag{11}$$

Proof. If we define $\hat{A}$ by (11) then it clearly satisfies (i) and (ii); moreover for $x, y \in L_0$ and $\alpha, \beta \in L_0^*$ we have

$$\begin{aligned}\hat{A}^*\omega^{L_1}((\alpha, x), (\beta, y)) &= \omega^{L_1}((A^{-1*}\alpha, Ax), (A^{-1*}\beta, Ay)) \\ &= (A^{-1*}\alpha)(Ay) - (A^{-1*}\beta)(Ax) = \alpha(y) - \beta(x) = \omega^{L_0}((\alpha, x), (\beta, y)).\end{aligned}$$

Thus $\hat{A}$ is a linear symplectomorphism.

Conversely, if $\hat{A}$ is a linear symplectomorphism obeying (i) and (ii), then it is necessarily of the from $\hat{A}(\alpha, x) = (B\alpha, Ax)$ for some linear $B\colon L_0^* \to L_1^*$. To see that $B = (A^{-1})^*$, we observe that the assumption that $\hat{A}$ is a linear symplectomorphism shows that, for all $\alpha \in L_0^*$ and $y \in L_0$,

$$\begin{aligned}\alpha(y) &= \omega^{L_0}((\alpha, 0), (0, y)) = \hat{A}^*\omega^{L_1}((\alpha, 0), (0, y)) = \omega^{L_1}((B\alpha, 0), (0, Ay)) \\ &= (B\alpha)(Ay) = (A^*B\alpha)(y)\end{aligned}$$

so $\hat{A}$ is a linear symplectomorphism only if $B = A^{-1*}$. □

We now show that the linear symplectic form ω^L on $L^* \oplus L$ gives a "normal form" for general symplectic vector spaces that contain L as a Lagrangian subspace.

Proposition 2.20. *Let (V, ω) be a symplectic vector space, and let L and M be Lagrangian subspaces such that $M \oplus L = V$. Then there is a linear symplectomorphism $T\colon V \to L^* \oplus L$ that maps L by the identity to L, and maps M to L^*.*

Proof. By Proposition 2.11, since M is an algebraic complement to $L^{\omega} = L$, the map $\theta^L_{\omega}\colon V \to L^*$ defined by $v \mapsto (\iota_v\omega)|_L$ restricts to M as an isomorphism from M to L^*. So define, for $m \in M$ and $\ell \in L$,

$$T(m+\ell) = (\theta^L_{\omega}m, \ell).$$

We find

$$\begin{aligned} T^*\omega^L(m_1+\ell_1, m_2+\ell_2) &= \omega^L((\theta^L_{\omega}m_1,\ell_1),(\theta^L_{\omega}m_2,\ell_2)) \\ &= \omega(m_1,\ell_2) - \omega(m_2,\ell_1) = \omega(m_1,\ell_2)+\omega(\ell_1,\ell_2)+\omega(m_1,m_2)+\omega(\ell_1,m_2) \\ &= \omega(m_1\ell_1, m_2+\ell_2), \end{aligned}$$

where in the second equality we have used the fact that L and M are both Lagrangian and hence $\omega(\ell_1,\ell_2) = \omega(m_1,m_2) = 0$. Thus T obeys the required properties. □

Corollary 2.21. *Let $(V_0,\omega_0),(V_1,\omega_1)$ be symplectic vector spaces of the same dimension, let $L_0, M_0 \le V_0$ and $L_1, M_1 \le V_1$ all be Lagrangian subspaces such that $M_0 \oplus L_0 = V_0$ and $M_1 \oplus L_1 = V_1$, and choose a linear isomorphism $A\colon L_0 \to L_1$. Then there is a unique linear symplectomorphism $\tilde{A}\colon V_0 \to V_1$ such that $\tilde{A}|_{L_0} = A$ and such tht $A(M_0) = M_1$.*

Proof. Let $T_0\colon V_0 \to L_0^* \oplus L_0$ be a linear symplectomorphism mapping L_0 by the identity to L_0 and M_0 to L_0^*, and similarly for $T_1\colon V_1 \to L_1^*\oplus L_1$. Letting $\hat{A}$ be as in Proposition 2.19, the map $\tilde{A} := T_1^{-1}\circ\hat{A}\circ T_0$ evidently satisfies the required properties. To show uniqueness, just note that if $\tilde{B}\colon V_0 \to V_1$ likewise satisfies the properties than the uniqueness statement in Proposition 2.19 shows that $\hat{A} = T_1\circ\tilde{B}\circ T_0^{-1}$ and hence that $\tilde{B} = \tilde{A}$. □

Corollary 2.22. *If (V_0,ω_0) and (V_1,ω_1) are symplectic vector spaces with Lagrangian subspaces L_0, L_1, and if $A\colon L_0 \to L_1$ is a linear isomorphism, then there is a linear symplectomorphism $\tilde{A}\colon V_0 \to V_1$ such that $\tilde{A}|_{L_0} = A$.*

Proof. Indeed, Proposition 2.15 finds Lagrangian complements M_0 and M_1 to L_0 and L_1, respectively, and then we can apply Corollary 2.21. □

2.3. **Complex structures.** There are various other kinds of geometric structures that one can put on a symplectic vector space (V,ω), and these interact with the linear symplectic form in important ways. The first such structure that we consider is:

Definition 2.23. Let V be a vector space over $\mathbb{R}$. A **complex structure** on V is a linear map $J\colon V \to V$ with the property that $J^2 = -1_V$ (where 1_V denotes the identity on V).

The reason for the name is that a complex structure J on V canonically makes V into a vector space over $\mathbb{C}$, by defining complex scalar multiplication by $(a+bi)v = av + bJv$. Conversely if V is already a vector space over $\mathbb{C}$, then setting $Jv = iv$ gives a complex structure in the sense of the above definition.

Clearly any even-dimensional vector space admits complex structures: if $\{e_1,\ldots,e_{2n}\}$ is a basis for V one can define a complex structure J on V by setting $Je_{2j-1} = e_{2j}$ and $Je_{2j} = -e_{2j-1}$ for $j \in \{1,\ldots,n\}$. Of course there are many bases and hence many possible J's that one could get by this prescription; in particular if J is a complex structure than so is $-J$. In the presence of a linear symplectic form ω on V it is useful to require J to interact appropriately with ω.

Definition 2.24. Let (V,ω) be a symplectic vector space and let J be a complex structure on V. We say that J is:

- **ω-tame** if, for all $v \in V\setminus\{0\}$, we have $\omega(v,Jv) > 0$.
- **ω-compatible** if J is ω-tame and moreover $\omega(Jv,Jw) = \omega(v,w)$ for all $v,w \in V$.

Another way of saying that J is ω-compatible is that the map $g_J\colon V \times V \to \mathbb{R}$ defined by

$$g_J(x,y) = \omega(x, Jy)$$

defines an inner product on V. Indeed, if J is ω-compatible then if $x \neq 0$ we have $g_J(x,x) = \omega(x, Jx) > 0$ by tameness, and moreover

$$g_J(y,x) = -\omega(Jx,y) = -\omega(J^2x, Jy) = \omega(x, Jy) = g_J(x,y).$$

Example 2.25. *On $\mathbb{R}^{2n} = \{(\vec{p},\vec{q})|\vec{p},\vec{q} \in \mathbb{R}^n\}$ with its standard linear symplectic structure $\omega_0 = \sum_{i=1}^n dp_i \wedge dq_i$, it's easy to see that setting $J_0(\vec{p},\vec{q}) = (-\vec{q},\vec{p})$ defines an ω_0-compatible complex structure. If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $(\vec{p},\vec{q}) \longleftrightarrow \vec{p} + i\vec{q}$, this J_0 is just given by the standard multiplication by i on $\mathbb{C}^n$.*

Based on Corollary 2.6 and Example 2.25 it's easy to see that there are ω-compatible complex structures on any symplectic vector space (V,ω): let A be a linear symplectomorphism from (V,ω) to $(\mathbb{R}^{2n},\omega_0)$ and define $J = A^{-1}J_0A$. In what follows I will give a less coordinate-dependent approach to constructing ω-compatible complex structures, which in fact describes *all* such structures in terms of Lagrangian subspaces.

First we observe:

Proposition 2.26. *If (V,ω) is a symplectic vector space, $L \leq V$ is a Lagrangian subspace, and J is an ω-compatible complex structure then $J(L)$ is also Lagrangian, and $J(L) \oplus L = V$.*

Proof. To see that $J(L)$ is Lagrangian, note that any two elements of $J(L)$ can be written as Jv and Jw for some $v,w \in L$, and then since J is ω-compatible and L is Lagrangian we have

$$\omega(Jv, Jw) = \omega(v,w) = 0.$$

So $J(L)$ is isotropic, and since its dimension, like that of L, is $\frac{1}{2}\dim V$ it follows that $J(L)$ is Lagrangian.

Since $dimJ(L) + \dim L = \dim V$, we will have $J(L) \oplus L = V$ if and only if $J(L) \cap L = \{0\}$. To check this, if we have $w \in J(L) \cap L$ then we can write $w = Jv$ where $v \in L$, and so $\omega(v, Jv) = \omega(v,w) = 0$ since L is Lagrangian and both v and w belong to L. Since J is ω-tame this forces v to be 0, and hence $w = Jv = 0$. $\square$

Here then is one way of characterizing all ω-compatible complex structures.

Theorem 2.27. *Let (V,ω) be a symplectic vector space, $L \leq V$ a Lagrangian subspace, and $M \leq V$ another Lagrangian subspace such that $M \oplus L = V$. Then for every inner product h* **on** *L, there is a unique ω-compatible complex structure on V such that $J(L) = M$ and $g_J|_{L\times L} = h$.*

Proof. We first prove uniqueness: if J is an ω-compatible complex structure with $J(L) = M$ and $g_J|_{L\times L} = h$ we will in fact produce a formula for h (which will also help with existence, since after this we'll just have to check that this formula satisfies the required properties). To do this, recall that, by Proposition 2.11, since M is an algebraic complement to $L^\omega = L$ we have an isomorphism $\theta^L_\omega\colon M \to L^*$ defined by $\theta^L_\omega(m) = \omega(m,\cdot)|_L$. Also, since the inner product h is a nondegenerate bilinear form on L, we get an isomorphism $\theta_h\colon L \to L^*$ defined by $\theta_h(\ell) = h(\ell,\cdot)$. So if J is is any ω-compatible almost complex structure with $J(L) = M$ and $g_J|_{L\times L}$, then for every $\ell, \ell' \in L$ we must have:

$$\begin{aligned}(\theta_h(\ell))(\ell') &= g_J(\ell,\ell') = g_J(\ell',\ell) = \omega(\ell', J\ell) \\ &= -\omega(J\ell,\ell') = -(\theta^L_\omega(J\ell))(\ell').\end{aligned}$$

So given that θ^L_ω is an isomorphism $M \to L^*$, J *must* send an arbitrary $\ell \in L$ to the point $-(\theta^L_\omega)^{-1}\theta_h(\ell) \in M$. But since $J^2 = -1_V$ and $M \oplus L = V$ this is enough to completely determine J: if $m \in M$ then J

must send m to $-J^{-1}m$, *i.e.* to $\theta_h^{-1}\theta_\omega^L(m)$. So if J satisfies the indicated properties, it must be given in block form with respect to the decomposition $V = M \oplus L$ as

$$\begin{pmatrix} 0 & -(\theta_\omega^L)^{-1}\circ\theta_h \\ \theta_h^{-1}\circ\theta_\omega^L & 0 \end{pmatrix}, \tag{12}$$

proving uniqueness. The proof of the theorem will be complete when we show that defining J by (12) yields an ω-compatible complex structure such that $J(L) = M$ and $g_J|_{L\times L} = h$. That $J(L) = M$ is immediate, and that $J^2 = -1_V$ follows by block matrix multiplication since the lower left block is the negative of the inverse of the upper right block. If $m, m' \in M$ and $\ell, \ell' \in L$ I will leave as an exercise the calculation

$$g_J(m+\ell, m'+\ell') = h\left(\theta_h^{-1}(\theta_\omega^L m), \theta_h^{-1}(\theta_\omega^L m')\right) + h(\ell', \ell). \tag{13}$$

Since h is an inner product and $\theta_h^{-1}\circ\theta_\omega^L\colon M \to L$ is an isomorphism, the above formula for g_J makes clear that g_J is symmetric and positive definite (thus an inner product, so J is ω-compatible) and that $g_J|_{L\times L} = h$, as desired. □

Exercise 2.28. *With J given by (12), prove the identity (13). (You should use that M and L are Lagrangian somewhere in the proof.)*

Corollary 2.29. *The space $\mathcal{J}_\omega(V)$ of ω-compatible almost complex structures on a symplectic vector space (V, ω) is contractible.*

Proof. Fix your favorite Lagrangian subspace $L \leq V$. Let $\mathfrak{LC}_L$ denote the set of Lagrangian subspaces M with $M \oplus L = V$. By Proposition 2.20, a choice of basepoint $M_0 \in \mathfrak{LC}_L$ allows us to identify the triple (V, L, M_0) with $(L^*\oplus L, L, L^*)$, and under this identification Exercise 2.18 identifies $\mathfrak{LC}_L$ with the space of symmetric linear maps $L^* \to L$. The latter has a natural, contractible topology (it's a finite-dimensional vector space), and we use this to topologize $\mathfrak{LC}_L$.

Also let $\mathfrak{Inn}_L$ denote the space of inner products on L. This is contractible with respect to its natural topology, since if h_0 is one inner product then $(t, h) \mapsto (1-t)h + th_0$ deformation retracts $\mathfrak{Inn}_L$ to the single point h_0.

In view of Proposition 2.26, there is a well-defined map $\mathcal{J}_\omega(V) \to \mathfrak{LC}_L \times \mathfrak{Inn}_L$ defined by $J \mapsto (J(L), g_J|_{L\times L})$, and Theorem 2.27 shows that this map is a bijection. In fact, from the explicit formula (12) one can see that it is a homeomorphism (details left to the reader). So $\mathcal{J}_\omega(V)$ is homeomorphic to the product of the contractible spaces $\mathfrak{LC}_L$ and $\mathfrak{Inn}_L$, and thus is contractible. □

When we move on to symplectic manifolds, where we have non-canonically isomorphic symplectic vector spaces at every point of a manifold, Corollary 2.29 will lead to the statement that the symplectic structure ω induces a canonical homotopy class of almost complex structures[8] via the ω-compatibility requirement. If one drops the compatibility requirement the relevant spaces are not contractible and this no longer works.

2.4. **Compatible triples.**

As we have discussed, by definition an ω-compatible complex structure on a symplectic vector space (V, ω) induces yet another geometric structure, namely an inner product $g_J(v, w) = \omega(v, Jw)$ on the (real) vector space V. In fact g_J and ω combine together in a standard way from the point of view of complex linear algebra, recalling that J can be considered to make V a complex vector space by identifying J with scalar multiplication by i.

Indeed, if V is a vector space over $\mathbb{C}$, recall that a **Hermitian inner product** on V is by definition a map $h\colon V \times V \to \mathbb{C}$ obeying

(i) $h(u, av + bw) = ah(u, v) + bh(u, w)$ for $a, b \in \mathbb{C}$, $u, v, w \in V$;

[8] *i.e.*, smoothly varying families of complex structure on each tangent space

(ii) $h(w,v) = \overline{h(v,w)}$ for $v, w \in V$;
(iii) For all $v \in V$, $h(v,v) \geq 0$ (in particular $h(v,v) \in \mathbb{R}$), with equality only if $v = 0$.

We can then decompose h into its real and imaginary parts as

$$h = g + i\omega$$

where both g and ω are maps from $V \times V$ to $\mathbb{R}$. Evidently (ii) above shows both that g is symmetric and ω is alternating, and (i) shows that both g and ω are bilinear (with respect to the $\mathbb{R}$-vector space structure of V). Together (i) and (ii) imply that

$$h(iv, iw) = ih(iv, w) = i\overline{h(w, iv)} = i(-i)\overline{h(w,v)} = h(v,w)$$

and hence that likewise $\omega(iv, iw) = \omega(v, w)$. Moreover (iii) shows that g is positive definite and so defines an inner product. Furthermore, we have, for $v, w \in V$,

$$g(v, iw) + i\omega(v, iw) = h(v, iw) = ih(v,w) = -\omega(v,w) + ig(v,w)$$

and thus

$$\omega(v,w) = -g(v, iw), \qquad g(v,w) = \omega(v, iw).$$

Thus a Hermitian inner product on a complex vector space V encodes a linear symplectic form ω on the underlying real vector space as its imaginary part; the endomorphism $J \colon V \to V$ given by multiplication by i is then an ω-compatible complex structure on V, and the real part of the Hermitian inner product can be recovered just from J and ω by the formula $g_J(v,w) = \omega(v, Jw)$. Alternatively, if one is given a real inner product g such that $g(Jv, Jw) = g(v,w)$ for all $v, w \in V$ then one can recover ω via $\omega(v,w) = -g(v, Jw) = g(Jv, w)$.

Definition 2.30. A **compatible triple** on a vector space V is a tuple (g, ω, J) where J is a complex structure on V, ω is a linear symplectic form such that J is ω-compatible, and $g = g_J$ is the inner product on V defined by $g(v,w) = \omega(v, Jw)$.

Clearly the tuple (g, ω, J) is a little redundant, since g can be recovered by a simple formula from ω and J; alternatively, ω can be recovered from g and J using $\omega(v,w) = g(Jv, w)$.

The foregoing discussion implies the following:

Proposition 2.31. *There is a one-to-one correspondence between:*
(i) *real vector spaces V equipped with compatible triples (g, ω, J), and*
(ii) *complex vector spaces V equipped with Hermitian inner products h.*

Namely, given data as in (i), we make V into a complex vector space by setting $(a + bi)v = av + bJv$ for $a, b \in \mathbb{R}$ and $v \in V$, and we define $h = g + i\omega$.

The standard model for a complex vector space with a Hermitian inner product is $\mathbb{C}^n$, which in keeping with what we have done so far we will regard as consisting of vectors $\vec{z} = \vec{p} + i\vec{q}$ where $\vec{p}, \vec{q} \in \mathbb{R}^n$, with inner product

$$h(\vec{w}, \vec{z}) = \sum_{j=1}^{n} \bar{w}_j z_j.$$

Writing this out in real and imaginary parts gives, if $\vec{w} = \vec{p}\,' + i\vec{q}\,'$ and $\vec{z} = \vec{p} + i\vec{q}$,

$$h(\vec{w}, \vec{z}) = \sum_j (p'_j - iq'_j)(p_j + iq_j) = \sum_j \left(p'_j p_j + q'_j q_j\right) + i \sum_j \left(p'_j q_j - q'_j p_j\right).$$

Thus the real part of h is just the standard dot product, and the imaginary part of h is the standard symplectic structure $\omega_0 = \sum_j dp_j \wedge dq_j$. This gives a motivation quite separate from Hamiltonian mechanics to think about the standard symplectic structure: it's the imaginary part of the standard Hermitian inner product on $\mathbb{C}^n$.

Corollary 2.9 shows that every symplectic vector space (V, ω) (say of dimension $2n$) is isomorphic to $\mathbb{R}^{2n}$ with its standard symplectic structure. If strengthen the structure on V to a compatible triple, we can refine this statement as follows.

Proposition 2.32. *Let (g, ω, J) be a compatible triple on a $2n$-dimensional real vector space V, and use J to regard V as an n-dimensional complex vector space. Then there is a complex-linear map $A\colon V \to \mathbb{C}^n$ such that $A^*\omega_0 = \omega$ and $(Av)\cdot(Aw) = g(v,w)$ for all $v, w \in V$.*

Proof. Regarding V as a complex vector space, we have a Hermitian inner product $h = g + i\omega$. A standard argument (with the Gram-Schmidt procedure) produces a basis $\{e_1, \ldots, e_n\}$ with respect to this Hermitian inner product. If we let $f_j = ie_j$, the linearity properties of h readily imply that $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is at the same time an orthonormal basis with respect to the real inner product g, and also a basis satisfying the conclusion of Corollary 2.9. We can then take for A the unique complex linear map $V \to \mathbb{C}^n$ that sends the basis $\{e_1, \ldots, e_n\}$ to the standard basis for $\mathbb{C}^n$ (and hence, by complex linearity, sends $\{f_1, \ldots, f_n\}$ to the result of multiplying each element of the standard basis by i). □

2.5. **The symplectic linear groups and related structures.** While Corollary 2.9 shows that any $2n$-dimensional vector space (V, ω) can be identified by an isomorphism with $(\mathbb{R}^{2n}, \omega_0)$, there are may ways of making this identification, and typically nothing that makes any particular one of these identifications better than any of the others. Somewhat more concretely, if $B\colon V \to \mathbb{R}^{2n}$ is one linear isomorphism with $B^*\omega_0 = \omega$, then the collection of *all* such isomorphisms consists of those maps that can be written as $A \circ B$ where $A\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a linear map with $A^*\omega_0 = \omega_0$. This partly motivates the study of the *symplectic linear group*[9]

$$Sp(2n) = \{A \in GL(2n; \mathbb{R}) | A^*\omega_0 = \omega_0\}.$$

That $Sp(2n)$ is indeed a group follows easily from the identity $A^*B^*\omega_0 = (B \circ A)^*\omega_0$ for all linear $A, B\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. (More broadly, for any symplectic vector space (V, ω) we could consider the group $Sp(V, \omega)$ of linear automorphisms preserving ω, but if $B\colon V \to \mathbb{R}^{2n}$ has $B^*\omega_0 = \omega$ then $Sp(V, \omega)$ is (non-canonically) isomorphic to $Sp(2n)$ via $A \mapsto BAB^{-1}$.)

Example 2.33. *If J is an ω_0-compatible complex structure then, by definition, $J^*\omega_0(v,w) = \omega_0(Jv, Jw) = \omega_0(v,w)$ for all $v, w \in \mathbb{R}^{2n}$, so $J \in Sp(2n)$. In particular by Example 2.25 this applies to the map $J_0(\vec{p}, \vec{q}) = (-\vec{q}, \vec{p})$.*

Example 2.34. *For any $t \in \mathbb{R} \setminus \{0\}$ it's easy to see that the maps $(\vec{p}, \vec{q}) \mapsto (t\vec{p}, \frac{1}{t}\vec{q})$ and $(\vec{p}, \vec{q}) \mapsto (\vec{p}, t\vec{p} + \vec{q})$ (represented in block matrix form by*

$$\begin{pmatrix} tI & 0 \\ 0 & t^{-1}I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & tI \\ 0 & I \end{pmatrix},$$

respectively, I denoting the $n \times n$ identity matrix) belong to $Sp(2n)$. Since t can be as large as one likes, this should make clear that $Sp(2n)$ is noncompact.

Let's now consider the other structures appearing in a compatible triple (g, ω, J) on a finite-dimensional vector space V. Proposition 2.32 shows that there is a linear map $V \to \mathbb{R}^{2n}$ that *simultaneously* identifies g with the dot product (which we'll call g_0), ω with ω_0, and J with the endomorphism $J_0\colon R^{2n} \to \mathbb{R}^{2n}$ given by multiplication by i (under the identification $(\vec{p}, \vec{q}) \sim \vec{p} + i\vec{q}$ of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$). So in studying automorphisms of g, ω, J, and/or the associated Hermitian metric

[9] In other branches of geometry you may find another group called the symplectic group and labeled with the same notation, namely a version of the unitary group for vector spaces over the quaternions. These groups are not the same—for instance one is compact and the other isn't—so this is an unfortunate overlap; the main thing that they have in common is that the complexifications of their Lie algebras are isomorphic.

$h = g + i\omega$ (whether together or separately) we may as well study automorphisms of g_0, ω_0, J_0, and $h_0 := g_0 + i\omega_0$.

The groups of such automorphisms each have names, most of which are likely familiar:

Definition 2.35.

- The orthogonal group is
$$O(2n) = \{A \in GL(2n;\mathbb{R}) | (Av)\cdot(Aw) = v\cdot w \text{ for all } v,w\in\mathbb{R}^{2n}\}.$$
- The symplectic linear group is
$$Sp(2n) = \{A \in GL(2n;\mathbb{R}) | \omega_0(Av,Aw) = \omega_0(v,w) \text{ for all } v,w\in\mathbb{R}^{2n}\}.$$
- The complex general linear group is
$$GL(n;\mathbb{C}) = \{A \in GL(2n;\mathbb{R}) | A(J_0v) = J_0(Av) \text{ for all } v\in\mathbb{R}^{2n}\}.$$
- The unitary group is
$$U(n) = \{A \in GL(2n;\mathbb{R}) | h_0(Av,Aw) = h_0(v,w) \text{ for all } v,w\in\mathbb{R}^{2n}\}.$$

We are abusing notation slightly with the last two: traditionally, $GL(n;\mathbb{C})$ and $U(n)$ would be regarded as groups of $n\times n$ *complex* matrices, but we are regarding them as groups of $2n\times 2n$ real matrices. This is consistent with the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. An $n\times n$ complex matrix can be written as $Z = X + iY$ where X and Y are real matrices, and then for $\vec{p},\vec{q}\in\mathbb{R}^n$ we have
$$Z(\vec{p}+i\vec{q}) = (X+iY)(\vec{p}+i\vec{q}) = (X\vec{p}-Y\vec{q}) + i(Y\vec{p}+X\vec{q}).$$
Accordingly we have a correspondence
$$X + iY \leftrightarrow \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$
between $n\times n$ complex matrices and (some) $2n\times 2n$ real matrices. The groups denoted $GL(n;\mathbb{C})$ and $U(n)$ indicated above are, strictly speaking, the images of the standard versions of these groups under this correspondence.

The relations between the constituent parts of a compatible triple (g,ω,J) and the Hermitian inner product $h = g + i\omega$ are reflected in the following.

Proposition 2.36. *The groups in Definition 2.35 obey*
$$O(2n)\cap Sp(2n) = O(2n)\cap GL(n;\mathbb{C}) = Sp(2n)\cap GL(n;\mathbb{C}) = U(n).$$

Proof. That $O(2n)\cap Sp(2n) = U(n)$ is immediate from the definitions since g_0 and ω_0 are, respectively, the real and imaginary parts of h_0.

Let us now show that $U(n)\subset GL(n;\mathbb{C})$. (Of course under the traditional definitions this is clear, but these aren't what we are using.) If $A\in U(n)$ and $v\in\mathbb{R}^{2n}$, then for all $w\in\mathbb{R}^{2n}$ we have
$$\begin{aligned} h_0(AJ_0v,Aw) &= h(J_0v,w) = g(J_0v,w) + i\omega(J_0v,w) \\ &= \omega(v,w) - ig(v,w) = -ih(v,w) = -ih(Av,Aw) = h(J_0Av,Aw). \end{aligned}$$
Taking real parts shows that $\theta_{g_0}(AJ_0v) = \theta_{g_0}(J_0Av)$ so since $\theta_{g_0}\colon \mathbb{R}^{2n}\to\mathbb{R}^{2n*}$ is an isomorphism we have $A\in GL(n;\mathbb{C})$.

So we have both $U(n)\subset O(2n)\cap GL(n;\mathbb{C})$ and $U(n)\subset Sp(2n)\cap GL(n;\mathbb{C})$; it remains to show the reverse inclusions. Given that we have already shown that $U(n) = O(2n)\cap Sp(2n)$, these reverse inclusions are equivalent to the statement that, if $A\in GL(2n;\mathbb{R})$ obeys $AJ_0 = J_0A$, then we have $g_0(Av,Aw) = g_0(v,w)$ for all v,w if and only if we have $\omega_0(Av,Aw) = \omega_0(v,w)$ for all v,w. But given the relations $g_0(v,w) = \omega_0(v,J_0w)$ and likewise $\omega_0(v,w) = g_0(J_0v,w)$ this is not hard: if $g_0(Av,Aw) = g_0(v,w)$ for all v,w then
$$\omega_0(Av,Aw) = g_0(J_0Av,Aw) = g_0(AJ_0v,Aw) = g_0(J_0v,w) = \omega_0(v,w),$$

and the argument for the converse is identical. □

Of course, in doing computations with linear operators on $\mathbb{R}^{2n}$ one typically uses the formalism of matrices, and it is often useful to rephrase Definition 2.35 in these terms. Regarding elements of $\mathbb{R}^{2n}$ as column vectors, one has $v \cdot w = v^T w$ (with the T denoting transpose, and 1×1 matrices identified with numbers), so $(Av)\cdot(Aw) = v^T A^T A w$ and $O(2n)$ consists of those matrices with $A^T A = I$ where I is the identity matrix.

The endomorphism J_0 of $\mathbb{R}^{2n}$ sends $(\vec{p}, \vec{q})$ to $(-\vec{q}, \vec{p})$, so in matrix notation we have

$$J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

A block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then belongs to $GL(n;\mathbb{C})$ if and only if it is invertible and commutes with J_0; we find

$$MJ_0 = \begin{pmatrix} B & -A \\ D & -C \end{pmatrix}, \qquad J_0 M = \begin{pmatrix} -C & -D \\ A & B \end{pmatrix},$$

so $M \in GL(n;\mathbb{C})$ iff M is invertible and can be written in the form $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$, consistently with the remarks about $n \times n$ complex matrices before Proposition 2.36.

Returning now to $Sp(2n)$, since $\omega_0(v,w) = g_0(J_0 v, w) = g_0(w, J_0 v)$, we see that $\omega_0(v,w) = w^T J_0 v$ while $\omega_0(Av, Aw) = (Aw)^T J_0 A v = w^T A^T J_0 A v$. Thus, since the ij entry of a matrix M is given by $e_i^T M e_j$ where $\{e_1, \ldots, e_{2n}\}$ is the standard basis:

Proposition 2.37. *$A \in Sp(2n)$ if and only if $A^T J_0 A = J_0$.*

Here are some noteworthy properties that follow from this:

Corollary 2.38. *If $A \in Sp(2n)$, then also $A^T \in Sp(2n)$.*

Proof. By Example 2.33, $J_0 \in Sp(2n)$. So if $A \in Sp(2n)$, *i.e.* if $A^T J_0 A = J_0$, then the fact that $Sp(2n)$ is a group shows that $A^T = J_0 A^{-1} J_0^{-1} \in Sp(2n)$. □

Corollary 2.39. *If $A \in Sp(2n)$ is a symplectic matrix and $\lambda \in \mathbb{C}$ is an eigenvalue of A then $\frac{1}{\lambda}$ is also an eigenvalue of A. In fact, J_0 maps the λ-eigenspace for A to the $\frac{1}{\lambda}$-eigenspace for A^T.*

Proof. Of course $\lambda \neq 0$ since A is invertible. The condition that $A \in Sp(2n)$ can be rewritten as $A^T J_0 = J_0 A^{-1}$. If v lies in the λ-eigenspace for A, then $A^{-1} v = \frac{1}{\lambda} v$ and hence

$$A^T(J_0 v) = J_0(A^{-1} v) = \frac{1}{\lambda} J_0 v.$$

This proves the second sentence of the propostion, which in turn implies the first since A and A^T have the same characteristic polynomials and hence also the same eigenvalues. □

One property of elements of $Sp(2n)$ that is *not* most easily seen using matrices is the following:

Proposition 2.40. *If $A \in Sp(2n)$ then* $\det A = 1$.

(From the formula $A^T J_0 A = J_0$ one would quickly deduce $\det A = \pm 1$, but it is hard to see from this why it is not possible for $\det A$ to be -1.)

Proof. That $A \in Sp(2n)$ means that $A^* \omega_0 = \omega_0$. It follows that $A^*(\omega_0^{\wedge n}) = \omega_0^{\wedge n}$. But $\omega_0^{\wedge n}$ is a nonzero alternating $(2n)$-linear form on $\mathbb{R}^{2n}$, and hence for any linear $B \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ it holds that $\det B$ is the unique number with $B^*(\omega_0^{\wedge n}) = (\det B)\omega_0^{\wedge n}$. (One can prove this by showing that the coefficient on $\omega_0^{\wedge n}$ in $B^*(\omega_0^{\wedge n})$ is, as a function of the columns of B, alternating and multilinear, and equals 1 when B is the identity, and the determinant is the only function having these properties.) Applying this with $B = A$ shows that $\det A = 1$. □

To further understand some relationships between $O(2n), Sp(2n)$, and $U(n)$ it is helpful to "review" the linear algebraic construction called **polar decomposition**.

If A is any $(2n)\times(2n)$ matrix with real coefficients, consider the matrices AA^T and A^TA. Observe that these matrices are both symmetric. Moreover the identities

$$(Av)\cdot(Av) = v\cdot(A^TAv), \qquad (A^Tv)\cdot(A^Tv) = v\cdot(AA^Tv)$$

make clear AA^T and A^TA are both nonnegative definite, and positive definite[10] if A is invertible.

Now rather generally if B is a symmetric positive definite $m\times m$ matrix over $\mathbb{R}$ there is a canonical way of taking arbitrary real powers of B: by the spectral theorem for symmetric matrices we can decompose $\mathbb{R}^m$ into an orthogonal direct sum of eigenspaces for B with all eigenvalues positive, and for any $s\in\mathbb{R}$ we can define B^s to be the linear operator on $\mathbb{R}^m$ which acts on the various λ-eigenspaces for B as multiplication by λ^s. One can verify (and it's a good exercise to think about how to get the details just right) that B^s depends continuously on the symmetric positive definite matrix B.

We now apply this with $B = AA^T$ for an arbitrary invertible matrix A:

Proposition 2.41. *If $A\in GL(2n;\mathbb{R})$, and $t>0$ define the* left and right singular spaces of A with singular value t *to be:*

$$L_t(A) = \ker(AA^T - t^2I), \qquad R_t(A) = \ker(A^TA - t^2I).$$

Then:

(i) *We have **orthogonal** direct sum decompositions*

$$\mathbb{R}^{2n} = \bigoplus_{t>0} L_t(A) = \bigoplus_{t>0} R_t(A).$$

(ii) *For each $t>0$, A maps $R_t(A)$ isomorphically to $L_t(A)$, with $\|Av\| = t\|v\|$ for all $v\in R_t(A)$.*

(iii) *The matrix $(AA^T)^{-1/2}A$ belongs to $O(2n)$.*

(iv) *If $A\in Sp(2n)$, then for all $t>0$ the standard complex structure J_0 maps $L_t(A)$ to $L_{1/t}(A)$.*

(v) *If $A\in Sp(2n)$, then for all $s\in\mathbb{R}$ the matrix $(AA^T)^sA$ also belongs to $Sp(2n)$. In particular, $(AA^T)^{-1/2}A\in U(n)$.*

Proof. Since AA^T and A^TA are both symmetric and positive definite, part (i) follows from the spectral theorem for symmetric matrices.

For part (ii), if $v\in R_t(A)$, so that $A^TAv = t^2v$, then evidently $AA^T(Av) = A(t^2v) = t^2Av$ so $Av\in L_t(A)$. Also

$$\|Av\|^2 = (Av)\cdot(Av) = v\cdot(A^TAv) = v\cdot(t^2v) = t^2\|v\|^2,$$

so $\|Av\| = t\|v\|$. This is enough to show that A maps $R_t(A)$ injectively into $L_t(A)$. But reversing the roles of A and A^T shows that A^T maps $L_t(A)$ injectively into $R_t(A)$, so the dimensions of these spaces must be equal and $A|_{R_t(A)}$ surjects to $L_t(A)$, completing the proof of (ii).

As for (iii), the map $(AA^T)^{-1/2}$ acts on $L_t(A)$ by scalar multiplication by $\frac{1}{t}$, so if $C = (AA^T)^{-1/2}A$, then part (ii) shows that, for each $v\in R_t(A)$, we have $Cv\in L_t(A)$ with $\|Cv\| = \|\frac{1}{t}Av\| = \|v\|$. Consequently if v and w both lie in the same right singular space $R_t(A)$, then

(14)
$$(Cv)\cdot(Cw) = \frac{1}{2}\left(\|C(v+w)\|^2 - \|Cv\|^2 - \|Cw\|^2\right) = \frac{1}{2}\left(\|v+w\|^2 - \|v\|^2 - \|w\|^2\right) = v\cdot w \quad (v,w\in R_t(A)).$$

[10] Recall that a matrix B is called positive definite if $v\cdot(Bv) > 0$ for all nonzero v. Evidently this implies that all eigenvalues of B are positive.

But now if v, w are arbitrary elements of $\mathbb{R}^{2n}$, by (i) there are distinct $t_1, \ldots, t_k > 0$ such that we can write

$$v = v_1 + \cdots + v_k, \quad w = w_1 + \cdots + w_k \quad (v_i, w_i \in R_{t_i}(A)),$$

and then $v \cdot w = \sum_i v_i w_i$ and $(Cv) \cdot (Cw) = \sum_i (Cv_i) \cdot (Cw_i)$ by the orthogonality of the direct sum decompositions in (i). So by (14) we have $(Cv) \cdot (Cw) = v \cdot w$ for all $v, w \in \mathbb{R}^{2n}$, confirming that $C \in O(2n)$.

Now suppose that $A \in Sp(2n)$, so $A^T J_0 A = J_0$. By Corollary 2.38, we also have $AJ_0A^T = J_0$. Thus $A^T J_0 = J_0 A^{-1}$, and $AJ_0 = J_0(A^T)^{-1}$. So if $v \in L_t(A)$ (*i.e.*, $AA^T v = t^2 v$), we find

$$AA^T J_0 v = AJ_0 A^{-1} v = J_0 (A^T)^{-1} A^{-1} v = J_0 (AA^T)^{-1} v = t^{-2} J_0 v,$$

proving (iv).

Finally we prove (v). It suffices to check that, for $A \in Sp(2n)$, we have $(AA^T)^s \in Sp(2n)$ for all s, since the fact that $Sp(2n)$ is closed under composition would then show that $(AA^T)^s A \in Sp(2n)$, and then (iii) would show that $(AA^T)^{-1/2} A \in Sp(2n) \cap O(2n)$, which by Proposition 2.36 is the same as $U(n)$. Write $B = AA^T$. Part (iv) shows that, for each $t > 0$, J_0 maps the t^2-eigenspace of B to the t^{-2}-eigenspace of B. Now for $s \in \mathbb{R}$ the t^2-eigenspace of B is the same as the t^{2s}-eigenspace of B^s, so this shows that, for each $\lambda, s \in \mathbb{R}$, J_0 maps the λ-eigenspace of B^s to the $\frac{1}{\lambda}$-eigenspace of B^s. Thus, at least for each v lying in an eigenspace of B^s, one has $B^s J_0 v = J_0 (B^s)^{-1} v$. But $\mathbb{R}^{2n}$ is the direct sum of the eigenspaces of B^s (namely the various $L_t(A)$), so this implies that $B^s J_0 = J_0 (B^s)^{-1}$, *i.e.* that $B^s J_0 B^s = J_0$. Since B^s is symmetric this is equivalent (via Proposition 2.37) to the statement that $B^s \in Sp(2n)$. □

The following is what is usually called the (left) polar decomposition—any invertible linear operator is the composition of a rotation, possibly a reflection, and an operator that dilates an orthogonal set of coordinate axes.

Corollary 2.42. *Every $A \in GL(2n; \mathbb{R})$ can be expressed as a product $A = PO$ where $P = (AA^T)^{1/2}$ is symmetric and positive definite, and $O \in O(2n)$. If $A \in Sp(2n)$ then $O \in U(n)$.*

Exercise 2.43. *Let ω be an arbitrary linear symplectic form on $\mathbb{R}^{2n}$, and let A be the $(2n) \times (2n)$ matrix determined by the property that $\omega(v, w) = (Av) \cdot w$ for all $v, w \in \mathbb{R}^{2n}$.*

(a) *Show that A is skew-symmetric and invertible.*
(b) *If $O = (AA^T)^{-1/2} A$ is the orthogonal part of the polar decomposition of A as in Corollary 2.42, prove, using part (a), that $O^2 = -I$. (Note: At some point in the proof you will use that a certain pair of matrices commute with each other; you need to show this carefully.)*
(c) *Prove that the complex structure O is ω-compatible.*

Corollary 2.44. *There is a strong deformation retraction $H\colon [0,1] \times GL(2n; \mathbb{R}) \to GL(2n; \mathbb{R})$ of $GL(2n; \mathbb{R})$ onto $O(2n)$, which restricts to a strong deformation retraction of $Sp(2n)$ onto $U(n)$.*

Proof. In view of Proposition 2.41, we set $H(s, A) = (AA^T)^{-s/2} A$. Evidently if $A \in O(2n)$, which is equivalent to the condition $AA^T = I$, we have $H(s, A) = A$ for all s. Obviously $H(0, \cdot)$ is the identity, and by (iii), $H(1, \cdot)$ has its image in $O(2n)$, so H indeed defines a strong deformation retraction of $GL(2n; \mathbb{R})$ onto $O(2n)$. By the last part of Corollary 2.41, the restriction of H to $[0,1] \times Sp(2n)$ has its image in $Sp(2n)$, and $H(1, \cdot)|_{Sp(2n)}$ has image in $U(n) = Sp(2n) \cap O(2n)$; this suffices to prove the last clause of the corollary. □

Corollary 2.45. *The inclusions $O(2n) \hookrightarrow GL(2n; \mathbb{R})$ and $U(n) \hookrightarrow Sp(2n)$ are homotopy equivalences.*

Motivated by this corollary, I'll now say a bit about the topology of $U(n)$. Since (as seen in Proposition 2.36) $U(n) \subset GL(n; \mathbb{C})$, we can regard elements of $U(n)$ as $n \times n$ complex matrices

$Z = X + iY$ rather than $2n \times 2n$ real matrices $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$. From this point of view $U(n)$ consists of the $n \times n$ complex matrices Z preserving the standard complex inner product on $\mathbb{C}^n$. The latter is given by $h(\mathbf{w}, \mathbf{z}) = \bar{\mathbf{w}}^T z$, so $h(Z\mathbf{w}, Z\mathbf{z}) = \bar{\mathbf{w}}\bar{Z}^T Z\mathbf{z}$ and $Z \in U(n)$ iff $\bar{Z}^T Z = I$.

We can regard $U(n)$ as embedded in $(\mathbb{C}^n)^n$ (send a matrix to the n-tuple consisting of its column vectors); when we speak of $U(n)$ as a topological space we are implicitly using this topology (any other reasonable embedding of $U(n)$ into, say, $\mathbb{C}^{n^2}$ or $(\mathbb{R}^{2n})^{2n}$ would yield the same topology). Since $\bar{Z}^T Z$ varies continuously with the complex matrix Z, $U(n)$ is evidently a closed subset of $(\mathbb{C}^n)^n$; moreover since the equation $\bar{Z}^T Z = I$ implies that each column of Z is a unit vector it follows that $U(n)$ is compact (unlike $Sp(2n)$). As an example, if $n = 1$, then Z is just a 1×1 matrix and the unitary condition just says that the only entry of Z lies on the unit circle, so $U(1) \cong S^1$.

Some quick information about the homotopy theory of $U(n)$ (and hence, by Corollary 2.45, of $Sp(2n)$) can be extracted from some basic Lie group theory (see, *e.g.* [L, Chapter 9] or [Wa, Chapter 3]) via the action of $U(n)$ on the unit sphere S^{2n-1} in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. One can check that this action is smooth and transitive, and hence that, combining [Wa, Theorems 3.58 and 3.62], the map

$$\begin{aligned} p\colon\ U(n) &\to S^{2n-1} \\ A &\mapsto Ae_1 \end{aligned}$$

(e_1 denoting the first standard basis vector) defines a fiber bundle, with fibers diffeomorphic to the "stabilizer" $\{A \in U(n) | Ae_1 = e_1\}$. Now the conditions that $A \in U(n)$ and $Ae_1 = e_1$ force both the first column and (since the columns of A are orthonormal) also the first row of A to be e_1, so A looks like

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{pmatrix}, \tag{15}$$

with $A' \in U(n-1)$. So the fibers of $p\colon\ U(n) \to S^{2n-1}$ are copies of $U(n-1)$.

By [Ha, Proposition 4.48 and Theorem 4.41], this implies that we have an exact sequence

$$\cdots \pi_{k+1}(S^{2n-1}) \to \pi_k(U(n-1)) \to \pi_k(U(n)) \to \pi_k(S^{2n-1}) \to \cdots$$

where the second map is induced by the map[11] that sends A' to the matrix A in (15), and the third map is induced by p. In particular $\pi_k(U(n-1)) \to \pi_k(U(n))$ is an isomorphism if $k < 2(n-1)$, and surjective if $k = 2(n-1)$. So the sequence of inclusion-induced maps

$$\pi_k(U(1)) \to \pi_k(U(2)) \to \cdots \to \pi_k(U(n-1)) \to \pi_k(U(n))$$

induces isomorphisms for $k = 0$ and $k = 1$. Since $U(1) \cong S^1$, we hence deduce that every $U(n)$ is path-connected, and that $\pi_1(U(n)) \cong \mathbb{Z}$ for all n.

In fact there's a fairly simple way to understand the map $\pi_1(U(n)) \to \mathbb{Z}$ (or, more directly, the map $\pi_1(U(n)) \to \pi_1(S^1)$) that induces this isomorphism: if $Z \in U(n)$ then taking the determinant of both sides of the equation $\bar{Z}^T Z = I$ shows that $|\det Z|^2 = 1$. Thus we have a map $\det\colon U(n) \to S^1$, and the isomorphism $\pi_1(U(n)) \cong \mathbb{Z}$ is just the map induced by det on π_1. Indeed for $n = 1$ det is essentially the identity, and under the inclusions $U(1) \to \cdots \to U(n-1) \to U(n)$ a loop generating $\pi_1(U(1))$ gets sent to a loop of matrices that rotates the nth copy of $\mathbb{C}$ and leaves the other factors fixed; the determinant of this loop evidently rotates once around the circle. This explains in fairly elementary fashion why we have a surjective map $\det_*\colon\ \pi_1(U(n)) \to \pi_1(S^1)$; that this map is injective is basically equivalent to the maps $\pi_1(U(n-1)) \to \pi_1(U(n))$ being surjective, which requires the above arguments with fiber bundles (or something similar).

[11] which below I will refer to, slightly inaccurately, as an "inclusion"

In any case, we have the following corollary:

Corollary 2.46. *The group $Sp(2n)$ is connected, and $\pi_1(Sp(2n)) \cong \mathbb{Z}$. An isomorphism $\pi_1(Sp(2n)) \to \mathbb{Z}$ is induced by, for each loop $\gamma\colon S^1 \to Sp(2n)$, defining a loop of unitary matrices*

$$\gamma_U(t) = (\gamma(t)\gamma(t)^T)^{-1/2}\gamma(t),$$

regarding these unitary matrices as complex $n \times n$ matrices, and then assigning to γ the degree of the map $S^1 \to S^1$ defined by $t \mapsto \det \gamma_U(t)$.

The number associated to γ in the corollary is called the *Maslov index* of γ and it arises rather frequently in symplectic topology; see the end of [MS, Section 2.2] for an axiomatic treatment and some other properties.

Exercise 2.47. *Let $\mathcal{L}_n$ denote the set of Lagrangian subspaces of the symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$, and fix an element $L_0 \in \mathcal{L}_n$.*

(a) *Prove that for each $L \in \mathcal{L}_n$ there is $A \in U(n)$ such that $A(L_0) = L$, and that the subgroup $\{A \in U(n) | A(L_0) = L_0\}$ is isomorphic to $O(n)$.*
(b) *(Not to be turned in) Convince yourself, perhaps by looking at material on homogeneous spaces in* [L] *or* [Wa]*, that part (a) implies that $\mathcal{L}_n$ can be given the structure of a smooth manifold diffeomorphic to the quotient of Lie groups $U(n)/O(n)$, and (using* [Ha, Theorem 4.41]*) that this implies there is an exact sequence*

$$\pi_1(O(n)) \to \pi_1(U(n)) \to \pi_1(\mathcal{L}_n) \to \pi_0(O(n)). \tag{16}$$

(c) *Show that $\pi_1(\mathcal{L}_n) \cong \mathbb{Z}$, and find an explicit example of a loop in $\mathcal{L}_n$ that represents a generator for $\pi_1(\mathcal{L}_n)$. (This generator should map via the last map in (16) to the nontrivial element of $\pi_0(O(n))$.)*

3. Introducing symplectic manifolds

3.1. **Almost symplectic and similar structures.** We now transition toward the more global theory of symplectic manifolds. Since we've just learned a significant amount of related linear algebra, we'll begin by considering a setting which incorporates this linear algebra at all of the points of smooth manifold (though it is missing an analytic condition that we'll impose later in order to likewise globalize the notions of Section 1).

Definition 3.1. Let M be a smooth manifold. An **almost symplectic structure** on M is a differential two-form $\omega \in \Omega^2(M)$ such that, for all $x \in M$, the bilinear form $\omega_x\colon T_xM \times T_xM \to \mathbb{R}$ is non-degenerate. The pair (M, ω) is then said to be an **almost symplectic manifold**.

Thus on an almost symplectic manifold (M, ω) we have an M-parametrized family of symplectic vector spaces (T_xM, ω_x). Recall that in the definition of the notion of a differential 2-form ω one requires that the "ω_x vary smoothly with x." Since the ω_x are defined on different vector spaces one should think through what this means, and there are various equivalent ways of formulating it: one can say for instance that in every local coordinate chart the pullback of ω by the inverse of the chart is a smooth differential form on an open subset of $\mathbb{R}^{2n}$ (observing that in a neighborhood of each point this is independent of the choice of coordinate chart), or one can say that for every pair of vector fields X, Y the function $\omega(X, Y)$ is smooth. Similar notions of a family of linear-algebraic objects defined on each T_xM "varying smoothly with x" will appear below, and in each case this should be defined similarly to the definition of smoothness of a differential form; I will implicitly leave to the reader the precise formulations.

In the spirit of Section 2.4 one can define the following similar notions:

Definition 3.2. Let M be a smooth manifold.

- An **almost complex structure** on M, denoted J, is a choice, for each $x \in M$, of a complex structure $J_x\colon T_xM \to T_xM$ which varies smoothly with x.
- A **Riemannian metric**[12] on M, denoted g, is a choice, for each $x \in M$, of an inner product $g_x\colon T_xM \to T_xM$ which varies smoothly with x.
- An **almost Hermitian structure** on M is a triple (g, ω, J) consisting of a Riemannian metric g, almost symplectic structure ω, and almost complex structure J such that, for each $x \in M$, (g_x, ω_x, J_x) is a compatible triple on T_xM.

We let $\mathscr{AS}(M)$, $\mathscr{AC}(M)$, $\mathscr{AE}(M)$, and $\mathscr{AH}(M)$ respectively denote the spaces of almost symplectic structures on M, almost complex structures on M, Riemannian metrics on M, and almost Hermitian structures on M, endowed with the C^∞ topology[13].

Thus by Proposition 2.31, giving an almost Hermitian structure is equivalent to (smoothly) making all of the T_xM into complex vector spaces using J and then defining a smooth family of Hermitian inner products (namely $h_x = g_x + i\omega_x$) on these complex vector spaces.

Probably the first question to ask about $\mathscr{AS}(M)$, $\mathscr{AC}(M)$, $\mathscr{AE}(M)$, and $\mathscr{AH}(M)$ is whether they are nonempty. For one (and only one) of them this is easy:

Proposition 3.3. *For any smooth manifold M, the space $\mathscr{AE}(M)$ of Riemannian metrics on M is nonempty and convex, and hence contractible.*

Proof. If $g, h \in \mathscr{AE}(M)$ then for all $t \in [0,1]$ and all $x \in M$ the map $tg_x + (1-t)h_x\colon T_xM \times T_xM \to \mathbb{R}$ inherits the properties of bilinearity, symmetry, and positive-definiteness from g_x and h_x and so defines an inner product; thus $tg + (1-t)h \in \mathscr{AE}(M)$. This proves convexity, and of course nonempty convex sets are contractible, so we will be done when we show how to construct just one Riemannian metric on M.

To do this, let $\{U_\alpha\}$ be an open cover of M by domains of smooth coordinate charts $\phi_\alpha\colon U_\alpha \to \mathbb{R}^m$, and for each index α define a Riemannian metric g_α on U_α by $g_\alpha(v,w) = (\phi_{\alpha*v})\cdot(\phi_{\alpha*}w)$. Also let $\{\chi_\alpha\}$ be a (locally finite) partition of unity subordinate to the cover $\{U_\alpha\}$. Then for each α, the product $\hat{g}_\alpha := \chi_\alpha g_\alpha$ extends by zero outside of U_α to define a smoothly varying family of symmetric bilinear forms on the various T_xM such that, for $0 \neq v \in T_xM$, we have $\hat{g}_\alpha(v,v) \geq 0$, with equality only if $\chi_\alpha(x) = 0$. Then $g = \sum_\alpha \hat{g}_\alpha$ (*i.e.*, modulo extension of the summands by zero, $g = \sum_\alpha(\chi_\alpha g_\alpha)$) is a Riemannian metric (it is well-defined and smooth because the partition of unity is locally finite, and it is positive definite because for each x there is some α for which $\chi_\alpha(x) > 0$ and hence $(\hat{g}_\alpha)_x$ is positive definite). □

Of course we cannot expect $\mathscr{AS}(M)$, $\mathscr{AC}(M)$, or $\mathscr{AH}(M)$ to be nonempty for arbitrary smooth M, as they are empty when $\dim M$ is odd. Even assuming that $\dim M$ is even, say $2n$, the observation that if $\omega \in \mathscr{AS}(M)$ then $\omega^{\wedge n}$ is a nowhere-vanishing top-degree form shows that $\mathscr{AS}(M)$ and $\mathscr{AH}(M)$ are empty if M is non-orientable; likewise one can show (try it) that an almost complex structure on M induces a preferred orientation and so if M is non-orientable then $\mathscr{AC}(M)$ is empty. This can in fact be viewed as the first in a sequence of (complicated, at least when n is not small) algebraic-topological obstructions to the existence of an almost complex structure; the next in the sequence is the statement that in a compact almost complex four-manifold the sum of the signature and Euler characteristic must be divisible by 4. While the description of $\mathscr{AS}(M)$, $\mathscr{AC}(M)$, and $\mathscr{AH}(M)$ is complicated, the following shows that, at least up to homotopy, this is one problem and not three:

[12] To be consistent with the other definitions one could call this an "almost Euclidean structure," but no one does.

[13] A sequence converges iff, for all k and in all local coordinate charts, its kth order partial derivatives converge uniformly on compact subsets.

Theorem 3.4. *The projection maps*

$$\begin{array}{ll} \mathscr{AH}(M) & \to \mathscr{AS}(M) \\ (g,\omega,J) & \mapsto \omega \end{array} \quad \textit{and} \quad \begin{array}{ll} \mathscr{AH}(M) & \to \mathscr{AC}(M) \\ (g,\omega,J) & \mapsto J \end{array}$$

are homotopy equivalences. Moreover the preimage of any point under either of these projections is contractible (and in particular nonempty).

Proof. The homotopy-theoretic aspects of the proof are the same in the two cases; we isolate these in the following lemma:

Lemma 3.5. *Let H, X, and E be topological spaces, with E contractible, and suppose we have continuous maps $p\colon H \to X$, $\hat{p}\colon H \to E \times X$, and $\mathfrak{h}\colon E \times X \to H$ such that $\mathfrak{h} \circ \hat{p} = 1_H$, $\pi \circ \hat{p} = p$, and $p \circ \mathfrak{h} = \pi$ where π is the projection from $E \times X$ to X. Then for each $x \in X$ the space $p^{-1}(\{x\})$ is contractible, and p is a homotopy equivalence from H to X.*

(Schematically, the assumption is that we have a diagram

$$\begin{tikzcd} H \arrow[rr, bend left, "\hat{p}"] \arrow[dr, "p"'] & & E \times X \arrow[ll, bend left, "\mathfrak{h}"] \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

which commutes except possibly for the loop from $E \times X$ to itself. We will apply this with $H = \mathscr{AH}(M)$, $E = \mathscr{AE}(M)$, and X equal to either $\mathscr{AS}(M)$ or $\mathscr{AJ}(M)$, with p the projection in the statement of the theorem and $\hat{p}$ the obvious lift $(g,\omega,J) \mapsto (g,\omega)$ or (g,J). In each case we will need to construct $\mathfrak{h}$, *i.e.* we will need to construct a rule sending an arbitrary pair (g,J) or (g,ω), generally not satisfying any compatibility conditions, to an almost Hermitian structure without changing J or ω.)

Proof of Lemma 3.5. For the statement about $p^{-1}(\{x\})$, the assumptions that $\pi \circ \hat{p} = p$ and $p \circ \mathfrak{h} = \pi$ imply that $\hat{p}$ and $\mathfrak{h}$ restrict as maps between $p^{-1}(\{x\})$ and $E \times \{x\}$. The composition of these maps that goes from $p^{-1}(\{x\})$ to $p^{-1}(\{x\})$ is the identity by assumption, while the composition that goes from $E \times \{x\}$ to $E \times \{x\}$ is homotopic to the identity because E is contractible and so all maps $E \times \{x\} \to E \times \{x\}$ are homotopic. Thus $\hat{p}$ restricts as a homotopy equivalence $p^{-1}(\{x\}) \simeq E \times \{x\}$, whence $p^{-1}(\{x\})$ is contractible.

Since (E being contractible) $\pi\colon E \times X \to X$ is a homotopy equivalence, to see that p is a homotopy equivalence it is enough to show that $\hat{p}\colon H \to E \times X$ is a homotopy equivalence. Half of what we need in this regard comes from the hypothesis that $\mathfrak{h} \circ \hat{p} = 1_H$. For the other composition, observe that there is only one homotopy class of maps $f : E \times X \to E \times X$ having the property that $\pi \circ f = \pi$. Indeed, if $\pi \circ f = \pi$ we can write $f(e,x) = (f_0(e,x), x)$ where $f_0\colon E \times X \to E$, and then if $A\colon [0,1] \times E \to E$ is a homotopy from 1_E to a constant map to a point $e_0 \in E$, we see that $(t,e,x) \mapsto (A(t, f_0(e,x)), x)$ defines a homotopy from f to $(e,x) \mapsto (e_0, x)$. The previous sentence applies with f equal either to $1_{E\times X}$ or to $\hat{p} \circ \mathfrak{h}$, whence these two maps are homotopic. So $\hat{p}$ and $\mathfrak{h}$ are homotopy inverses, so p is a homotopy equivalence. □

Turning now to geometry, we need to construct maps $\mathscr{AE}(M) \times \mathscr{AJ}(M) \to \mathscr{AH}(M)$ and $\mathscr{AE}(M) \times \mathscr{AS}(M) \to \mathscr{AH}(M)$ suitable for use as $\mathfrak{h}$ in Lemma 3.5. We do the case of $\mathscr{AJ}(M)$ first.

In this case we have $p(g,\omega,J) = J$, $\hat{p}(g,\omega,J) = (g,J)$, and $\mathfrak{h}$ should send a pair (g,J) where g is an arbitrary Riemannian metric and J an arbitrary almost complex structure to an almost Hermitian structure (g', ω', J) (the two J's should coincide because we want $p \circ \mathfrak{h} = \pi$), in such a way that in the special case that (g,J) is already part of a Hermitian structure (g,ω,J), $\mathfrak{h}$ sends (g,J) to this (g,ω,J) (because we want $\mathfrak{h}\circ\hat{p} = 1_{\mathscr{AH}(M)}$). Now generally the latter condition applies iff $g(Jv,Jw) = g(v,w)$ for all v and w, and then ω is determined by g and J, so to construct $\mathfrak{h}$ we need to convert a general Riemannian metric g to one which is J-invariant in this sense. The typical way of doing such things is *averaging*, which is made easier by the fact that $J^2 = -1$: given $g \in \mathscr{AE}(M)$ we define

$$g_J(v,w) = \frac{1}{2}(g(v,w)+g(Jv,Jw)), \qquad \omega_J(v,w) = g_J(Jv,w) = \frac{1}{2}(g(Jv,w)-g(v,Jw)).$$

and so define $\mathfrak{h}$ by $\mathfrak{h}(g,J) = (g_J,\omega_J,J)$. By construction, the image of $\hat{p}$ consists precisely of (g,J) having the property that $g_J = g$, so $\mathfrak{h}\circ\hat{p} = 1_{\mathscr{AH}(M)}$, and the remaining hypotheses of Lemma 3.5 manifestly hold, so that lemma implies the part of the theorem that concerns $\mathscr{AJ}(M)$.

Turning now to $\mathscr{AS}(M)$, we make use of Exercise 2.43. Given $(g,\omega) \in \mathscr{AE}(M)\times\mathscr{AS}(M)$, if we define a family of operators $A = (A_x)_{x\in M}$ on the tangent spaces T_xM by the condition that $\omega(v,w) = g(Av,w)$ (in other words, $A = \theta_g^{-1}\theta_\omega$), then the family of operators $J^g = (J^g_x)_{x\in M}$ on the various T_xM given by $(AA^T)^{-1/2}A$ will define an element of $\mathscr{AJ}(M)$. (Here the transpose is to be computed with respect to an orthonormal basis for g; to say things in a more basis-independent way—which perhaps makes it clearer that the J^g_x depend smoothly on x and continuously on g,ω—we can write $A^T = \theta_g^{-1}A^*\theta_g$.) Writing in general $\hat{g}_{\omega,J}(v,w) = \omega(v,Jw)$, we then define our desired map $\mathfrak{h}\colon \mathscr{AE}(M)\times\mathscr{AS}(M) \to \mathscr{AH}(M)$ by $\mathfrak{h}(g,\omega) = (\hat{g}_{\omega,J_g},\omega,J_g)$. Exercise 2.43 shows that $(\hat{g}_{\omega,J^g},\omega,J_g)$ is indeed an almost Hermitian structure. Clearly $p\circ\mathfrak{h} = \pi$, so we just need to check that $\mathfrak{h}\circ\hat{p}$ is the identity, *i.e.* that if (g,ω,J) is a compatible triple then $J^g = J$ and $\hat{g}_{\omega,J} = g$. The latter condition is clear from the definition of a compatible triple. For the former, note that in this case the family of operators A is just equal to J, and that $J^T = -J = J^{-1}$ (by Proposition 2.31 it suffices to check this latter condition in the case of the standard Hermitian structure on $\mathbb{C}^n$, and there it is clear), so the formula $J^g = (AA^T)^{-1/2}A$ just simplifies to $J^g = J$. □

3.2. **Symplectic structures.** We now turn to symplectic structures as opposed to just almost symplectic ones. There are various ways of phrasing the definition; for instance:

Definition 3.6. Let ω be an almost symplectic structure on a $2n$-dimensional smooth manifold M. We say that ω is a **symplectic structure** on M if every point $x\in M$ has a neighborhood U together with a smooth coordinate chart $\phi\colon U\to\mathbb{R}^{2n}$ such that $\phi^*\omega_0 = \omega|_U$, where $\omega_0 = \sum_{j=1}^n dp_j\wedge dq_j$ is the standard symplectic form on $\mathbb{R}^{2n}$. The pair (M,ω) is then said to be a **symplectic manifold**.

So a symplectic manifold (M,ω) admits a *symplectic atlas* $\{\phi_\alpha\colon U_\alpha\to\mathbb{R}^{2n}\}$, where the U_α form an open cover of M and each ϕ_α is a coordinate chart that pulls back the standard symplectic form to ω. So the transition functions $\phi_\beta\circ\phi_\alpha^{-1}\colon \phi_\alpha(U_\alpha\cap U_\beta)\to\phi_\beta(U_\alpha\cap U_\beta)$ will obey

$$(\phi_\beta\circ\phi_\alpha^{-1})^*\omega_0 = \phi_\alpha^{-1*}(\phi_\beta^*\omega_0) = \phi_\alpha^{-1*}\omega = \omega_0$$

(suppressing notation for restrictions). If one so desired, one could remove all explicit mention of ω from the definition of a symplectic manifold, just saying that a symplectic manifold is a smooth manifold equipped with an atlas $\{\phi_\alpha\colon U_\alpha\to\mathbb{R}^{2n}\}$ whose transition functions $\phi_\beta\circ\phi_\alpha^{-1}$ all pull back $\omega_0 = \sum_j dp_j\wedge dq_j$ to itself; from this one could recover ω by the piecewise formula $\omega|_{U_\alpha} = \phi_\alpha^*\omega_0$, noting that the condition on the transition functions amounts to the statement that $(\phi_\alpha^*\omega_0)|_{U_\alpha\cap U_\beta} = (\phi_\beta^*\omega_0)|_{U_\alpha\cap U_\beta}$.

Another equivalent formulation of the definition of a symplectic manifold arises from the following fundamental result:

Theorem 3.7 (Darboux's Theorem). *A necessary and sufficient condition for an almost symplectic manifold (M,ω) to be symplectic is that ω be closed: $d\omega = 0$.*

Thus a symplectic manifold is precisely a pair (M,ω) where ω is a closed, non-degenerate two-form on M. For example, if $\dim M = 2$, any nowhere-vanishing two-form on M (the existence of which is equivalent to M being orientable) is a symplectic structure. I will give the (easy) proof that the condition is necessary here, postponing the deeper fact that it is sufficient until after further discussion. Already in the case that $\dim M = 2$ the latter should seem tricky—a general nowhere-vanishing form on a surface looks locally like $f(p,q)dp \wedge dq$ for an arbitrary positive function f, and Darboux's theorem says that we can choose coordinates in such a way that $f \equiv 1$.

Proof of necessity. The key point here is that, for each $x \in M$, the alternating three-form $(d\omega)_x$ depends only on the restriction of ω to a neighborhood of x. Thus if $\{\phi_\alpha\colon U_\alpha \to \mathbb{R}^{2n}\}$ is a symplectic atlas it is enough to check that $d(\omega|_{U_\alpha}) = 0$ for each α. But by the naturality of the exterior derivative operator, $d(\omega|_{U_\alpha}) = d\phi_\alpha^*\omega_0 = \phi_\alpha^* d\omega_0 = 0$ since direct inspection of $\omega_0 = \sum_j dp_j \wedge dq_j$ shows that it is closed. □

If you're not used to this, it is worth thinking about why this argument cannot be modified to show that ω is exact—after all, $\omega_0 = d\left(\sum_j p_j dq_j\right)$ is exact, and pullbacks of exact forms are exact. The issues is that exactness, unlike closedness, is not a local condition, so all that the above reasoning shows is that for each α we can write $\omega|_{U_\alpha} = d\eta_\alpha$ for some $\eta_\alpha \in \Omega^1(U_\alpha)$, and in order to conclude that ω is exact one would need to be able to choose the various η_α so that they agree on overlaps.

The easy half of Darboux's theorem is enough to give the following obstruction to the existence of a symplectic structure:

Proposition 3.8. *If (M,ω) is a compact symplectic manifold then $H^2(M;\mathbb{R}) \neq 0$. In fact, if $\dim M = 2n$ there is $a \in H^2(M;\mathbb{R})$ such that $a^n \neq 0$.*

(Here a^n means the n-fold product of a with itself under the cup product.)

Proof. Since ω is closed, it represents a class in the second de Rham cohomology of M, which the de Rham theorem identifies with $H^2(M;\mathbb{R})$; denote this class as a. Recall also that under the de Rham isomorphism, cup product of cohomology classes is the operation induced by wedge product of differential forms; thus $a^n = [\omega^{\wedge n}]$. Now $\omega^{\wedge n}$ is a nowhere-vanishing top-degree form on M, so it induces an orientation, and with respect to this orientation the evaluation of a^n on the fundamental class of M is then given by

$$\langle a^n, [M]\rangle = \int_M \omega^{\wedge n} > 0.$$

Thus $a^n \neq 0$. □

Thus for instance no sphere of dimension greater than 2 admits a symplectic structure (since these all have $H^2 = 0$), nor for that matter does any manifold of the form $S^k \times M$ where M is compact and $k \geq 3$ (since, even if M has nontrivial H^2, the product $S^k \times M$ will not have any classes in H^2 with nonzero nth power where $\dim S^k \times M = 2n$).

In just the same way as we have gone from the notion of an almost symplectic manifold to that of a symplectic manifold, one can specialize the notion of an almost complex manifold to that of a **complex manifold**. An almost complex manifold (M,J) is said to be complex if M is covered by coordinate charts $\phi\colon U \to \mathbb{C}^n$ that identify the complex structures J_x on the tangent spaces T_xM with the standard complex structure J_0 (multiplication by i) on $T_{\phi(x)}\mathbb{C}^n = \mathbb{C}^n$, in the sense that the derivative ϕ_* of ϕ obeys $\phi_* \circ J_x = J_0 \circ \phi_*$ for all x. Equivalently a complex structure on a smooth manifold M is given by an atlas for which the transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ are holomorphic (as maps between open subsets of $\mathbb{C}^n$).

Complex manifolds comprise a rich subject that is generally beyond the scope of these notes; for now we just note that, while Theorem 3.4 shows that the manifolds that admit almost symplectic structures are the same as the ones that almost complex ones, the situation is quite different if one is comparing (genuine, not almost) symplectic and complex structures. Here is a simple example. Fix any $n \geq 2$, and let M be the quotient of $\mathbb{C}^n \setminus \{0\}$ by the equivalence relation transitively generated by identifying any complex vector $\vec{z}$ with $2\vec{z}$. (So $\vec{w}$ and $\vec{z}$ belong to the same equivalence class iff there is $k \in \mathbb{Z}$ with $\vec{w} = 2^k\vec{z}$.) So we have a covering map $\pi \colon \mathbb{C}^n \setminus \{0\} \to M$, and it's not hard to check that there is a complex structure J on M uniquely characterized by the statement that $\pi_* \circ J_0 = J \circ \pi_*$ (just use local inverses to π to identify the sets in a cover of M with open subsets of $\mathbb{C}^n$, and take J equal to J_0 under this identification; that this prescription is consistent on overlaps uses the fact that the covering transformations $\vec{z} \mapsto 2^k\vec{z}$ are holomorphic). It's also not hard to see that M is diffeomorphic to $S^{2n-1} \times S^1$, since one can obtain M by starting with the annular region $\{\vec{z} \in \mathbb{C}^n | 1 \leq \|\vec{z}\| \leq 2\}$ and then identifying the boundary components by the obvious scaling. So $S^{2n-1} \times S^1$ admits a complex structure, whereas Proposition 3.8 shows that it does not admit a symplectic structure.

There are also examples of symplectic manifolds that do not admit complex structures, though these are somewhat harder to explain, largely because it would require a lengthy digression to explain cases in which one can tell that a given smooth manifold does not support a complex structure.

One can likewise sharpen the requirements in the definitions of Riemannian metrics and of almost Hermitian structures. A manifold equipped with a Riemannian metric for which there are coordinate charts on which the metric coincides with the standard dot product is called a Euclidean manifold, or sometimes a Bieberbach manifold after the person who classified them, showing that they are quite constrained: any compact Euclidean manifold is the quotient of a torus by a finite group of isometries. As for variations on the definition of an almost Hermitian structure (g, ω, J), if one asked for coordinate charts in which all three of g, ω, and J were standard, this would lead to something even more restricted than a Euclidean manifold that as far as I know does not have a standard name. A **Kähler manifold** is by definition an almost Hermitian manifold (M, g, ω, J) with the properties both that (M, ω) is a symplectic manifold and that (M, J) is a complex manifold—so this requires the existence of charts in which ω is standard and charts in which J is standard. These need not be the same charts, so that there may not be charts in which the "Kähler metric" g is standard; indeed the study of possible curvatures of Kähler metrics on a given complex manifold has been a flourishing area for many decades, and if there were charts in which g was standard then the curvature would be zero. Kähler manifolds feature prominently in algebraic geometry over $\mathbb{C}$, since, as we will discuss later, a smooth algebraic variety embedded in $\mathbb{C}P^N$ has a natural Kähler structure. The cases intermediate between almost Hermitian and Kähler also have names: an almost Hermitian manifold (M, g, ω, J) is **Hermitian** if (M, J) is complex, and **almost Kähler** if (M, ω) is symplectic. By Theorem 3.4, the space of almost Kähler structures with ω equal to a given symplectic structure is contractible (and in particular nonempty). We will later see examples of symplectic structures not associated to any Kähler structure.

4. The Moser technique and its consequences

4.1. **Cartan's magic formula.** Before proving Darboux's theorem we will discuss a formula from the theory of differential forms that is useful both in the proof of Darboux's theorem and in many other aspects of symplectic geometry. This is a formula for the **Lie derivative** $L_V\omega$ of a differential form ω along a vector field V on a smooth manifold. To define this, first recall that a vector field V on a smooth manifold M give rise under suitable ("completeness") assumptions to a $\mathbb{R}$-parametrized family of diffeomorphisms $\psi^{V,t}\colon M \to M$, depending smoothly on t and characterized by the properties that, for $x \in M$ and $t \in \mathbb{R}$,

$$\psi^{V,0}(x) = x, \qquad \frac{d}{dt}\psi^{V,t}(x) = V(\psi^{V,t}(x)). \tag{17}$$

(cf. [L, Chapter 17], or Definition 1.6 when $M = \mathbb{R}^k$). Thus $t \mapsto \psi^{V,t}(x)$ is the unique solution to the ODE system determined by V that passes through x at time zero. In general one needs assumptions on V because of the possibility of solutions diverging in finite time, but without any assumptions on V one can at least say that if $K \subset U \subset M$ with K compact and U open there is $\epsilon > 0$ and a unique smooth family of maps $\psi^{V,t}\colon K \to U$ defined for $t \in (-\epsilon, \epsilon)$ that obeys (17). In particular we do not need a completeness assumption on V to make the following definition:

Definition 4.1. Let M be a smooth manifold, V a vector field on M, and $\omega \in \Omega^k(M)$. The **Lie derivative of ω along** V is the differential k-form $L_V\omega$ whose value at $x \in M$ is given by

$$(L_V\omega)_x = \left.\frac{d}{dt}\right|_{t=0} \left(\psi^{V,t*}\omega\right)_x.$$

In other words,

$$(L_V\omega)_x(v_1, \ldots, v_k) = \left.\frac{d}{dt}\right|_{t=0} \omega_{\psi^{V,t}(x)}\left(\psi^{V,t}_* v_1, \ldots, \psi^{V,t}_* v_k\right).$$

Proposition 4.2. *The maps $L_V\colon \Omega^k(M) \to \Omega^k(M)$ (as k varies) obey the following properties:*

(i) *For $k = 0$, and $f \in C^\infty(M) = \Omega^0(M)$, we have $L_V f = df(V)$.*

(ii) *If $\{\omega_\alpha\}$ is a locally finite[14] collection of differential k-forms, then $\{L_V\omega_\alpha\}$ is also locally finite and*
$$L_V\left(\sum_\alpha \omega_\alpha\right) = \sum_\alpha L_V\omega_\alpha.$$

(iii) *For all $\omega \in \Omega^k(M)$, $L_V(d\omega) = d(L_V\omega)$.*

(iv) *For $\omega \in \Omega^k(M), \theta \in \Omega^\ell(M)$, we have $L_V(\omega \wedge \theta) = (L_V\omega) \wedge \theta + \omega \wedge L_V\theta$.*

Moreover, if $L\colon \Omega^*(M) \to \Omega^*(M)$ is any map obeying (i)-(iv) above, then $L = L_V$.

Proof. For (i), we find by the chain rule

$$(L_V f)(x) = \left.\frac{d}{dt}\right|_{t=0} f(\psi^{V,t}(x)) = (df)_x\left(\left.\frac{d}{dt}\right|_{t=0} \psi^{V,t}(x)\right) = df_x(V)$$

for all x.

(ii) essentially follows from pullback being a linear map; to be a little careful about locally-finite-but-infinite sums, first note that if ω_α vanishes identically on a neighborhood U of a point x, then $(\omega_\alpha)_{\psi^{V,t}(x)}$ will be zero for all sufficiently small t, and so $L_V\omega_\alpha$ will vanish at x. This reasoning holds at all points of U, showing that if $\omega_\alpha|_U \equiv 0$ then $(L_V\omega_\alpha)|_U \equiv 0$, and in particular that the local finiteness of $\{\omega_\alpha\}$ implies that of $\{L_V\omega_\alpha\}$. The equation $L_V\left(\sum_\alpha \omega_\alpha\right) = \sum_\alpha L_V\omega_\alpha$ can then be checked on each member of a cover of M by open sets on which all but finitely many ω_α (and hence also $L_V\omega_\alpha$) vanish identically, and there the equation follows from linearity of pullbacks.

(iii) follows from the naturality of the exterior derivative d: we find

$$L_V(d\omega) = \left.\frac{d}{dt}\right|_{t=0} \psi^{V,t*}d\omega = \left.\frac{d}{dt}\right|_{t=0} d\psi^{V,t*}\omega = dL_V\omega.$$

[14] *i.e.*, each point has a neighborhood on which all but finitely many ω_α are identically zero

For (iv), for r,s close to zero let $\Phi(r,s)=\psi^{V,r*}\omega\wedge\psi^{V,s*}\theta$. Also define $\delta\colon \mathbb{R}\to\mathbb{R}^2$ by $\delta(t)=(t,t)$. Then using the chain rule and the fact that $\psi^{V,0}$ is the identity,

$$\begin{aligned} L_V(\omega\wedge\theta) &= \left.\frac{d}{dt}\right|_{t=0}\Phi(\delta(t)) = \left.\frac{\partial\Phi}{\partial r}\right|_{\delta(0)}\left.\frac{\partial r}{\partial t}\right|_0 + \left.\frac{\partial\Phi}{\partial s}\right|_{\delta(0)}\left.\frac{\partial s}{\partial t}\right|_0 \\ &= \left.\frac{d}{dr}\right|_{r=0}\psi^{V,r*}\omega\wedge\theta + \omega\wedge\left.\frac{d}{ds}\right|_{s=0}\psi^{V,s*}\theta = (L_V\omega)\wedge\theta+\omega\wedge(L_V\theta). \end{aligned}$$

The last statement of the proposition is based on the observation that any k-form ω on M can be written as a locally finite sum of the form

$$\omega=\sum_\alpha f_\alpha dg_{\alpha,1}\wedge\cdots\wedge dg_{\alpha,k}$$

for some smooth functions $f_\alpha, g_{\alpha,j}$. Indeed if M is covered by a single coordinate patch one can do this by taking the $g_{\alpha,j}$ to be coordinate functions x_{i_j}; in general one can patch together such local expressions for ω using cutoff functions and partitions of unity. Given this, if L satisfies properties (i)-(iv), then the values of $Lf_\alpha, Lg_{\alpha,j}$ are determined by (i); then the values of $Ldg_{\alpha,j}$ are determined by (iii); then the values of $f_\alpha dg_{\alpha,1}\wedge\cdots\wedge dg_{\alpha,k}$ are determined by induction and (iv); and finally the value of $L\omega$ is determined by (i). Thus any two maps obeying (i)-(iv) are equal. □

While the Lie derivative of a zero-form is readily given by item (i) of Proposition 4.2, it would seem to be harder to describe the Lie derivative of a higher-degree differential form. However the following makes this straightforward:

Theorem 4.3 (Cartan's magic formula). *For any vector field V on a smooth manifold M and any differential form ω we have*

$$L_V\omega = d\iota_V\omega+\iota_V d\omega.$$

Proof. We just need to check that defining $L\omega = d\iota_V\omega+\iota_V d\omega$ (for all ω) leads to an operator obeying (i)-(iv). Now if $f\in\Omega^0(M)$ then $\iota_V f=0$ (there are no nontrivial degree-(-1) differential forms), so we have $Lf=\iota_V df = df(V)$, confirming (i). Condition (ii) on compatibility with sums is trivial. For condition (iii), observe that, using twice that $d\circ d=0$,

$$d\circ L = d\circ(d\circ\iota_V+\iota_V\circ d) = d\circ\iota_V\circ d = (d\circ\iota_V+\iota_V\circ d)\circ d = L\circ d.$$

Finally for the Leibniz rule for L we require the Leibniz rule for the interior product ι_V, which says (similarly to the Leibniz rule for d) that, if $\omega\in\Omega^k(M)$, then $\iota_V(\omega\wedge\theta)=(\iota_V\omega)\wedge\theta+(-1)^k\omega\wedge(\iota_V\theta)$. So we get, if $\omega\in\Omega^k(M)$ and $\theta\in\Omega^\ell(M)$:

$$\begin{aligned} L(\omega\wedge\theta) &= d\left((\iota_V\omega)\wedge\theta+(-1)^k\omega\wedge(\iota_V\theta)\right)+\iota_V\left((d\omega)\wedge\theta+(-1)^k\omega\wedge d\theta\right) \\ &= (d\iota_V\omega)\wedge\theta+(-1)^{k-1}\iota_V\omega\wedge d\theta+(-1)^k d\omega\wedge\iota_V\theta+(-1)^{2k}\omega\wedge d\iota_V\theta \\ &\quad+(\iota_V d\omega)\wedge\theta+(-1)^{k+1}d\omega\wedge\iota_V\theta+(-1)^k\iota_V\omega\wedge d\theta+(-1)^{2k}\omega\wedge\iota_V d\theta \\ &= (L\omega)\wedge\theta+\omega\wedge L\theta, \end{aligned}$$

as desired. □

There are a couple of directions in which it is useful to modestly extend Theorem 4.3. First, instead of just computing $\frac{d}{dt}\psi^{V,t*}\omega$ at $t=0$, it is helpful to have a formula at other values of t. Second, we will often want to consider the flows of **time-dependent** vector fields $\mathbb{V}=(V_t)_{t\in\mathbb{R}}$ (so for each t V_t is a smooth vector field, and the dependence on t is also smooth). We will write these time-dependent flows as $\psi^{\mathbb{V},t}$, and they are characterized by the properties that $\psi^{\mathbb{V},0}=1_M$ and, for all $x\in M$, $\frac{d}{dt}(\psi^{\mathbb{V},t}(x))=V_t(\psi^{\mathbb{V},t}(x))$. The existence and uniqueness theory for time-independent vector fields V carries over without change to the time-dependent case (and the latter can even be formally dedeuced from the former by considering vector fields on $\mathbb{R} \times M$ rather than M): $\psi^{\mathbb{V},t}$ is unique and a smooth if it exists, and it exists for all t if the V_t are supported in a fixed compact subset; more generally for any neighborhood U of any compact set K there is $\epsilon_K > 0$ such that the restriction of $\psi^{\mathbb{V},t}$ to K is well-defined for $t \in (-\epsilon_K, \epsilon_K)$ as a map to U for all $t \in (-\epsilon_K, \epsilon_K)$. (In other words, for each $x \in K$ there is a (necessarily unique) solution $\gamma_x \colon (\epsilon_K \to \epsilon_K) \to U$ to the initial value problem $\gamma_x(0) = x$, $\dot{\gamma}_x(t) = V_t(\gamma_x(t))$; by definition, $\psi^{\mathbb{V},t}(x)$ is then equal to $\gamma_x(t)$.)

The following accomplishes these extensions simultaneously:

Proposition 4.4. *Let $\mathbb{V} = (V_t)_{t\in\mathbb{R}}$ be a time-dependent vector field on a smooth manifold M, and $\omega \in \Omega^k(M)$. Then, as functions of t,*

$$\frac{d}{dt}\psi^{\mathbb{V},t*}\omega = \psi^{\mathbb{V},t*}L_{V_t}\omega \tag{18}$$

at every point of M and every value of t at which $\psi^{\mathbb{V},t}$ is well-defined.

Proof. We use the same basic strategy as in the proof of Cartan's magic formula. For any fixed t and on any open subset of M where $\psi^{\mathbb{V},t}$ is well-defined, we first claim that both sides of (18), when considered as functions of ω, satisfy properties (ii), (iii), and (iv) in the list in Proposition 4.2. For property (ii) (additivity) this should be clear since $\psi^{\mathbb{V},t*}$ is a linear map (and can be evaluated locally, so the generalization from finite sums to locally finite sums poses no problem). Likewise the fact that $d \circ \psi^{\mathbb{V},t*} = \psi^{\mathbb{V},t*} \circ d$ implies that both sides of (18) obey property (iii) (compatibility with d). As for property (iv) (the Leibniz rule), that the right side of (18) satisfies it follows immediately from the same property for the Lie derivative L_{V_t}, together with the fact that the pullback $\psi^{\mathbb{V},t*}$ respects wedge products. That the left hand side obeys the Leibniz rule follows from the same argument as was used to prove the corresponding fact for the Lie derivative: given differential forms ω and θ, let $\Phi(r,s) = \psi^{\mathbb{V},r*}\omega \wedge \psi^{\mathbb{V},s*}\theta$ and use the multivariable chain rule to express the left-hand side of (18), which is $\frac{d}{dt}\Phi(t,t)$, as a sum of two terms.

As in the proof of Proposition 4.2, a general k-form ω can be written as a locally finite sum $\omega = \sum_\alpha f_\alpha dg_{\alpha,1} \wedge \cdots \wedge dg_{\alpha,k}$ for some smooth functions $f_\alpha, g_{\alpha,1}, \ldots, g_{\alpha,k}$. Consequently if one has a collection of maps $L \colon \Omega^k(M) \to \Omega^k(M)$, defined for all k, which obeys conditions (ii), (iii), and (iv), then repeated application of these properties shows that this collection is completely determined by how it acts on $\Omega^0(M)$. So it remains only to show that the two sides of (18) coincide when $\omega \in \Omega^0(M)$. In this case, evaluating the left-hand side at a point x yields

$$\frac{d}{dt}\left(\omega(\psi^{\mathbb{V},t}(x))\right) = (d\omega)_{\psi^{\mathbb{V},t}(x)}(V_t)$$

and, using Theorem 4.3, evaluating the right-hand side at x yields

$$(\iota_{V_t}d\omega)_{\psi^{\mathbb{V},t}(x)} = (d\omega)_{\psi^{\mathbb{V},t}(x)}(V_t).$$

So the two sides of (18) coincide when ω is a 0-form, and hence (using conditions (ii),(iii),(iv)) when ω is any differential form. $\square$

Exercise 4.5. *Suppose that $\lambda \in \Omega^k(M)$ and that V is a vector field on M with the property that $\iota_V d\lambda = \lambda$. Prove that $\psi^{V,t*}d\lambda = e^t d\lambda$ for all t such that $\psi^{V,t}$ is defined.*

Exercise 4.6. *Suppose that $\mathbb{V} = (V_t)_{t\in[0,1]}$ is a time-dependent vector field such that the time-one flow $\psi^{\mathbb{V},1}$ is defined on all of M. Construct another time-dependent vector field $\mathbb{W} = (W_t)_{t\in[0,1]}$ such that the time-one flow $\psi^{\mathbb{W},1}$ is equal to $(\psi^{\mathbb{V},1})^{-1}$. **Suggestion**: Arrange that, whenever $\gamma \colon [0,1] \to M$ obeys $\dot{\gamma}(t) = V_t(\gamma(t))$, the "reversed" curve $\bar{\gamma}(t) = \gamma(1-t)$ obeys $\dot{\bar{\gamma}}(t) = W_t(\bar{\gamma}(t))$.*

4.2. **The Moser trick.** The formulas of the preceding section allow one to prove a variety of "stability" results in symplectic geometry, in which an apparent change to an object defined on a manifold does not alter the underlying structure because the change can be accounted for by a diffeomorphism of the manifold. Darboux's theorem, stating that any closed non-degenerate two-form is locally equivalent by a diffeomorphism to the standard symplectic form ω_0, will be one example of this.

Suppose we have two two-forms $\omega_0, \omega_1 \in \Omega^2(M)$ which are both closed and non-degenerate; let's make the additional assumption that all of the forms $\omega_t := (1-t)\omega_0 + t\omega_1$ for $t \in [0,1]$ (so the ω_t linearly interpolate between ω_0 and ω_1) are also non-degenerate. We will find soon that, under appropriate hypotheses, it is possible to find a smooth family of diffeomorphisms $\psi_t \colon M \to M$ such that $\psi_0 = 1_M$ and $\psi_t^*\omega_t = \omega_0$. So intuitively speaking, all of the forms become equivalent after relabeling the points of M appropriately.

I'll first point out a hypothesis that will certainly be needed to make this work. The forms ω_t are all closed, so the represent classes $[\omega_t]$ in the de Rham cohomology which we identify with $H^2(M;\mathbb{R})$. If the ψ_t described in the previous paragraph existed, they would act on $H^2(M;\mathbb{R})$ (by $[\eta] \mapsto [\psi_t^*\eta]$ for any closed $\eta \in \Omega^2(M)$), and by homotopy invariance of cohomology this action will be the same for all t. But then since ψ_0 is the identity this action is the identity for all t, so for all closed $\eta \in \Omega^2(M)$ we'd have $[\psi_t^*\eta] = [\eta]$. Applying this with $\eta = \omega_t$, which are supposed to satisfy $\psi_t^*\omega_t = \omega_0$, we get $[\omega_0] = [\psi_t^*\omega_t] = [\omega_t]$. In particular for $t = 1$ we see that $[\omega_1] = [\omega_0]$, *i.e.* that the form $\omega_1 - \omega_0$ must be *exact*, say $\omega_1 - \omega_0 = d\alpha$ where $\alpha \in \Omega^1(M)$. Conversely if $\omega_1 - \omega_0 = d\alpha$ we see that $\omega_t = \omega_0 + t d\alpha$ so $[\omega_t] = [\omega_0]$ for all t.

So we impose this hypothesis from now on, and ask, supposing that the family of closed two-forms

$$\omega_t = \omega_0 + t d\alpha \quad (t \in [0,1])$$

is non-degenerate for every t, whether there is a smooth family of diffeomorphisms $\psi_t \colon M \to M$ with $\psi_0 = 1_M$ and $\psi_t^*\omega_t = \omega_0$ for all t. If M is compact, we will see that the answer is automatically yes.

Whether or not M is compact, the strategy for constructing ψ_t will be to posit[15] that it is the flow $\psi^{\mathbb{V},t}$ of a time-dependent vector field $\mathbb{V} = (V_t)_{t\in[0,1]}$ and then use the results of the previous section to find conditions on $\mathbb{V}$ that will ensure that $\psi^{\mathbb{V},t*}\omega_t = \omega_0$. Compactness of M will simplify matters somewhat in that it will mean that the flow of $\mathbb{V}$ exists for all time independently of what $\mathbb{V}$ we choose, but in some cases (like the proof of Darboux's theorem) we won't want to work on a compact manifold and then we'll have to be more careful about ODE existence questions.

Since $\psi^{\mathbb{V},0}$ is by definition the identity, we will have $\psi^{\mathbb{V},t*}\omega_t = \omega_0$ for all t if and only if $\frac{d}{dt}\left(\psi^{\mathbb{V},t*}\omega_t\right) = 0$. But Proposition 4.4 and Theorem 4.3 essentially tell us how to compute $\frac{d}{dt}\left(\psi^{\mathbb{V},t*}\omega_t\right)$: whenever the expressions below are well-defined, we have[16]

$$\begin{aligned}\frac{d}{dt}\left(\psi^{\mathbb{V},t*}\omega_t\right) &= \psi^{\mathbb{V},t*}L_{V_t}\omega_t + \psi^{\mathbb{V},t*}\frac{d\omega_t}{dt} = \psi^{\mathbb{V},t*}\left(d\iota_{V_t}\omega_t + \iota_{V_t}d\omega_t + \alpha\right) \\ &= \psi^{\mathbb{V},t*}d\left(\iota_{V_t}\omega_t + \alpha\right).\end{aligned}$$

Note that we have used the fact that $d\omega_t = 0$ to eliminate one of the terms. Now due to the assumption that the ω_t are all non-degenerate, for all t the equation $\iota_{V_t}\omega_t = \alpha$ can be solved uniquely for the vector field V_t (pointwise, in terms of earlier notation, we have $V_t(x) = \theta^{-1}_{(\omega_t)_x}(\alpha_x)$).

[15]Actually, though we won't need this, *any* smooth family of diffeomorphisms starting at the identity is the flow of some time-dependent vector field, so this isn't a restrictive assumption.

[16]One can justify the first equality by the usual technique of considering the function of two variables $(r,s) \mapsto \psi^{\mathbb{V},r*}\omega_s$ and applying the multivariable chain rule.

So we have found a time-dependent vector field $\mathbb{V} = (V_t)_{t\in[0,1]}$ which, modulo questions about whether and where the flow of $\mathbb{V}$ exists, has the property that $\frac{d}{dt}\psi^{\mathbb{V},t*}\omega_t = 0$. We summarize this as follows:

Proposition 4.7. *Let M be a smooth manifold, $\omega_0 \in \Omega^2(M)$ a closed two-form, and $\alpha \in \Omega^1(M)$ be such that, for all $t \in [0,1]$, the form $\omega_t = \omega_0 + td\alpha$ is non-degenerate. Let $\mathbb{V} = (V_t)_{t\in[0,1]}$ be the unique time-dependent vector field obeying the equation $\iota_{V_t}\omega_t = -\alpha$ for all t, and let $U \subset M$ be the set on which the flow $\psi^{\mathbb{V},t}$ is well-defined (as a map to M) for all $t \in [0,1]$. Then $\left(\psi^{\mathbb{V},t*}\omega_t\right)\big|_U = \omega_0|_U$ for all $t \in [0,1]$.*

The set U in Proposition 4.7 consists of those $x \in M$ with the property that the initial value problem $\begin{cases} \dot{\gamma}(t) = V_t(\gamma(t)) \\ \gamma(0) = x \end{cases}$ has a solution $\gamma\colon [0,1] \to M$. In general the most that can be said about U is that it is an open subset[17] (possibly the empty set), and that $\psi^{\mathbb{V},1}$ then defines a diffeomorphism from U to its image (which is also an open subset of M). However if the V_t are all supported in a fixed compact subset then $U = M$ and the $\psi^{\mathbb{V},t}$ are diffeomorphisms of M, leading to the following corollary:

Corollary 4.8. *Let M be a smooth manifold, $\omega_0 \in \Omega^2(M)$ a closed two-form, and $\alpha \in \Omega^1(M)$ be such that, for all $t \in [0,1]$, the form $\omega_t = \omega_0 + td\alpha$ is non-degenerate. Assume moreover that α has compact support. Then there is a smooth path of diffeomorphisms $\psi_t\colon M \to M$ such that $\psi_0 = 1_M$ and $\psi_t^*\omega_t = \omega_0$ for all t.*

Proof. The vector field V_t in Proposition 4.7 is characterized uniquely by the property that $\iota_{V_t}\omega_t = -\alpha$, V_t vanishes at any given point if and only if α vanishes there. Thus the V_t have the same, compact, support as α, and so their flow is defined on all of M. So the result holds with $\psi_t = \psi^{\mathbb{V},t}$. □

This leads quickly to a classification of volume forms on closed surfaces:

Proposition 4.9. *Let M be a connected, oriented, compact surface without boundary, and let $\omega_0, \omega_1 \in \Omega^2(M)$ be nowhere-vanishing. Then there is an orientation-preserving diffeomorphism $\psi\colon M \to M$ such that $\psi^*\omega_1 = \omega_0$ **if and only if** $\int_M \omega_0 = \int_M \omega_1$.*

Proof. The forward implication follows directly from the change-of-variables formula, which assuming that $\psi^*\omega_1 = \omega_0$ with ψ an orientation-preserving diffeomorphism yields that

$$\int_M \omega_0 = \int_M \psi^*\omega_1 = \int_{\psi(M)} \omega_1 = \int_M \omega_1.$$

For the converse, first note that ω_0 and ω_1 are closed (since their exterior derivatives lie in $\Omega^3(M)$ which is $\{0\}$ since $\dim M = 2$). Recall that the map $[\omega] \mapsto \int_M \omega$ (which is well-defined by Stokes' theorem) is an isomorphism $H^2(M;\mathbb{R}) \cong \mathbb{R}$. Thus the assumption that $\int_M \omega_0 = \int_M \omega_1$ implies that $[\omega_1 - \omega_0] = 0 \in H^2(M;\mathbb{R})$, and hence that there is $\alpha \in \Omega^1(M)$ such that $\omega_1 = \omega_0 + d\alpha$. So the result will follow immediately from Corollary 4.8 as soon as we show that the two-forms $\omega_t = \omega_0 + td\alpha = (1-t)\omega_0 + t\omega_1$ are non-degenerate for each $t \in [0,1]$. Now ω_0 and ω_1 are nowhere-vanishing, so they each define an orientation on M (declare a basis $\{e_1, e_2\}$ for any T_xM to be positive iff $\omega_0(e_1,e_2)$ or $\omega_1(e_1,e_2)$ is positive). The orientations defined by ω_0 and ω_1 must be the same, because $\int_M \omega_0$ and $\int_M \omega_1$ have the same sign. If $\{e_1, e_2\}$ is an oriented basis for T_xM with respect to this common orientation induced by both ω_0 and ω_1, then for all $t \in [0,1]$ we will

[17] It's perhaps worth emphasizing that when I say that M is a smooth manifold I mean a smooth manifold without boundary; if ∂M were nonempty than U might not be open, as you should be able to convince yourself.

have $\omega_t(e_1,e_2) = (1-t)\omega_0(e_1,e_2) + t\omega_1(e_1,e_2) > 0$ (as both summands are nonnegative, and at least one is positive for every t). This shows that ω_t does not vanish at any point, and hence that it is non-degenerate since its rank, being even and nonzero, must be two. So the hypotheses of Corollary 4.8 are satisfied, yielding a smooth path of diffeomorphisms $\psi_t\colon M \to M$ with $\psi_0 = 1_M$ and $\psi_t^*\omega_t = \omega_0$. So the result holds with $\psi = \psi_1$ (which is orientation-preserving since it is homotopic to the orientation-preserving diffeomorphism ψ_0). □

To interpret Proposition 4.9, note that giving a nowhere-vanishing two-form ω on a surface M is basically the same as giving an orientation of M together with a (smooth, in an appropriate sense) measure μ_ω on M defined by $\mu_\omega(E) = \int_E \omega$ (where the orientation induced by ω is used to determine the sign of the integral, so that $\mu_\omega(E) \geq 0$). The change of variables formula then says that, for a diffeomorphism ψ, we have $\mu_{\psi^*\omega}(E) = \mu_\omega(\psi(E))$, so the statement that $\psi^*\omega_1 = \omega_0$ is equivalent to the statement that $\mu_{\omega_0}(E) = \mu_{\omega_1}(\psi(E))$. So Corollary 4.9 says that (under its hypotheses) we can find a diffeomorphism $\psi\colon M \to M$ such that $\mu_{\omega_0}(E) = \mu_{\omega_1}(\psi(E))$ for **all** measurable sets E as soon as one satisfies the obviously-necessary condition that $\mu_{\omega_0}(M) = \mu_{\omega_1}(M)$. (No reference is made to orientations in the previous sentence since, if we are only concerned with measures, if necessary we can replace ω_0 by $-\omega_0$ since these determine the same measure, and we can apply Corollary 4.9 to ω_1 and whichever of ω_0 and $-\omega_0$ induces the same orientation as ω_1.)

To prove results like Darboux's theorem we will need to apply Proposition 4.7 in cases where M is noncompact, leading to concerns about whether the flow $\psi^{\mathbb{V},t}$ exists for all $t \in [0,1]$. Our general aim will be to arrange for the flow to exist at least on a small open set around some prescribed subset $K \subset M$ (in the case of Darboux's theorem K will be a singleton). Here is a general statement giving conditions for this to work:

Proposition 4.10 (Relative Moser Stability I). *Let M be a smooth manifold equipped with closed, non-degenerate forms $\omega_0, \omega_1 \in \Omega^2(M)$, and let $K \subset M$ be a closed subset such that:*

- (i) *For all $x \in K$, $(\omega_0)_x$ and $(\omega_1)_x$ are equal as bilinear forms on T_xM.*
- (ii) *There is a neighborhood U of K and $\alpha \in \Omega^1(U)$ such that $\omega_1 - \omega_0 = d\alpha$ everywhere on U, and such that $\alpha_x = 0$ (as a linear functional on T_xM) for all $x \in K$.*

Then there are neighborhoods U_0 and U_1 of K and a diffeomorphism $\psi\colon U_0 \to U_1$ such that $\psi|_K = 1_K$ and $\psi^*\omega_1 = \omega_0$.

Remark 4.11. As we will explain later (see Corollary 4.15), in the case that K is a smooth compact submanifold of M (as will be true in all of our applications), condition (i) implies condition (ii), so one only actually needs to check condition (i).

Proof. As usual let $\omega_t = (1-t)\omega_0 + t\omega_1$. Consider the set

$$S = \{(t,x) \in [0,1] \times M | (\omega_t)_x \text{ is non-degenerate}\}.$$

Since in local coordinates around any $x_0 \in M$ the linear maps $\theta_{(\omega_t)_x}$ are represented by skew-symmetric matrices $M(t,x)$ that vary smoothly with t and x, with the condition that $(\omega_t)_x$ be non-degenerate equivalent to the condition that this matrix be invertible, and since the set of invertible matrices is open, we see that S (being locally given as the preimage of the set of invertible matrices under $(t,x) \mapsto M(t,x)$) is likewise **open**. Moreover if $x \in K$ then $(\omega_t)_x = (\omega_0)_x$ for all t (by condition (i) in the statement of the proposition), and $(\omega_0)_x$ is non-degenerate, so our set S contains $[0,1] \times K$. But then by the Tube Lemma S likewise contains $[0,1] \times U'$ for some neighborhood U' of K. Intersecting with U if necessary, we may as well assume that U' is contained in the subset U of condition (ii) in the statement of the proposition.

Thus on U' we have forms $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + t\,d\alpha$, each of which is closed and non-degenerate. Proposition 4.7 then says that, defining V_t by $\iota_{V_t}\omega_t = -\alpha$, the flow $\psi^{\mathbb{V},t}$ will pull back

ω_t to ω_0 wherever this flow is defined. Condition (ii) says that $\alpha_x = 0$ for all $x \in K$, and hence that $V_t(x) = 0$ for all $x \in K$. So this flow *is* defined on K, and is equal to the identity there, since if $x \in K$ then the constant map $\gamma(t) = x$ will obey $\gamma(0) = x$ and $\dot{\gamma}(t) = V_t(\gamma(t))$. If we let U_0 be the subset of U' on which $\psi^{\mathbb{V},t}$ is defined (as a map to U') for all $t \in [0,1]$, then U_0 will thus be an open neighborhood of K and the proposition will hold with $\psi = \psi^{\mathbb{V},1}$ and $U_1 = \psi(U_0)$. □

We are now positioned to prove Darboux's theorem; the main point is to apply relative Moser stability to the case that $K = \{\vec{0}\} \subset \mathbb{R}^{2n}$. In fact in this case we don't need to formally assume all of the hypotheses of Proposition 4.10:

Proposition 4.12. *Let $U \subset \mathbb{R}^{2n}$ be an open set containing $\vec{0}$, and let $\omega_0, \omega_1 \in \Omega^2(U)$ both be closed and non-degenerate. Then there are neighborhoods $U_0, U_1 \subset U$ of $\vec{0}$ and a diffeomorphism $\psi\colon U_0 \to U_1$ such that $\psi(\vec{0}) = \vec{0}$ and $\psi^*\omega_1 = \omega_0$.*

Proof. First we reduce to the case that condition (i) in Proposition 4.10 holds: by Corollary 2.6 there is $A \in GL(2n,\mathbb{R})$ such that $A^*\left((\omega_1)_{\vec{0}}\right) = (\omega_0)_{\vec{0}}$ (as bilinear forms on $T_{\vec{0}}\mathbb{R}^{2n} = \mathbb{R}^{2n}$). Thus setting $K = \{\vec{0}\}$, the forms ω_0 and $A^*\omega_1$ obey condition (i) in Proposition 4.10. If we find $\tilde{\psi}$ with $\tilde{\psi}(\vec{0}) = 0$ and $\tilde{\psi}^*(A^*\omega_1) = \omega_0$ (on some neighborhood of $\vec{0}$) then the map $\psi = A \circ \tilde{\psi}$ will have $\psi(\vec{0}) = \vec{0}$ and $\psi^*\omega_1 = \omega_0$ (on some neighborhood of $\vec{0}$). This justifies replacing ω_1 by $A^*\omega_1$, reducing us to the case that $(\omega_1)_{\vec{0}} = (\omega_0)_{\vec{0}}$.

We then just need to check that condition (ii) in Proposition 4.10 holds (which we'll do without appealing to Remark 4.11 since we're in an easy case). Now a ball B aroud 0 has $H^2(B;\mathbb{R}) = \{0\}$, so at least upon restricting to this ball ω_1 and ω_0 are cohomologous, say $\omega_1 - \omega_0 = d\alpha_0$. This α_0 can be written in coordinates as $(\alpha_0)_x = \sum_{i=1}^{2n} f_i(x)dx_i \wedge dx_j$ for some smooth functions $f_i(x)$; if we define a new one-form α by $\alpha_x = \sum_{i=1}^{2n}(f_i(x) - f_i(\vec{0})dx_i$ then we will have $d\alpha = d\alpha_0 = \omega_1 - \omega_0$ and $\alpha_{\vec{0}} = 0$. The result now immediately follows by applying Proposition 4.10 on a small ball B to the set $K = \{\vec{0}\}$. □

End of the proof of Darboux's theorem (Theorem 3.7). We are to show that, if $\omega \in \Omega^2(M)$ is closed and non-degenerate and $x \in M$, then there is a coordinate chart $\phi\colon U \to \phi(U) \subset \mathbb{R}^{2n}$ with $x \in U$ such that $\phi^*\omega_0 = \omega$, where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$ (and we suppress notation for restrictions). To do this, first choose an arbitrary smooth coordinate chart $f\colon V \to f(V) \subset \mathbb{R}^{2n}$ with $x \in V$ and $f(x) = \vec{0}$. Then $f(V)$ is a neighborhood of the origin, and we obtain a closed, non-degenerate form $\omega_1 = (f^{-1})^*\omega \in \Omega^2(f(V))$. We can then apply Proposition 4.12 to get neighborhoods $U_0, U_1 \subset f(V)$ of $\vec{0}$ and a diffeomorphism $\psi\colon U_0 \to U_1$ with $\psi^* f^{-1*}\omega = \omega_0$, and hence $(\psi^{-1} \circ f)^*\omega_0 = \omega$. So the result holds with $\phi = \psi^{-1} \circ f$, which defines a coordinate chart on the neighborhood $f^{-1}(U_1) = f^{-1}(\psi(U_0))$ of x in M. □

Having proven Darboux's theorem, we are entitled to refer to closed and non-degenerate two-froms on smooth manifolds as "symplectic forms," understanding this term in the strong sense that around each point there should be local coordinates in which the form is identified with the standard symplectic form $\sum_j dp_j \wedge dq_j$.)

4.3. **Tubular neighborhood theorems.** Relative Moser stability can be used to show that a closed non-degenerate two-form fits into a standard model in neighborhoods of various types of submanifolds; the Darboux theorem is then the special case when the submanifold is a point but other cases are important, especially the case where the submanifold is Lagrangian. Before explaining this we should recall what kind of model there is for the neighborhood of a submanifold just at the level of differential topology.

Let P be a smooth k-dimensional submanifold of a smooth m-dimensional manifold M. Then at each $x \in P$ we have an inclusion of vector spaces $T_xP \leq T_xM$. As a set, we define the *normal bundle* $\nu_M P$ to P in M to consist of pairs $(x, [v])$ where $[v] \in \frac{T_xM}{T_xP}$. Using charts for M that are adapted to P (*i.e.*, P intersects each chart in an open subset of $\mathbb{R}^k \times \{\vec{0}\}$) it is not hard to put smooth coordinate charts on $\nu_M P$, making it a smooth manifold, with the projection $\pi\colon (x, v) \mapsto x$ being a smooth map. Indeed, if constructed appropriately, these charts make $\nu_M P$ into a vector bundle over P, *i.e.* (in addition to the fact that they form a smooth atlas for $\nu_M P$) each of the charts has domain equal to $\pi^{-1}(U)$ for some $U \subset P$ open, and, for some coordinate chart $\phi\colon U \to \mathbb{R}^k$ for P, sends $\pi^{-1}(U)$ to $\phi(U) \times \mathbb{R}^{m-k}$ by a map which restricts to each $\pi^{-1}(\{x\}) = \frac{T_xM}{T_xP}$ as a linear isomorphism to $\{\phi(x)\} \times \mathbb{R}^{m-k}$. See, *e.g.*, [L, Chapter 5] for an introduction to vector bundles.

It is perhaps more intuitive to regard elements of $\nu_M P$ as consisting of actual tangent vectors to M rather than elements of the quotient space $\frac{T_xM}{T_xP}$. One can do this, non-canonically, by choosing a Riemannian metric g for M, and forming the space $\nu_M^g P$ consisting of pairs (x, v) with $x \in P$ and v in the g-orthogonal complement $T_xP^{\perp_g}$ of T_xP in T_xM. Since the quotient projection $T_xM \to T_xM/T_xP$ restricts to $T_xP^{\perp_g}$ as an isomorphism, the map $(x, v) \mapsto (x, [v])$ defines an isomorphism of vector bundles (in particular, a diffeomorphism) between $\nu_M^g P$ and $\nu_M P$.

Now a copy of P exists naturally in any vector bundle E over P: if $\pi\colon E \to P$ is the bundle projection, so that for each $x \in P$ the fiber $E_x := \pi^{-1}(\{x\})$ is a vector space, the *zero section* consisting of the zero elements of the various vector spaces E_x is a smooth submanifold of E that is diffeomorphic to P. I will continue to denote this zero section as P (rather than some other symbol like 0_E). If $x \in P$, note that we have two distinguished transverse, complementary-dimensional submanifolds of E passing through x, namely P and the fiber E_x (in which x appears as the zero element), so $T_xE = T_xP \oplus T_0E_x$, *i.e.*, since the tangent space to a vector space such as E_x at any point is equal to the vector space itself, $T_xE = T_xP \oplus E_x$. In the special case that $E = \nu_M^g P$, so that $E_x = T_xP^{\perp_g}$, we thus have $T_x \nu_M^g P = T_xP \oplus T_xP^{\perp_g} = T_xM$ at all points x in the zero section.

The *tubular neighborhood theorem* asserts that this is a universal local model for how a smooth manifold P appears as a submanifold of other smooth manifolds. I'll quote the following version that is a little more specific than what one sometimes sees:

Theorem 4.13 (Tubular Neighborhood Theorem, p. 346, Theorem 20 and its proof in [S])**.** *Let P be a smooth compact submanifold of a smooth manifold M, and choose a Riemannian metric g on M. For each $\epsilon > 0$ let $U_\epsilon = \{(x, v) \in \nu_M^g P | g(v, v) < \epsilon^2\}$. Then there is a neighborhood U of P in M, a value $\epsilon > 0$, and a diffeomorphism $\Phi\colon U \to U_\epsilon$ which restricts to P as the identity map to the zero section, and whose derivative at any $x \in P$ is the identity map from $T_xU = T_xM$ to $T_x \nu_M^g P = T_xP \oplus T_xP^{\perp_g} = T_xM$.*

Now the set $U_\epsilon \subset \nu_M^g P$ in Theorem 4.13 obviously deformation retracts to the zero section by the homotopy $r_t(x, v) = (x, tv)$, so by applying the diffeomorphism in Theorem 4.13 we immediately obtain the important fact that a smooth compact submanifold P always has a neighborhood U which deformation retracts to it. In particular the map $H^k(U; \mathbb{R}) \to H^k(P; \mathbb{R})$ induced by restriction of differential forms is an isomorphism. Considering how to see this at the level of differential forms rather than their cohomology classes leads to the following, which justifies Remark 4.11 about the relation between the two conditions in Proposition 4.10.

Proposition 4.14. *Let P be a compact smooth submanifold of a smooth manifold M, and suppose that $\eta \in \Omega^k(M)$ has the properties that $d\eta = 0$ and that $\eta_x = 0$ for all $x \in P$. Then there is a neighborhood U of P, and a differential form $\alpha \in \Omega^{k-1}(U)$ such that $d\alpha = \eta$ and $\alpha_x = 0$ for all $x \in P$.*

Proof. By Theorem 4.13 it's enough to show this when $M = U_\epsilon \subset \nu_M^g P$. For $t \in [0, 1]$ define $r_t\colon U_\epsilon \to U_\epsilon$ by $r_t(x, v) = (x, tv)$. Thus the r_t with $t > 0$ are all diffeomorphisms with r_1 equal to the identity; on the other hand r_0 has image P, on which $\eta \equiv 0$, so $r_0^*\eta = 0$. Thus we have

$$\eta = \int_0^1 \frac{d}{dt} r_t^*\eta dt,$$

and the plan is to compute $\frac{d}{dt} r_t^*\eta$.

Before doing this, let us note that, on any vector bundle E over P, there is a *tautological vector field T* whose value at a point v in the fiber E_x over $x \in P$ is the element v itself, where we are using the canonical identification of E_x with the subspace T_vE_x of T_vE. This tautological vector field (restricted to U_ϵ) is related to the homotopy r_t as follows:

$$\frac{d}{dt} r_t(x,v) = \frac{d}{dt}(x,tv) = v = \frac{1}{t} T(r_t(x,v)) \text{ for } t \neq 0.$$

So for any $\epsilon > 0$, the restriction of the isotopy r_t to the time interval $[\epsilon, 1]$ is generated by the time-dependent vector field $(\frac{1}{t}T)_{t\in[\epsilon,1]}$. (It's not a coincidence that this does not extend to $t = 0$, since r_0 isn't a diffeomorphism.) So Theorem 4.3 and Proposition 4.4 and the assumption that $d\eta = 0$ show that, for $t > 0$,

$$\frac{d}{dt} r_t^*\eta = r_t^* L_{\frac{1}{t}T}\eta = r_t^* d\iota_{\frac{1}{t}T}\eta = d\left(r_t^*\iota_{\frac{1}{t}T}\eta\right).$$

Now for $w_1, \ldots, w_{k-1} \in T_{(x,v)}U_\epsilon$ we see that, for $t > 0$,

$$\begin{aligned}\left(r_t^*\iota_{\frac{1}{t}T}\eta\right)_{(x,v)}(w_1,\ldots,w_{k-1}) &= (\iota_{\frac{1}{t}T}\eta)_{(x,tv)}(r_{t*}w_1,\ldots,r_{t*}w_{k-1}) \\ &= \eta_{(x,tv)}(v, r_{t*}w_1,\ldots,r_{t*}w_{k-1}).\end{aligned}$$

Note that the right-hand side is bounded (indeed converges to zero, since η vanishes along P) as $t \to 0$. This shows that if, for all $t \in [0,1]$, we define $\alpha_t \in \Omega^{k-1}(U_\epsilon)$ by

$$(\alpha_t)_{(x,v)}(w_1,\ldots,w_{k-1}) = \eta_{(x,tv)}(v, r_{t*}w_1,\ldots,r_{t*}w_{k-1}),$$

then for all $t > 0$ we have $\frac{d}{dt} r_t(x,v) = d\alpha_t$. But then by continuity considerations this equality must hold at $t = 0$ as well.

So we obtain (by the Leibniz rule for differentiating under integral signs)

$$d\left(\int_0^1 \alpha_t dt\right) = \int_0^1 \frac{d}{dt} r_t^*\eta dt = \eta$$

and so the result holds with $\alpha = \int_0^1 \alpha_t dt$, noting that $\alpha_{(x,0)} = 0$ for all $x \in P$ as a direct consequence of the definition of α and the fact that $\eta_{(x,0)} = 0$ for all $x \in P$. □

Together with what we have already done, this immediately gives a more straightforward version of relative Moser stability (proven by Weinstein in 1971) in the case that the set K is a smooth compact submanifold.

Corollary 4.15 (Relative Moser Stability II). *Let M be a smooth manifold with compact smooth submanifold P, and let ω_0, ω_1 be two closed non-degenerate two-forms on M such that, for all $x \in P$, $(\omega_0)_x = (\omega_1)_x$ are equal as bilinear forms on T_xM. Then there are neighborhoods U_0, U_1 of P and a diffeomorphism $\psi\colon U_0 \to U_1$ such that $\psi|_P = 1_P$ and $\psi^*\omega_1 = \omega_0$.*

Proof. Referring to Proposition 4.10, we are assuming that the set K in its statement is a compact submanifold, and that assumption (i) in its statement holds. We can apply Proposition 4.14 with $\eta = \omega_1 - \omega_0$ to deduce that assumption (ii) in Proposition 4.10 also holds, and hence that its conclusion does as well. □

We now turn to symplectic versions of the tubular neighborhood theorem. The objective will be to prove a theorem stating that, given data consisting of symplectic manifolds $(M,\omega),(M',\omega')$ and compact submanifolds $P\subset M, P'\subset M'$ obeying some hypotheses, that there are neighborhoods U of P in M and U' of P' in M' and a symplectomorphism[18] $F\colon U\to U'$ such that $F(P)=P'$. Whatever the hypotheses are, they should concern information that would be accessible to observers in the submanifolds P and P', respectively, who are capable of looking toward the rest of their ambient manifolds (in arbitrary directions in T_xM for $x\in P$, or T_xM' for $x\in P'$) but who are not able to leave P or P'. After formulating and proving a general result we will see what it implies in cases where P,P' are particular kinds of submanifolds, *e.g.* Lagrangian or symplectic, but for now we will not impose any such hypothesis.

If a symplectomorphism F as in the previous paragraph is to exist, then certainly it must restrict as a diffeomorphism $f\colon P\to P'$, and taking its derivative at any $x\in P$ would give a *linear* symplectomorphism $\hat{f}_x\colon T_xM\to T_{f(x)}M'$ whose restriction to the subspace T_xP coincides with the derivative $(f_*)_x$. Thanks to Corollary 4.15, the data of f and $\hat{f}=(f_x)_{x\in P}$ turn out to suffice to produce the desired map F. To explain some notation in the following, $TM|_P$ (and similarly for $TM'|_{P'}$) means the restriction of the tangent bundle of M to P, *i.e.* the vector bundle over P whose fiber at $x\in P$ is the tangent space T_xM. Evidently the restrictions of ω to the various T_xM for $x\in P$ make $TM|_P$ into a *symplectic vector bundle*, *i.e.* a vector bundle together with a family of *linear* symplectic forms on the various fibers that vary smoothly with $x\in P$ when expressed in terms of local trivializations.

Theorem 4.16 (General Weinstein Neighborhood Theorem). *Let $(M,\omega),(M',\omega')$ be symplectic manifolds with smooth compact submanifolds $P\subset M, P'\subset M'$. Assume that there is a commutative diagram*

$$\begin{tikzcd} TM|_P \arrow[r, "\hat{f}"] \arrow[d] & TM'|_{P'} \arrow[d] \\ P \arrow[r, "f"] & P' \end{tikzcd}$$

where f is a diffeomorphism, $\hat{f}$ is an isomorphism of symplectic vector bundles[19]*, and $\hat{f}|_{T_xP}=(f_*)_x$ for all $x\in P$. Then there are neighborhoods $U\subset M, U'\subset M'$ of P and P', respectively, and a diffeomorphism $F\colon U\to U'$ such that $F^*\omega'=\omega$ and $F|_P=f$.*

Proof. Choose an arbitrary Riemannian metric g on M, giving rise to the normal bundle ν_M^gP. Since, for each $x\in P$, $\hat{f}$ restricts to a linear isomorphism $T_xM\to T_{f(x)}M'$ that sends T_xP to $T_{f(x)}P'$, applying $\hat{f}$ to both sides of the direct sum decomposition $TM|_P=TP\oplus\nu_M^gP$ gives a similar direct sum decomposition $TM'|_{P'}=TP'\oplus\hat{f}(\nu_M^gP)$. Since each fiber $\hat{f}(\nu_M^gP_x)$ is complementary to $T_{f(x)}P'$ in $T_{f(x)}M'$, we can use partitions of unity to construct a Riemannian metric g' on M' such that the *orthogonal* complement $\nu_{M'}^{g'}P'$ to TP' in $TM'|_{P'}$ is equal to $\hat{f}(\nu_M^gP)$. In other words we are constructing g' such that $\hat{f}$ restricts to ν_M^gP as a map to $\nu_{M'}^{g'}P'$.

Using Theorem 4.13 for the outer two squares and the hypothesis for the middle square, we then have, for appropriate neighborhoods $U^{\nu_M},U^{\nu_{M'}}$ of the zero sections in the normal bundles

[18]*i.e.*, a diffeomorphism satisfying $\psi^*\omega'=\omega$

[19]*i.e.* its restriction to each T_xM is an isomorphism to $T_{f(x)}M'$, with $\omega'(\hat{f}v,\hat{f}(w))=\omega(v,w)$ for all $v,w\in T_xP$

$\nu_M^g P$, $\nu_{M'}^{g'} P'$, and neighborhoods U_0 of P, U_1 of P', commutative diagrams

$$\begin{tikzcd} U_0 \arrow[r] & U^{\nu_M} \arrow[r, "\hat{f}"] \arrow[d] & U^{\nu_{M'}} \arrow[r] \arrow[d] & U_1 \\ P \arrow[u, hook] \arrow[r, equal] & P \arrow[r, "f"] & P' \arrow[r, equal] & P' \arrow[u, hook] \end{tikzcd} \tag{19}$$

where the outer vertical arrows are inclusions, all maps in the top row are diffeomorphisms, and (due to the condition on the derivative in Theorem 4.13) the derivative of the composition of the maps in the top row at any $x \in P$ sends T_xM to $T_{f(x)}M'$ via the linear symplectomorphism $\hat{f}|_{T_xM}$. Let $F_0 \colon U_0 \to U_1$ be the composition of the maps in the top row of (19). So F restricts to $P \subset U$ as the diffeomophism $f : P \to P'$, and for all $x \in P$ we have $(F_0^*\omega')_x = \omega_x$ as bilinear forms on T_xM. So Corollary 4.15 gives a diffeomorphism $\psi \colon V_0 \to V_1$ between neighborhoods of P such that $\psi|_P = 1_P$ and $\psi^*(F_0^*\omega') = \omega$. Since $\psi^* \circ F_0^* = (F_0 \circ \psi)^*$, the theorem then holds with $U = V_0$, $U' = F_0(V_1)$, and $F = F_0 \circ \psi$. □

We now consider what Theorem 4.16 says about various particular classes of submanifolds. If (M, ω) is a symplectic manifold, we say that a submanifold $P \subset M$ is a **Lagrangian submanifold** if, for all $x \in P$, T_xP is a Lagrangian subspace of the symplectic vector space (T_xM, ω_x). Similarly one defines symplectic, isotropic, and coisotropic submanifolds P as those whose tangent spaces T_xP are, respectively, symplectic, isotropic, or coisotropic subspaces of (T_xM, ω_x) at all $x \in P$ (recall Definition 2.13). For any submanifold P, one has a "symplectic orthogonal complement" TP^ω to the subbundle $TP \subset TM$ (the fiber of TP^ω at x is $(T_xP)^{\omega_x}$), and P is thus symplectic if $TP^\omega \cap TP$ is the zero section (equivalently, if $TM = TP \oplus TP^\omega$), isotropic if $TP \subset TP^\omega$, coisotropic if $TP^\omega \subset TP$, and Lagrangian if $TP = TP^\omega$.

Before proceeding I'll note that one should not expect a general submanifold P to fall into any of these four classes unless it has a particularly good reason to. One good reason would be based on dimension: if $\dim P = 1$ then P is isotropic, and if $\dim P = \dim M - 1$ then P is coisotropic since, for all x, $(T_xP)^{\omega_x}$ is one-dimensional and hence isotropic. In general, for each $x \in P$, ω_x restricts to T_xP as a linear two-form of some even rank between $\max\{0, 2\dim P - \dim M\}$[20] and $\dim P$, inclusive; if one imagines $\omega_x|_{T_xP}$ to be a "random" such two-form then one would expect the rank to be as large as possible ($\dim P$ or $\dim P - 1$, depending on whether $\dim P$ is odd or even) at most points x.[21] However assuming that the dimension and codimension of P are both at least two—which given that $\dim M$ is even can be seen to be equivalent to the statement that there are at least two even numbers ranging from $\max\{0, 2\dim P - \dim M\}$ to $\dim P$—one would also expect a codimension-one subset of P (corresponding to the vanishing of the determinant of some matrix depending on x) at which the rank of $\omega_x|_{T_xP}$ falls below its maximal possible value. The case that P is symplectic corresponds to $\dim P$ being even and this codimension-one subset happening to be empty—this is in principle possible, and is robust to small perturbations of P so that in an appropriate topology the space of symplectic submanifolds is open, but it should not be considered typical. The case that P is isotropic or coisotropic corresponds to the rank of $\omega_x|_{T_xP}$ being as small as possible (0 in the isotropic case, $2\dim P - \dim M$ in the coisotropic case) at every $x \in P$, which is a quite unusual situation when the dimension and codimension of P are both at least 2.

[20] The $2\dim P - \dim M$ comes from noticing that the subspace of T_xP on $\omega_x|_{T_xP}$ vanishes is $T_xP \cap T_xP^\omega$, whose dimension is at most $\dim T_xP^\omega = \dim M - \dim P$, so that $\omega_x|_{T_xP}$ has rank at least $\dim P - (\dim M - \dim P)$.

[21] One can turn this sort of reasoning about random 2-forms into rigorous statements about generic embeddings into (M, ω) using the jet transversality theorem, which appears in [Hi, Section 3.2].

While symplectic and (especially) isotropic/coisotropic/Lagrangian submanifolds are somewhat special objects, they do arise in a variety of natural situations, and the fact that they are special is already an indication that when they arise they may have significant things to tell us. Let's first consider Theorem 4.16 in the case that $P \subset (M, \omega)$ and $P' \subset (M', \omega')$ are symplectic submanifolds. In this case, $(P, \omega|_P)$ and $(P', \omega'|_{P'})$ are themselves symplectic manifolds, and we have direct sum decompositions of vector bundles

$$TM|_P = TP \oplus TP^{\omega} \qquad TM'|_{P'} = TP' \oplus TP'^{\omega'}.$$

Now the hypothesis of Theorem 4.16 says that there is a diffeomorphism $f\colon P \to P'$ and an isomorphism of symplectic vector bundles $\hat{f}\colon TM|_P \to TM'|_{P'}$ such that $\hat{f}|_{TP} = f_*$. In particular $\hat{f}$ maps TP to TP', and then since it is an isomorphism of symplectic vector bundles (so in particular $\omega'(\hat{f}v, \hat{f}w) = 0$ iff $\omega(v, w) = 0$) we see that $\hat{f}$ maps TP^{ω} to $TP'^{\omega'}$. Also since $f_* = \hat{f}|_{TP}$ we see that for $v, w \in T_xP$ it holds that $\omega'_{f(x)}(f_*v, f_*w) = \omega_x(v, w)$, *i.e.* that $f\colon P \to P'$ is a symplectomorphism. So in the case that the submanifolds in question are symplectic, the data in the hypothesis of Theorem 4.16 amount to a *symplectomorphism* from $P \to P'$ that is covered by an isomorphism of the symplectic normal bundles TP^{ω} and $TP^{\omega'}$. So applying that theorem gives:

Theorem 4.17 (Symplectic Neighborhood Theorem). *Let (M, ω) and (M', ω') be symplectic manifolds, containing compact symplectic submanifolds P and P', respectively. Suppose that there is a symplectomorphism $f\colon (P, \omega|_P) \to (P', \omega'|_{P'})$, and an isomorphism of symplectic normal bundles $\hat{f}\colon TP^{\omega} \to TP'^{\omega'}$ mapping each T_xP^{ω} to $T_{f(x)}P'^{\omega'}$. Then there are neighborhoods $U \subset M$ of P and $U' \subset M'$ of P' and a symplectomorphism $F\colon U \to U'$ such that $F|_P = f$.*

The hypotheses of Theorem 4.17 are especially easy to check in the case that $\dim P = \dim P' = 2$ and $\dim M = \dim M' = 4$. In this case, assuming for simplicity that P and P' are also connected, Corollary 4.9 implies[22] that a symplectomorphism $f\colon P \to P'$ exists iff P and P' are diffeomorphic and have the same area, *i.e.* $\int_P \omega = \int_{P'} \omega'$. As for the symplectic bundle isomorphism $\hat{f}\colon TP \to TP'^{\omega'}$ covering f, because these bundles have rank 2, $\hat{f}$ can be constructed provided merely that one has an isomorphism of *oriented* bundles $\hat{f}_0$, since if $\hat{f}_0$ sends $T_xP^{\omega} \to T_{f(x)}P'^{\omega'}$ by an orientation-preserving linear isomorphism there will be a smooth positive function $r\colon P \to (0, \infty)$ such that $\omega'_{f(x)}(\hat{f}_0v, \hat{f}_0w) = r(x)\omega_x(v, w)$ for all v and w, and then $\hat{f} = \frac{1}{\sqrt{r}}\hat{f}_0$ will be the desired symplectic bundle isomorphism. Moreover rank-2 oriented vector bundles over surfaces are classified by their Euler classes (in H^2) which can be turned into numbers by evaluating on the fundamental class (this requires orienting the surface, but in our context an orientation is given by the symplectic form). Thus the desired map $\hat{f}$ exists as long as the Euler numbers of the symplectic normal bundles TP^{ω} and $TP'^{\omega'}$ are equal. Finally, since (using that our submanifolds are symplectic) the symplectic normal bundles are complementary to the tangent bundles and so are isomorphic to the normal bundles as defined in the usual way, we can appeal to the fact that the Euler number of the normal bundle to an oriented surface in an oriented four-manifold is just the self-intersection number of that surface, *i.e.* the signed count of intersections between two generic representatives of the homology class of the surface. (This last fact follows easily from the tubular neighborhood theorem if one recalls the characterization of the Euler class in terms of zeros of a generic section, since then one can take the two representatives to be the zero section and the image of a generic section under the tubular neighborhood map.) Putting all this together we obtain:

Corollary 4.18. *Let (M, ω) and (M', ω') be symplectic four-manifolds, and let $P \subset M, P' \subset M'$ be connected, compact, two-dimensional symplectic submanifolds. Then there is a symplectomorphism*

[22] To apply Corollary 4.9 one should first choose an orientation-preserving diffeomorphism $g\colon P \to P'$ and then apply the corollary with $\omega_0 = \omega$ and $\omega_1 = g^*\omega'$.

$F: U \to U'$ between appropriate neighborhoods U of P and U' of P' such that F restricts as a symplectomorphism $P \to P'$ if and only if P and P' have the same genus, area, and self-intersection.

We now turn to Lagrangian submanifolds; some of the work we did in Section 2.2 pays off in the following, which shows that, in the Lagrangian case, much of what is required in Theorem 4.16 holds automatically, for reasons of linear algebra.

Proposition 4.19. *Let P and P' be Lagrangian submanifolds of symplectic manifolds (M,ω) and (M',ω') and suppose that $f: P \to P'$ is a diffeomorphism. Then there is a symplectic bundle isomorphism $\hat{f}: TM|_P \to TM'|_{P'}$ such that, for each $x \in P$, $\hat{f}$ restricts to T_xP as the linear isomorphism $f_*: T_xP \to T_{f(x)}P'$.*

Proof. Choose compatible almost complex structures J on M and J' on M'. By Proposition 2.26, for each $x \in P$ the subspace $J(T_xP) \leq T_xM$ is a Lagrangian complement to T_xP, and likewise each $J'(T_{f(x)}P') \leq T_{f(x)}M'$ is a Lagrangian complement to $T_{f(x)}P'$. Corollary 2.22 then gives, for each $x \in P$, a unique linear symplectomorphism $\hat{f}_x: T_xM \to T_{f(x)}M'$ that coincides with f_* on T_xP and maps $J(T_xP)$ to $J'(T_{f(x)}P')$. One then obtains $\hat{f}$ by assembling the various $\hat{f}_x$ together as a map $TM|_P \to TM'|_{P'}$; one can check that this is smooth by inspecting the proof of Corollary 2.22 and using that $\theta_{\omega_x}, \theta_{\omega_{f(x)}}, J|_{T_xM}, J'|_{T_{f(x)}M'}$ all depend smoothly on x, and $\hat{f}$ manifestly obeys the other required properties. □

A little more conceptually, what is happening here is that the short exact sequence (10), applied simultaneously for all $x \in P$, sets up a canonical isomorphism between the normal bundle to P and the cotangent bundle T^*P; we use the almost complex structure J to identify the normal bundle to P with a specific complement $J(TP)$ to TP in $TM|_P$. So at the linear-algebraic level $TM|_P$ just looks like $TP \oplus T^*P$, and a diffeomorphism $P \to P'$ induces a bundle isomorphism $TP \oplus T^*P \cong TP' \oplus T^*P'$.

Corollary 4.20 (Lagrangian Neighborhood Theorem)**.** *Let P and P' be compact Lagrangian submanifolds of symplectic manifolds (M,ω) and (M',ω'), respectively, and let $f: P \to P'$ be a diffeomorphism. Then there are neighborhoods U of P and U' of P' and a symplectomorphism $F: U \to U'$ such that $F|_P = f$.*

Proof. This follows directly from Proposition 4.19 and Theorem 4.16. □

If one has a single compact Lagrangian submanifold Q of a symplectic manifold (M,ω), one can then go looking for other symplectic manifolds containing Q as a Lagrangian submanifold, and use any of these as a model for M in a neighborhood of Q. The most widely used such model is the **cotangent bundle** of Q, denoted T^*Q and defined as the vector bundle over Q whose fiber at q is the dual T_q^*Q to T_qQ, and we will now explain how to put a symplectic structure on T^*Q which makes the zero section into a Lagrangian submanifold. (We already saw a hint of this in Definition 2.17.)

Let Q be any smooth manifold. If $U \subset Q$ is the domain of a coordinate chart, say with coordinate functions $q_1,\ldots,q_n$, then for each $q \in U$ the cotangent space T_q^*Q acquires a basis $\{(dq_1)_q,\ldots,(dq_n)_q\}$, so each $p \in T_q^*Q$ can be written as $\sum_i p_i(p,q)(dq_i)_q$ for some $p_1(p,q),\ldots,p_n(p,q) \in \mathbb{R}$. The functions $p_1,\ldots,p_n,q_1,\ldots,q_n$ then define a smooth coordinate system for $T^*U \subset T^*Q$. So $\sum_{i=1}^n dp_i \wedge dq_i$ defines a symplectic form on T^*U. Moreover one can show with a little effort that different local coordinates $(q_1,\ldots,q_n)$ on P would yield the same two-form $\sum_{i=1}^n dp_i \wedge dq_i$, and so one can piece together the two-forms just defined to a 2-form ω_{can} on T^*Q, which is evidently closed and non-degenerate, and has the zero section (where all the p_i are zero) as a Lagrangian submanifold.

While the most obvious way of trying to show that $\sum_i dp_i \wedge dq_i$ is independent of the coordinates q_i involves computations with Jacobians of coordinate transformations, there is a more informative way that gives a manifestly coordinate-independent description of ω_{can}. In fact we will show that there is a one-form $\lambda_{can} \in \Omega^1(T^*Q)$ that is equal within each T^*U to $\sum_i p_i dq_i$, and then ω_{can} will just be equal to $d\lambda_{can}$. To do this, let $\pi\colon T^*Q \to Q$ denote the bundle projection (acting by $(p,q) \mapsto q$), and for any $q \in Q$ and $p \in T^*_qQ$, *i.e.* for any $(p,q) \in T^*Q$, and for $v \in T_{(p,q)}T^*Q$, define

$$(\lambda_{can})_{(p,q)}(v) = p(\pi_* v).$$

(This makes sense, since $\pi_* v \in T_qQ$ so $p(\pi_* v)$ is defined, and thus the resulting one-form λ_{can} is defined everywhere on T^*Q without using coordinates.) If we choose local coordinates $(p_1,\ldots,p_n,q_1,\ldots,q_n)$ as above, so the $p_1,\ldots,p_n$ parametrize p within T^*_qQ by writing p as $\sum_i p_i(p,q)(dq_i)_q$, and if $v = \sum_i (a_i(\partial_{p_i})_{(p,q)} + b_i(\partial_{q_i})_{(p,q)})$ then $\pi_* v = \sum_i b_i(\partial_{q_i})_q$. So

$$p(\pi_* v) = \sum_i p_i(p,q)b_i = \left(\sum_i p_i(p,q)(dq_i)_{(p,q)}\right)(v),$$

so λ_{can} is indeed equal locally to $\sum_i p_i dq_i$. Thus $\omega_{can} = d\lambda_{can}$ is a symplectic form on T^*Q that makes the zero section into a Lagrangian submanifold, and Corollary 4.20 yields:

Corollary 4.21. *If Q is a compact Lagrangian submanifold of a symplectic manifold (M,ω) then there are neighborhoods U of Q in M and U' of the zero-section in T^*Q and a symplectomorphism $F\colon U \to U'$ such that $F|_Q = 1_Q$, where the cotangent bundle T^*Q is endowed with its standard symplectic structure $d\lambda_{can}$.*

Exercise 4.22. *Let ω_0 be the standard area form on S^2 (so for $x \in S^2$ and $v,w \in T_xS^2 = \{x\}^\perp \subset \mathbb{R}^3$ we have $\omega_x(v,w) = x \cdot (v \times w)$). Let $M = S^2 \times S^2$ and let ω be the product symplectic form induced by ω_0 (formally $\omega = \pi_1^*\omega_0 + \pi_2^*\omega_0$ where $\pi_1,\pi_2\colon M \to S^2$ are the two projections). You can use without proof that $H_2(M;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, with generators[23] $A := [S^2 \times \{y\}]$ and $B := [\{x\} \times S^2]$ for arbitrary $x,y \in S^2$.*

(a) *If $\alpha\colon S^2 \to S^2$ is the antipodal map, prove that $L = \{(x,\alpha(x))|x \in S^2\}$ is a Lagrangian submanifold of M. What is the homology class of L, in terms of A and B?*
(b) *Prove that if m,n are positive integers there is a symplectic submanifold of M representing the class $mA + nB$. (Suggestion: Start with a union of "horizontal" and "vertical" spheres $S^2 \times \{x_i\}, \{y_j\} \times S^2$, which gives a singular object due to the intersections between the spheres, and develop a technique for fusing together the spheres near each these intersections which yields a smooth, symplectic submanifold.)*

Exercise 4.23. *Let (M,ω) be a symplectic manifold and let $C \subset M$ be a compact,* ***coisotropic*** *submanifold.*

(a) *Let J be an ω-compatible almost complex structure, inducing the Riemmannian metric $g_J(v,w) = \omega(v,Jw)$, and for all $x \in C$ let $E_x = J(T_xC^\omega)$ and let V_x be the g_J-orthogonal complement of $E_x \oplus T_xC^\omega$ in T_xM. Prove that $T_xM = E_x \oplus T_xC$, that $T_xC = T_xC^\omega \oplus V_x$, and that $E_x \oplus T_xC^\omega$ and V_x are both symplectic subspaces of T_xM.*
(b) *Suppose that (M',ω') is another symplectic manifold containing C as a coisotropic submanifold, and that $\omega'(v,w) = \omega(v,w)$ for all $v,w \in TC$. Show, using Theorem 4.16, that there are neighborhoods U,U' of C in M and M', respectively, and a symplectomorphism $F\colon U \to U'$ that restricts to the identity on C. (Suggestion: Argue similarly to the proof of Proposition*

[23]To fix signs (a little pedantically), endow S^2 with the orientation induced by ω, yielding a fundamental class $[S^2] \in H_2(S^2;\mathbb{Z})$, and take A and B to be the images of $[S^2]$ under the maps $S^2 \to M$ defined respectively by $z \mapsto (z,y)$ and $z \mapsto (x,z)$.

4.19, applying Corollary 2.22 to the symplectic vector spaces $E_x \oplus T_x C^\omega$ from part (a). Finding the space that plays the same role as $J' T_{f(x)} P'$ in the proof of Proposition 4.19 may require somewhat more care.)

5. Flows on symplectic manifolds

We now return to some of the themes of Section 1, now in the more general setting of symplectic manifolds. One of the main results of that Section, Corollary 1.15, was that if $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ is a smooth function whose Hamiltonian vector field X_H (defined by $\iota_{X_H}\omega_0 = -dH$) has a well-defined flow $\psi^{X_H,T}$, then this flow preserves the standard symplectic form ω_0 in the sense that $\psi^{X_H,T*}\omega_0 = \omega_0$. Proposition 4.4 allows us to generalize and refine this as follows:

Proposition 5.1. *Let (M,ω) be a symplectic manifold, and let $\mathbb{V} = \{V_t\}_{t\in\mathbb{R}}$ be a time-dependent vector field whose flow $\psi^{\mathbb{V},t}$ exists for all $t \in [0,T]$. Then the following are equivalent:*

(i) *$\psi^{\mathbb{V},t*}\omega = \omega$ for all $t \in [0,T]$;*
(ii) *For all $t \in [0,T]$ the one-form $\iota_{V_t}\omega$ is **closed**.*

Proof. By Theorem 4.3, Proposition 4.4, and the fact that $d\omega = 0$, we have

$$\frac{d}{dt}\psi^{\mathbb{V},t*}\omega = \psi^{\mathbb{V},t*}L_{V_t}\omega = \psi^{\mathbb{V},t*}(d\iota_{V_t}\omega + \iota_{V_t}d\omega) = \psi^{\mathbb{V},t*}d\iota_{V_t}\omega.$$

Condition (i) is equivalent to the statement that $\frac{d}{dt}\psi^{\mathbb{V},t*}\omega = 0$ for all t, and condition (ii) is equivalent to the statement that $d\iota_{V_t}\omega = 0$ for all t. So since $\psi^{\mathbb{V},t*}$ is invertible the result immediately follows. □

Definition 5.2. A time-dependent vector field $\mathbb{V} = \{V_t\}$ on a symplectic manifold (M,ω) is said to be a **symplectic** vector field if each one-form $\iota_{V_t}\omega$ is closed, and a **Hamiltonian** vector field if each one-form $\iota_{V_t}\omega$ is exact.

Conversely, if $H\colon I \times M \to \mathbb{R}$ is a smooth function (where $I \subset \mathbb{R}$ is some interval), the **Hamiltonian vector field** of H is the time-dependent vector field $\mathbb{X}_H = \{X_{H_t}\}$ characterized by the property that $\iota_{X_{H_t}}\omega = -dH_t$ for all t, where we write H_t for the function $H(t,\cdot)$ from M to $\mathbb{R}$. The **Hamiltonian flow** associated to H is then $\{\psi^{\mathbb{X}_H,t}\}$ (provided that this flow exists).

We will often use the notation ϕ_H^t in place of $\psi^{\mathbb{X}_H,t}$. When a smooth function H is used in this role it is often called a "Hamiltonian."[24]

The following is immediate from Proposition 5.1 and Definition 5.2:

Corollary 5.3. *Let (M,ω) be a symplectic manifold. Any Hamiltonian vector field on M is a symplectic vector field, and if a diffeomorphism $\phi\colon M \to M$ arises as the time-t flow of a symplectic vector field then ϕ is a symplectomorphism (*i.e. *$\phi^*\omega = \omega$).*

Example 5.4. *On $\mathbb{R}^2$ with coordinates (p,q) and standard symplectic form $\omega_0 = dp\wedge dq$, the function $H(t,p,q) = q$ has Hamiltonian vector field given by $X_{H_t} = -\partial_p$ for all t. So the resulting Hamiltonian flow is $\phi_H^t(p,q) = (p-t,q)$.*

Example 5.5. *Consider the symplectic manifold $(T^*S^1, d\lambda_{can})$, as defined in the previous section. This can be viewed as the cylinder $\mathbb{R}\times S^1$, with coordinates p and q but with q only defined modulo 2π, and with the symplectic form again given by $dp\wedge dq$. (The canonical one-form λ_{can} is given by $p\,dq$.) The vector field $-\partial_p$ is symplectic since $\iota_{-\partial_p}(dp\wedge dq) = -dq$; however since q is only defined modulo*

[24] In classical mechanics the Hamiltonian is a special function that describes the total energy of a physical system as a function of the current state of the system, but usage in pure mathematics allows any time-dependent smooth function on a symplectic manifold to be called a Hamiltonian regardless of whether it has any particular physical significance.

2π this one-form is not exact and so the resulting flow $(p,q)\mapsto(p-t,q)$ is ***not*** *Hamiltonian. It is instructive to go back to the proof of Proposition 1.14 and think about why the same argument shows that, for $t\neq 0$, the map on $\mathbb{R}\times S^1$ given by $(p,q)\mapsto(p-t,q)$ could never arise from a Hamiltonian flow, even though it does preserve area. Of course this is related to $\mathbb{R}\times S^1$ having nontrivial first de Rham cohomology (so that closed doesn't imply exact for one-forms), and hence nontrivial fundamental group.*

Example 5.6. *In* either *of the two previous examples we can consider the time-independent Hamiltonian $H(t,p,q)=p$, which has Hamiltonian vector field ∂_q, giving rise to a Hamiltonian flow $\phi_H^t(p,q)=(p,q+t)$ which translates the q coordinate. In the case of Example 5.5 where q is only defined modulo 2π this flow rotates the cylinder T^*S^1; thus on the cylinder the obvious rotations are Hamiltonian but the translations are not.*

More generally consider either $\mathbb{R}^{2n}$ or the cotangent bundle T^*T^n of the n-dimensional torus, in each case having coordinates $\rho_1,\ldots,\rho_n,\theta_1,\ldots,\theta_n$, with the θ_j only being defined modulo 2π in the case of T^*T^n. (I'm changing names for the coordinates in anticipation of the end of example 5.7.) In either case the standard symplectic form is $\omega=\sum_j d\rho_j\wedge d\theta_j$. A Hamiltonian of the form $H(t,\vec{\rho},\vec{\theta})=\sum_j a_j\rho_j$ (where $a_j\in\mathbb{R}$) has associated Hamiltonian vector field $\sum_j a_j\partial_{\theta_j}$ and hence induces the flow $\phi_H^t(\vec{\rho},\vec{\theta})=(\vec{\rho},\vec{\theta}+t\vec{a})$, with $\vec{\theta}+t\vec{a}$ being understood modulo $(2\pi\mathbb{Z})^n$ in the case of T^*T^n.

Example 5.7. *On $\mathbb{R}^{2n}$ with its standard symplectic structure $\omega_0=\sum_j dp_j\wedge dq_j$ consider the Hamiltonian*

$$H(t,\vec{p},\vec{q})=\sum_j c_j(p_j^2+q_j^2)$$

where $c_j\in\mathbb{R}$. We have $dH_t=\sum_j 2c_j(p_jdp_j+q_jdq_j)$ and hence

$$X_{H_t}=\sum_j 2c_j(p_j\partial_{q_j}-q_j\partial_{p_j}).$$

So finding the flow of this vector field amounts to solving (with arbitrary initial conditions) the ODE system

$$\begin{cases}\dot{p}_j=-2c_jq_j\\ \dot{q}_j=2c_jp_j\end{cases},$$

or equivalently, writing $z_j=p_j+iq_j\in\mathbb{C}$,

$$\dot{z}_j=2ic_jz_j.$$

This has general solution $z_j(t)=e^{2ic_jt}z_j(0)$, so identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ the flow is given by

$$\phi_H^t(z_1,\ldots,z_n)=\left(e^{2ic_1t}z_1,\ldots,e^{2ic_nt}z_n\right),$$

i.e. *the flow separately rotates each factor $\mathbb{C}$ of $\mathbb{C}^n$ with angular speed, $2c_j$ for the jth factor.*

In fact this example can be understood as a version of Example 5.6 by making use of polar coordinates. Let r_j and θ_j be the usual polar coordinates on (the complement of 0 *in) the jth copy of $\mathbb{C}$ in $\mathbb{C}^n$ (so r_j is valued in $(0,\infty)$ and θ_j in $\mathbb{R}/2\pi\mathbb{Z}$). From what you know about integrating in polar coordinates you should be able to convince yourself that $dp_j\wedge dq_j=r_jdr_j\wedge d\theta_j$ (or you could just compute this directly, after expressing r_j and θ_j locally in terms of p_j,q_j). From the point of view of symplectic geometry, a better way to express this formula is*

$$dp_j\wedge dq_j=d\left(\frac{1}{2}r_j^2\right)\wedge d\theta_j.$$

This symplectically identifies $(\mathbb{C}\setminus\{0\})^n$ (with $\omega_0=\sum_j dp_j\wedge dq_j$) with the open subset of T^*T^n (with symplectic form $\sum_j d\rho_j\wedge d\theta_j$) consisting of points with all $\rho_j>0$, with ρ_j corresponding to the *function $\frac{1}{2}r_j^2 = \frac{1}{2}(p_j^2 + q_j^2)$. Under this identification our Hamiltonian on $(\mathbb{C}\setminus\{0\})^n$ can be written as $H(t,\vec{\rho},\vec{\theta}) = \sum_j 2c_j\rho_j$, and so we immediately read off from Example 5.6 that the Hamiltonian flow will act on each $\mathbb{C}\setminus\{0\}$ factor by rotating θ_j with angular speed $2c_j$. To make this identification with a subset of T^*T^n we of course had to remove points with some $z_j = 0$, but by continuity considerations if we are trying to determine how ϕ_H^t acts on $\mathbb{C}^n$ it suffices to determine it on $(\mathbb{C}\setminus\{0\})^n$.*

Exercise 5.8. *Let $D^* = \{z \in \mathbb{C} | 0 < |z| < 1\}$, endowed with the standard symplectic structure $dp \wedge dq$ where $z = p + iq$. Using the fact that $dp \wedge dq = d(\frac{1}{2}r^2) \wedge d\theta$, give an explicit formula for a symplectomorphism $\phi\colon D^* \to D^*$ that "turns D^* inside out" in the sense that if $\{z_n\}$ is a sequence in D^* with $\lim_{n\to\infty} z_n = 0$ then $\lim_{n\to\infty} |\phi(z_n)| = 1$.*

All of the examples so far have involved Hamiltonian functions $H\colon \mathbb{R}\times M \to \mathbb{R}$ that do not depend on the $\mathbb{R}$ factor; such Hamiltonians are called **autonomous**; we will accordingly refer to smooth functions $H\colon M \to \mathbb{R}$ as "autonomous Hamiltonians" and use X_H and $\phi_H^t(M)$ to denote the Hamiltonian vector field and flow resulting from considering H as a function on $\mathbb{R}\times M$ that does not depend on the $\mathbb{R}$ parameter. One reason to sometimes consider non-autonomous Hamiltonians is that, as we will discuss later, the composition of two flows generated by autonomous Hamiltonians is typically not generated by an autonomous Hamiltonian, but is generated by a non-autonomous one. For a while though we will continue focusing on autonomous Hamiltonians; physically this corresponds to a system whose environment is not changing in a way that influences the evolution of the system (or perhaps better said, in which everything that is influencing the evolution of the system is being treated as part of the system).

One important fact about autonomous Hamiltonian flows is the following; in physics the first statement is interpreted as the law of conservation of energy.

Proposition 5.9. *Let (M,ω) be a symplectic manifold and let $H\colon M \to \mathbb{R}$ be an autonomous Hamiltonian with flow ϕ_H^t. Then for all $x \in M$ we have $H(\phi_H^t(x)) = x$. Moreover if $C \subset M$ is a coisotropic submanifold such that $H|_C$ is constant, then $\phi_H^t(C) \subset C$.*

The second sentence is almost a special case of the third, applied with $C = H^{-1}(\{c\})$; this is coisotropic assuming c is a regular value of H so that $H^{-1}(\{c\})$ is a codimension-one submanifold.

Proof. By the chain rule and the definition of the Hamiltonian vector field X_H we have

$$\frac{d}{dt}H(\phi_H^t(x)) = (dH)_{\phi_H^t(x)}(X_H) = -\omega_{\phi_H^t(x)}(X_H, X_H) = 0,$$

so the function $t \mapsto H(\phi_H^t(x))$ is constant, proving the second sentence. For the statement about coisotropic submanifolds, it suffices to show that for all $x \in C$ we have $X_H(x) \in T_xC$, since a path (such as $t \mapsto \phi_H^t(x_0)$ for some x_0) that is everywhere tangent to C will remain in C. But for all $v \in T_xC$ the assumption that $H|_C$ is constant shows that $(dH)_x(v) = 0$, and so $\omega_x(X_H, v) = -dH_x(v) = 0$, whence $X_H(x) \in T_xC^{\omega}$. That C is coisotropic then implies that $X_H(x) \in T_xC$. $\square$

Example 5.10. *For any $r_1, \ldots, r_n > 0$, the set*

$$C = \{(z_1, \ldots, z_n) \,|\, |z_1| = r_1, \ldots, |z_n| = r_n\}$$

is a Lagrangian (in particular coisotropic) submanifold of $\mathbb{C}^n$, and a Hamiltonian of the form $H(z_1, \ldots, z_n) = \sum_j a_j|z_j|^2$ is constant on C, so we can infer without any computation that the Hamiltonian flow of H will map C to itself. This is of course consistent with what we found in Example 5.7.

Here is another class of examples of Hamiltonian flows on cotangent bundles.

Example 5.11. *Let $h\colon Q \to \mathbb{R}$ be any smooth function on a smooth manifold Q, and extend this to a smooth function $H\colon T^*Q \to \mathbb{R}$ by $H(p,q) = h(q)$ when $p \in T^*_qQ$. In the standard local coordinates $(p_1,\ldots,p_n,q_1,\ldots,q_n)$ on T^*Q (with $(q_1,\ldots,q_n)$ local coordinates on Q) we have $dH = \sum_j \frac{\partial h}{\partial q_j}dq_j$ and hence $X_H = -\sum_j \frac{\partial h}{\partial q_j}\partial_{p_j}$. So Hamilton's equations (within $T^*Q|_U$ for a coordinate chart $(q_1,\ldots,q_n)\colon U \to \mathbb{R}^n$) become*

$$\dot{p}_j = -\frac{\partial h}{\partial q_j}$$
$$\dot{q}_j = 0$$

This implies that, for $p \in T^*_xQ$,

$$\phi^t_H(p,q) = (p - t(dh)_q, q).$$

One way in which this is consistent with Proposition 5.9 is that the individual fibers T^*_xP inside T^*P are Lagrangian submanifolds, and our Hamiltonian H takes the constant value $h(x)$ on each of these fibers, and we indeed see from the formula that ϕ^t_H maps each T_x*P to itself.

Identifying Q with the zero-section of T^*Q as usual, note that we have

$$\phi^1_H(Q) = \left\{(-(dh)_q, q) | q \in Q\right\} = \mathrm{Im}(-dh)$$

where in the last equality the one-form $-dh$ is being regarded as a map $Q \to T^*Q$ (some references refer to this set as the "graph" of $-dh$ instead of the image). Now Q is a Lagrangian submanifold of T^*Q, and so since ϕ^1_H is a symplectomorphism the submanifold $\phi^1_H(Q)$ is likewise Lagrangian. So we have learned that the image of any exact one-form on Q is a Lagragian submanifold of T^*Q.

So far we have been considering the flow associated to one Hamiltonian; it is also interesting to consider how different flows interact with each other. First we prove the following, which is closely related to Section 1.4

Proposition 5.12. *Let (M,ω) and (M',ω') be symplectic manifolds, let $H\colon I \times M \to \mathbb{R}$ be a Hamiltonian with flow $\{\phi^t_H\}_{t\in I}$, and let $\psi\colon M \to M'$ be a symplectomorphism. Then the smooth function $K\colon I \times M' \to \mathbb{R}$ defined by*

$$K(t,x) = H(t,\psi^{-1}(x)) \quad \textit{has Hamiltonian flow} \quad \phi^t_K = \psi \circ \phi^t_H \circ \psi^{-1}.$$

In particular (in the case $(M',\omega') = (M,\omega)$), the group of diffeomorphisms obtained from Hamiltonian flows forms a normal subgroup of the group of all symplectomorphisms.

Proof. Writing as usual $K_t = K(t,\cdot)$ and $H_t = H(t,\cdot)$ we have $K_t = H_t \circ \psi^{-1}$ and so $dK_t = \psi^{-1*}dH_t$. So, just as in the proof of Proposition 1.20, we find for $x \in M$ and $v \in T_xM$,

$$\begin{aligned}\omega_x(X_{K_t}(x), v) &= -(\psi^{-1*}dH_t)_x(v) = -(dH_t)_{\psi^{-1}(x)}(\psi^{-1}_*v) \\ &= (\iota_{X_{H_t}}\omega)_{\psi^{-1}(x)}(\psi^{-1}_*v) = \omega_{\psi^{-1}(x)}(X_{H_t}(\psi^{-1}(x)), \psi^{-1}_*v) = \omega_x(\psi_*(X_{H_t}(\psi^{-1}(x)), v)\end{aligned}$$

(in the last equality we used that ψ is a symplectomorphism), and so $X_{K_t}(x) = \psi_*(X_{H_t}(\psi^{-1}(x)))$ for all t and x.

It follows that if $\gamma_{x_0}\colon [0,T] \to M$ obeys $\dot{\gamma}_{x_0}(t) = X_{H_t}(\gamma_{x_0}(t))$ for all $t \in [0,T]$ and $\gamma_{x_0}(0) = x_0$ (*i.e.* if $\gamma_{x_0}(t) = \phi^t_H(x_0)$), then $\psi \circ \gamma_{x_0}$ has $\psi \circ \gamma_{x_0}(0) = \psi(x_0)$ and $\frac{d}{dt}(\psi \circ \gamma_{x_0}(t)) = X_{K_t}(\psi \circ \gamma_{x_0}(t))$, whence $\psi \circ \gamma_{x_0}(t) = \phi^t_K(\psi(x_0))$. So we've shown that, for all $x_0 \in M$, $\psi(\phi^t_H(x_0)) = \phi^t_K(\psi(x_0))$, which is equivalent to the statement of the proposition. □

Definition 5.13. If $F,G\colon M \to \mathbb{R}$ are two smooth functions on a symplectic manifold, the **Poisson bracket** of F and G is the function $\{F,G\}\colon M \to \mathbb{R}$ defined by

$$\{F,G\} = \omega(X_F, X_G).$$

Example 5.14. *On $\mathbb{R}^{2n}$ with its standard symplectic structure $\omega_0 = \sum_j dp_j \wedge dq_j$ we can consider F and G each equal to one of the standard coordinate functions $p_1, \ldots, p_n, q_1, \ldots, q_n$; since $X_{\partial_{p_j}} = \partial_{q_j}$ and $X_{\partial_{q_j}} = -\partial_{p_j}$ we find*

$$\{p_j, p_k\} = \{q_j, q_k\} = 0, \quad \{p_j, q_k\} = \omega_0(\partial_{q_j}, -\partial_{p_k}) = \delta_{jk} \text{ (Kronecker delta)}.$$

Remark 5.15. Note that since the product rule for differentiation gives $d(GH) = GdH + HdG$, one likewise has $X_{GH} = GX_H + HdG$ and hence $\{F, GH\} = G\{F, H\} + H\{F, G\}$.

Observe also that

$$dG(X_F) = \omega(X_F, X_G) = \{F, G\} = -\omega(X_G, X_F) = -dF(X_G). \tag{20}$$

Example 5.16. *Again on $\mathbb{R}^{2n}$ with its standard symplectic structure, define*

$$F(\vec{p}, \vec{q}) = \frac{1}{2}(p_1^2 + q_1^2), \quad G(\vec{p}, \vec{q}) = \frac{1}{2}\sum_{j=1}^{n}(p_j^2 + q_j^2), \quad H(p, q) = q_1.$$

Using Example 5.7, we have

$$X_F = p_1\partial_{q_1} - q_1\partial_{p_1}, \quad X_G = \sum_j (p_j\partial_{q_j} - q_j\partial_{p_j}).$$

So by (20),

$$\{F, G\} = dG(X_F) = -p_1q_1 + q_1p_1 = 0, \quad \{F, H\} = \{G, H\} = dq_1(p_1\partial_{q_1} - q_1\partial_{p_1}) = p_1.$$

Here is a connection between the Poisson bracket of two Hamiltonians and the flows that they generate.

Proposition 5.17. *Let $F, G \colon M \to \mathbb{R}$ be two autonomous Hamiltonians on a symplectic manifold, inducing flows $\{\phi_F^t\}, \{\phi_G^t\}$. Then the following are equivalent:*

(i) *$\{F, G\}(x) = 0$ for all $x \in M$.*
(ii) *G is conserved along the flow of F, in the sense that $G \circ \phi_F^t = G$.*
(iii) *F is conserved along the flow of G, in the sense that $F \circ \phi_G^t = F$.*

Moreover any of (i)-(iii) implies:

$$(iv) \text{ For all } s, t, x \text{ such that both sides are defined, } \phi_F^s(\phi_G^t(x)) = \phi_G^t(\phi_F^s(x)),$$

and (iv) implies that $\{F, G\}$ is constant on each connected component of M, and equal to zero if either F or G is compactly supported.

Example 5.18. *Returning to Example 5.16, Example 5.7 shows that (with $z_j = p_j + iq_j$) $\phi_F^t(z_1, z_2, \ldots, z_n) = (e^{it}z_1, z_2, \ldots, z_n)$ and that $\phi_G^t(z_1, z_2, \ldots, z_n) = (e^{it}z_1, e^{it}z_2, \ldots, e^{it}z_n)$, while we have $\phi_H^t(z_1, z_2 \ldots, z_n) = (z_1 - t, z_2, \ldots, z_n)$. We see that ϕ_F^s and ϕ_G^t commute, consistently with the fact that $\{F, G\} = 0$, while ϕ_H^s does not commute either with ϕ_F^t or ϕ_G^t, consistently with the fact that $\{F, H\}$ and $\{G, H\}$ are nonzero (and indeed nonconstant).*

Proof of Proposition 5.17. That (i) is equivalent both to (ii) and to (iii) follows easily from (20), since

$$\frac{d}{dt}G \circ \phi_F^t = dG(X_F) = \{F, G\}, \qquad \frac{d}{dt}F \circ \phi_G^t = dF(X_G) = -\{F, G\}.$$

To see that (ii) implies (iv), we apply Proposition 5.12 with $H = G$ and $\psi = (\phi_F^s)^{-1}$ to see that (for fixed s) the path of diffeomorphisms $t \mapsto (\phi_F^s)^{-1} \circ \phi_G^t \circ \phi_F^s$ is, on general grounds, the flow of the Hamiltonian $G \circ \phi_F^s$, which is just equal to G assuming (ii). Thus (ii) implies that $(\phi_F^s)^{-1} \circ \phi_G^t \circ \phi_F^s = \phi_G^t$ which is equivalent to (iv).

Conversely if $\phi_F^s \circ \phi_G^t = \phi_G^t \circ \phi_F^s$ then (for fixed s) Proposition 5.12 implies that the Hamiltonians G and $G \circ \phi_F^s$ generate the same Hamiltonian flow, and hence that $X_G = X_{G\circ\phi_F^s}$. Taking the interior product of these vector fields with ω shows that $d(G \circ \phi_F^s - G) = 0$, and then taking the derivative with respect to s shows that $d\left(dG(X_F)\right) = 0$, which is equivalent to the Poisson bracket $\{F, G\}$ being constant on every component of M. It remains only to show that if a Poisson bracket $\{F, G\}$ between two *compactly supported* functions F and G is constant then this constant must be zero. The proof of this depends on whether M is compact or not; if it is not compact then the fact that F is compactly supported implies that there is a nonempty open set on which X_F is identically zero and so $\{F, G\}$ must also be zero on that open set, and hence must be identically zero if it is constant.

If instead M is compact we use an argument with Stokes' theorem and the algebra of interior products to show that the integral (with respect to the volume form ω^n if M is $2n$-dimensional) of $\{F, G\}$ over each component of M is always zero (regardless of whether F and G obey (iv)). First note that $\{F, G\} = \iota_{X_F} dG$ and, by the Leibniz rule for interior products, and the fact that $\Omega^{2n+1}(M) = \{0\}$,

$$0 = \iota_{X_F}(dG \wedge \omega^n) = \{F, G\}\omega^n - dG \wedge \iota_{X_F}\omega^n = \{F, G\}\omega^n - n dG \wedge (\iota_{X_F}\omega) \wedge \omega^{n-1}.$$

Hence for every connected component M_0 of M we get

$$\int_{M_0} \{F, G\}\omega^n = -n \int_{M_0} dG \wedge dF \wedge \omega^{n-1} = -n \int_{M_0} d\left(G dF \wedge \omega^{n-1}\right) = 0$$

by Stokes' theorem. So given (iv), $\{F, G\}$ is a locally constant function that integrates to zero over every component of M, so $\{F, G\} = 0$. □

Exercise 5.19. *Prove (without using Exercise 5.20) that if $\{F, G\}$ is constant then (iv) holds.*

Proposition 5.17 shows that $\{F, G\}$ being zero (or constant) has a simple interpretation; the following suggests how $\{F, G\}$ should be interpreted when it is not constant, and is important when one considers actions of nonabelian groups on symplectic manifolds.

Exercise 5.20. *Prove that the Hamiltonian vector field of $\{F, G\}$ is given by*

$$X_{\{F,G\}} = [X_F, X_G]$$

where the right-hand side denotes the commutator $X_F \circ X_G - X_G \circ X_F$ (under the identification of vector fields as differential operators on $C^\infty(M)$). (Suggestion: By the nature of the quantities involved it's enough to prove the identity on a coordinate chart in which the symplectic form is given by $\sum_j dp_j \wedge dq_j$, and in this case you can compute everything in terms of the partial derivatives of F and G and see that the two sides of the equation agree. If you know something about Lie derivatives of vector fields you might try to find a more conceptual, coordinate-free proof.)

Corollary 5.21. *If (M, ω) is a symplectic manifold then the Poisson bracket $\{\cdot, \cdot\}$ on M satisfies the Jacobi identity: for $F, G, H \in C^\infty(M)$ we have*

$$\{\{F, G\}, H\}\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = 0.$$

Proof. Recall that when a vector field V is being regarded as a derivation, for any $f \in C^\infty(M)$ the notation Vf means the same thing as $df(V)$. So the Poisson bracket $\{f, g\}$ is the same as $X_f g$. With this in mind, we see that

$$\begin{aligned}\{\{F, G\}, H\} &= X_{\{F,G\}}H = [X_F, X_G]H = X_F(X_G H) - X_G(X_F H) = X_F(\{G, H\}) - X_G(\{F, H\}) \\ &= \{F, \{G, H\}\} - \{G, \{F, H\}\}.\end{aligned}$$

Since $\{\cdot, \cdot\}$ is skew-symmetric this can be rearranged to give $\{\{F, G\}, H\}\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = 0$. □

We'll now show show how to compose Hamiltonian flows; in particular this confirms that the diffeomorphisms arising as time-one maps ϕ_H^1 form a subgroup of the group of symplectomorphisms.

Proposition 5.22. *Let (M,ω) be a symplectic manifold, and let $H\colon [0,1]\times M\to\mathbb{R}$ and $K\colon [0,1]\times M\to\mathbb{R}$ be such that the flows ϕ_H^t and ϕ_K^t are defined for all $t\in[0,1]$. Then the Hamiltonian $G\colon [0,1]\times M\to\mathbb{R}$ defined by*

$$G(t,x)=(-H+K)(t,\phi_H^t(m)) \quad \textit{obeys} \quad \phi_G^t=(\phi_H^t)^{-1}\circ\phi_K^t.$$

Remark 5.23. Assuming the proposition, setting $K=0$ shows that the Hamiltonian $\bar{H}(t,m):=-H(t,\phi_H^t(m))$ has $\phi_{\bar{H}}^t(m)=(\phi_H^t)^{-1}(m)$. Then replacing H in the proposition by $\bar{H}$ shows that $H\#K(t,m):=H(t,m)+K(t,(\phi_H^t)^{-1}(m))$ has $\phi_{H\#K}^t=\phi_H^t\circ\phi_K^t$. Note that if H and K are both autonomous (independent of t), then $H\#K$ is autonomous iff $\{H,K\}=0$, in which case $H\#K=K\#H=H+K$. If H and K do not Poisson commute, on the other hand, the Hamiltonian $H+K$ should not be expected to have any particular geometric significance.

Proof.

Lemma 5.24. *If $\mathbb{V}=\{V_t\}_{t\in[0,1]}$ is a time-dependent vector field with flow $\psi^{\mathbb{V},t}$ for $t\in[0,1]$, then the time-dependent vector field $\mathbb{W}=\{W_t\}_{t\in[0,1]}$ defined by $W_t(x)=-(\psi_*^{\mathbb{V},t})^{-1}\left(V_t(\psi^{\mathbb{V},t}(x))\right)$ obeys $\psi^{\mathbb{W},t}=(\psi^{\mathbb{V},t})^{-1}$ for all $t\in[0,1]$.*

Proof of Lemma 5.24. Temporarily define a time-dependent vector field X_t by the property that, for all x, $\frac{d}{dt}((\psi^{\mathbb{V},t})^{-1}(x))=X_t(\psi^{\mathbb{V},t})^{-1}(x))$ (in other words, $X_t=\frac{d}{dt}(\psi^{\mathbb{V},t})^{-1}\circ\psi^{\mathbb{V},t}$; this formula makes clear that X_t exists). We will see that $X_t=W_t$. We find using that chain rule that, for all x,

$$0=\frac{d}{dt}\left((\psi^{\mathbb{V},t})^{-1}(\psi^{\mathbb{V},t}(x))\right)=X_t((\psi^{\mathbb{V},t})^{-1}(\psi^{\mathbb{V},t}(x)))+(\psi^{\mathbb{V},t})_*^{-1}\left(V_t(\psi^{\mathbb{V},t}(x))\right),$$

and then solving for X_t shows that indeed $X_t(x)=-(\psi_*^{\mathbb{V},t})^{-1}\left(V_t(\psi^{\mathbb{V},t}(x))\right)=W_t(x)$. □

Proceeding to the proof of Proposition 5.22, we have $G_t=\phi_H^{t*}(K_t-H_t)$, so for all $x\in M$ and $v\in T_xM$ we obtain

$$\begin{aligned}(\iota_{X_{G_t}}\omega)_x(v)&=-d(\phi_H^{t*}(K_t-H_t))_x(v)=-\left(\phi_H^{t*}(d(K_t-H_t))\right)_x(v)\\&=(-d(K_t-H_t))_{\phi_H^t(x)}(\phi_{H*}^t v)=\omega_{\phi_H^t(x)}(X_{K_t}-X_{H_t},\phi_{H*}^t v)=\omega_x((\phi_{H*}^t)^{-1}(X_{K_t}-X_{H_t}),v)\end{aligned}$$

where the last equation uses the fact that ϕ_H^t is a symplectomorphism. So by the non-degeneracy of ω we see that

$$X_{G_t}(x)=(\phi_{H*}^t)^{-1}((X_{K_t}-X_{H_t})(\phi_H^t(x))).$$

The chain rule and Lemma 5.24 then give:

$$\begin{aligned}\frac{d}{dt}\left((\phi_H^t)^{-1}\circ\phi_K^t(x)\right)&=-(\phi_{H*}^t)^{-1}(X_{H_t}(\phi_H^t((\phi_H^t)^{-1}(\phi_K^t(x)))))+(\phi_{H*}^t)^{-1}(X_{K_t}(\phi_K^t(x)))\\&=(\phi_{H*}^t)^{-1}\left((X_{K_t}-X_{H_t})(\phi_K^t(x))\right)=X_{G_t}((\phi_H^t)^{-1}(\phi_K^t(x))).\end{aligned}$$

So indeed the time-t Hamiltonian flow generated by G is $(\phi_H^t)^{-1}\circ\phi_K^t$. □

6. Exact symplectic manifolds

Many of the more basic examples of symplectic manifolds fit into the following class:

Definition 6.1. An **exact symplectic manifold** is a pair (M,λ) where M is a smooth manifold and $\lambda\in\Omega^1(M)$ has the property that $d\lambda$ is non-degenerate.

Of course $d\lambda$ is closed, so (by Darboux's theorem) if (M,λ) is an exact symplectic manifold then $(M,d\lambda)$ is a symplectic manifold. An argument with Stokes' theorem (essentially given in the proof of Proposition 3.8; we'll repeat it now) shows that an exact symplectic manifold can never be compact (and without boundary): if $d\lambda$ is non-degenerate and $\dim M = 2n$ then $(d\lambda)^n$ defines a volume form, but if the manifold without boundary M is compact then

$$\int_M (d\lambda)^n = \int_M d(\lambda \wedge (d\lambda)^{n-1}) = 0,$$

contradicting the fact that the integral of a volume form (with respect to the orientation that it induces) is always positive.

On the other hand the standard symplectic structure $\omega = \sum_j dp_j \wedge dq_j$ on $\mathbb{R}^{2n}$ is certainly exact; standard choices of one-form λ with $d\lambda = \omega_0$ are $\lambda_0 = \sum_j p_j dq_j$ and $\lambda_1 = \frac{1}{2}\sum_j(p_j dq_j - q_j dp_j)$. The latter form λ_1 connects nicely to complex geometry: if we view $\mathbb{R}^{2n}$ as $\mathbb{C}^n$ with complex coordinates $z_j = p_j + iq_j$ then $(\lambda_1)_{\vec{z}}(\vec{v}) = \frac{1}{2}\mathrm{Im}(h_0(\vec{z},\vec{v}))$ where h_0 is the standard Hermitian inner product.

The simpler-looking form $\lambda_0 = \sum_j p_j dq_j$ arises naturally from thinking of $\mathbb{R}^{2n}$ as the cotangent bundle $T^*\mathbb{R}^n$. As we explained at the end of Section 4, for any smooth manifold Q the cotangent bundle T^*Q carries a canonical one-form $\lambda_{can} \in \Omega^1(T^*Q)$ given by, for $(p,q) \in T^*Q$ (*i.e.*, for $q \in Q$ and $p \in T_q^*Q$) and for $v \in T_{(p,q)}T^*Q$, setting

$$(\lambda_{can})_{(p,q)}(v) = p(\pi_* v)$$

where $\pi\colon T^*Q \to Q$ is the projection $(p,q) \mapsto q$. We saw earlier that, if $q_1,\ldots,q_n$ are local coordinates on Q with associated coordinates $p_1,\ldots,p_n$ on the cotangent fibers, then λ_{can} is given locally by $\sum_j p_j dq_j$. So λ_0 is indeed the canonical one-form on $T^*\mathbb{R}^n$.

Here is another feature of the canonical one-form:

Proposition 6.2. *Let Q be a smooth manifold and let $\theta \in \Omega^1(Q)$. Regard θ as a map $\theta\colon Q \to T^*Q$ (sending q to (θ_q, q) where $\theta_q \in T_q^*M$ is the cotangent vector given by evaluating θ at q). Then we have the following equality of one-forms on Q:*

$$\theta^*\lambda_{can} = \theta.$$

Proof. Evidently $\pi \circ \theta = 1_Q$, so $\pi_* \circ \theta_*$ is also the identity. So for $w \in T_qQ$ we have

$$(\theta^*\lambda_{can})_q(w) = (\lambda_{can})_{(\theta_q,q)}(\theta_* w) = \theta_q(\pi_*\theta_* w) = \theta_q(w).$$

This holds for all $q \in Q$ and $w \in T_qQ$, so indeed $\theta^*\lambda_{can} = \theta$. □

Returning to the general context, there are various equivalence relations that one can put on the class of exact symplectic manifolds. Note first that if (M,λ) and (M,λ') are two exact symplectic manifolds then they give the same symplectic structure (*i.e.* $d\lambda = d\lambda'$) iff $\lambda - \lambda' \in \Omega^1(M)$ is *closed*. I will generally use a slightly finer relation that deems (M,λ) and (M,λ') to be essentially the same if $\lambda - \lambda'$ is *exact*. Of course if $H^1(M;\mathbb{R}) = \{0\}$ this is just the same as saying that $d\lambda = d\lambda'$ (so for instance the forms λ_0 and λ_1 on $\mathbb{R}^{2n}$ from earlier would be considered as giving the same exact symplectic structure on $\mathbb{R}^{2n}$), but in general it's stronger; for example on T^*S^1 with fiber coordinate p and $(\mathbb{R}/2\pi\mathbb{Z})$-valued position coordinate q, the canonical one-form pdq would be considered different from the one-form $(p+1)dq$ even though they induce the same symplectic structure $dp \wedge dq$. If one takes this viewpoint the appropriate notion of isomorphism between exact symplectic manifolds is the following:

Definition 6.3. An **exact symplectomorphism** from an exact symplectic manifold (M,λ) to an exact symplectic manifold (M',λ') is a diffeomorphism $\psi\colon M \to M'$ such that there exists $f\colon M \to \mathbb{R}$

obeying

$$\psi^*\lambda' - \lambda = df.$$

(Some authors impose the stronger requirement that f be compactly supported.)

Proposition 6.4. *Let (M,λ) be an exact symplectic manifold, and let $H\colon \mathbb{R}\times M\to\mathbb{R}$ be smooth. Then, for all t such that the Hamiltonian flow $\phi_H^t\colon M\to M$ is defined, $\phi_H^t\colon M\to M$ is an exact symplectomorphism.*

Proof. As usual we apply Cartan's Magic Formula:

$$\begin{aligned}\frac{d}{dt}\phi_H^{t*}\lambda = \phi_H^{t*}L_{X_{H_t}}\lambda &= \phi_H^{t*}\left(d\iota_{X_{H_t}}\lambda + \iota_{X_{H_t}}d\lambda\right)\\ &= \phi_H^{t*}\left(d\iota_{X_{H_t}}\lambda - dH_t\right) = d\left(\phi_H^{t*}(\iota_{X_{H_t}}\lambda - H_t)\right) = dg_t\end{aligned}$$

where $g_t\colon M\to M$ is given by $g_t = (\iota_{X_{H_t}}\lambda - H_t)\circ\phi_H^t$. So integrating yields $\phi_H^{t*}\lambda - \lambda = d\left(\int_0^t g_s ds\right)$. □

Definition 6.5. If (M,λ) is an exact symplectic manifold, an **exact Lagrangian submanifold** is a submanifold $L\subset M$ with $\dim L = \frac{1}{2}\dim M$ such that $\lambda|_L$ is an exact one-form.

Note that L being just a Lagrangian submanifold (without the word "exact") is equivalent to $d(\lambda|_L) = d\lambda|_L = 0$, so exact Lagrangian submanifolds are examples of Lagrangian submanifolds, and the notions are equivalent if $H^1(L;\mathbb{R}) = \{0\}$. Also note that if $\lambda' - \lambda$ is exact (on M), say $\lambda' = \lambda + dh$, so that according to our conventions (M,λ') and (M,λ) are equivalent, then L is an exact Lagrangian submanifold of (M,λ') iff it is an exact Lagrangian submanifold of (M,λ) (if $\lambda|_L = df$ then $\lambda'|_L = d(f + h|_L)$).

Corollary 6.6. *If (M,λ) is an exact symplectic manifold, L is an exact Lagrangian submanifold, and ϕ_H^t is the time-t flow of some Hamiltonian $H\colon \mathbb{R}\times M\to\mathbb{R}$, then $\phi_H^t(L)$ is also an exact Lagrangian submanifold.*

Proof. Let $\iota\colon L\to M$ denote the inclusion, so $\iota^*\lambda$ is exact, say $\iota^*\lambda = df$ where $f\colon L\to\mathbb{R}$. Since the exactness or non-exactness of a differential form is unaffected by pullback by diffeomorphisms and $\phi_H^t\circ i\colon L\to\phi_H^t(L)$ is a diffeomorphism, $\phi_H^t(L)$ will be exact iff $(\phi_H^t\circ\iota)^*\lambda$ is exact. By Proposition 6.4, we can write $\phi_H^{t*}\lambda - \lambda = dh$, and hence

$$(\phi_H^t\circ\iota)^*\lambda = \iota^*(\lambda + dh) = d(f + \iota^*h).$$

□

Example 6.7. *In the exact symplectic manifold (T^*Q,λ_{can}), the zero section Q is an exact Lagrangian submanifold, as is each of the cotangent fibers T_q^*Q. Indeed in each case λ_{can} is identically zero on the submanifold in question.*

Hence by Corollary 6.6 the image of either the zero section or of a cotangent fiber under a Hamiltonian flow is also an exact Lagrangian submanifold. If Q is compact, the nearby Lagrangian conjecture *asserts that the only compact exact Lagrangian submanifolds $L\subset T^*Q$ are those obtained as images of the zero section under a Hamiltonian flow. This question has generated a lot of recent activity; the current state of the art on the general question is that L is simple homotopy equivalent to the zero section* [AK18]. *The full conjecture is known in a very small handful of low-dimensional cases, most recently for $Q = T^2$* [DGI16].

Example 6.8. *If $L\subset\mathbb{R}^2$ is any compact Lagrangian submanifold (*i.e.* any simple closed curve) I claim that L **cannot** be an exact Lagrangian submanifold (with respect to any λ with $d\lambda = dp\wedge dq$). Indeed* *by the Jordan Curve Theorem L is the boundary of some compact region U, and we have by Stokes' theorem*

$$\int_L \lambda = \int_U d\lambda = \text{Area}(U) > 0$$

but if $\lambda|_L$ were exact another application of Stokes' theorem would show that $\int_L \lambda = 0$.

I'm also not going to be able to give you any examples of compact exact Lagrangian submanifolds of $\mathbb{R}^{2n}$ (with its standard symplectic structure), because a famous theorem of Gromov [G85, 2.3.B_2] *says that no such exist. Interestingly, there are symplectic structures on $\mathbb{R}^{2n}$ which do admit compact exact Lagrangian submanifolds (see* [AL, Chapter X]*); by Gromov's theorem it follows that these are not symplectomorphic to the standard symplectic structure (bearing in mind that $\mathbb{R}^{2n}$ has $H^1 = H^2 = \{0\}$ so in this case there's no difference between a symplectic structure up to symplectomorphism and an exact symplectic structure up to exact symplectomorphism).*

If one drops compactness assumptions there are of course plenty of exact Lagrangian submanifolds in $\mathbb{R}^{2n}$; for instance since $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ one can obtain some from Example 6.7.

Example 6.9. *Returning to the example of a cotangent bundle (T^*Q, λ_{can}), if $\theta \in \Omega^1(Q)$ we can consider the image $\Gamma_\theta = \{(\theta_q, q) | q \in Q\}$ of θ (considering θ as a map $Q \to T^*Q$). Since θ is a diffeomorphism to its image Γ_θ, the latter will be a Lagrangian (resp. exact Lagrangian) submanifold if and only if $\theta^*\lambda_{can}$ is closed (resp. exact). But by Proposition 6.2 we have $\theta^*\lambda_{can} = \theta$, so we arrive at the simple statement that Γ_θ is Lagrangian iff θ is closed, and exact Lagrangian iff θ is exact.*

Combining Example 6.9 with the Lagrangian Neighborhood Theorem (Corollary 4.20) makes it possible to describe the collection of Lagrangian submanifolds that are close to (in an appropriate sense) a given compact Lagrangian submanifold L of a symplectic manifold (M, ω). To indicate what close should mean we can rephrase matters in terms of maps (for which there are various choices of norms with which one could measure closeness) instead of submanifolds: given L, let's say that a different submanifold $L' \subset M$ is C^k-close to L (where $k \in \mathbb{N}$) if there exists a diffeomorphism $g\colon L \to L' \subset M$ that is close in the C^k metric to the inclusion $\iota\colon L \to M$ when we regard both g and ι as maps from L to M. (For ease of exposition I'm not formalizing what two maps from L to M being "close in the C^k metric" means, but it is not hard to do this, perhaps with the assistance of a Riemannian metric on M.)

By Corollary 4.20, there is a symplectomorphism $F\colon U \to U'$ from a neighborhood of U of L in M to a neighborhood U' of (the zero section) L in T^*L, restricting as the identity on L. If the submanifold $L' \subset M$ is (sufficiently) C^0-close to L then L' will be contained in U, and hence will map to some submanifold $F(L') \subset U' \subset T^*L$. Since F is a symplectomorphism, $F(L')$ will be Lagrangian iff L' is. Now we are not assuming that the symplectic form ω on M is exact, so we can't speak of exact Lagrangian submanifolds of M. We can however ask if L' is Hamiltonian isotopic to L (*i.e.*, if we can write $L' = \phi_H^1(L)$ for some Hamiltonian $H\colon [0,1] \times M \to \mathbb{R}$). If this is the case, and if moreover we have $\phi_H^t(L) \subset U$ for all $t \in [0,1]$, then by Proposition 5.12 we can write $F(\phi_H^t(L)) = \phi_K^t(L)$ for an appropriate Hamiltonian $K\colon [0,1] \times U' \to \mathbb{R}$; in particular (taking $t = 1$) $F(L')$ will be Hamiltonian isotopic to the zero section of T^*L and hence will be an exact Lagrangian submanifold of T^*L by Example 6.7.

Now suppose L' is C^1-close to L, not just C^0-close. Then (here and below we leave appropriate ϵ-δ formulations to the reader) $F(L')$ likewise will be C^1-close to the zero-section L, as submanifolds of T^*L, *i.e.* there will be a diffeomorphism $g\colon L \to F(L')$ that is C^1 close to the inclusion ι. Letting $\pi\colon T^*L \to L$ be the cotangent bundle projection, so in particular $\pi \circ \iota = 1_L$, it then follows that $\pi \circ g$ is C^1-close to the identity $L \to L$ which (for a sufficiently stringent definition of closeness) can be seen by an argument with the implicit function theorem and the compactness of L to imply that $\pi \circ g\colon L \to L$ is a diffeomorphism. Since $g\colon L \to F(L')$ is a diffeomorphism, this implies that

$\pi|_{F(L')}\colon F(L') \to L$ is a diffeomorphism. If $\theta_{L'}\colon L \to F(L') \subset T^*L$ is the inverse to $\pi|_{F(L')}$, then evidently $\theta_{L'}$ is a smooth map from L to T^*L obeying $\pi \circ \theta = 1_L$, *i.e.* $\theta_{L'}$ is a one-form on L, and $F(L')$ is just the image of $\theta_{L'}$. None of this paragraph has assumed that L' (and hence $F(L')$) is Lagrangian, but we can now read off from Example 6.9 that L' is Lagrangian iff $\theta_{L'}$ is closed. Moreover, using the previous paragraph, if L' is Hamiltonian isotopic to L by a Hamiltonian isotopy that remains in U, then $\theta_{L'}$ is exact. We summarize this as follows:

Proposition 6.10. *Fix a compact submanifold L of a symplectic manifold (M,ω). The Lagrangian submanifolds $L' \subset M$ that are sufficiently C^1-close to L are precisely those submanifolds that are sent by a Lagrangian neighborhood map as in Corollary 4.20 to the image in T^*L of a* ***closed*** *one-form on L. If such a submanifold is Hamiltonian isotopic to L by a Hamiltonian isotopy that remains sufficiently C^0-close to L, then the corresponding one-form on L is exact.*

6.1. **Fixed points and the Arnold conjecture.** Proposition 6.10 can be regarded as an early piece of evidence for the Arnold conjecture, which has motivated many developments in symplectic topology. This conjecture concerns fixed points of Hamiltonian diffeomorphisms of compact symplectic manifolds. Questions about fixed points of symplectomorphisms connect to questions about Lagrangian submanifolds as follows. If (M,ω) is a symplectic manifold let us endow the product $M \times M$ with the symplectic form $\Omega := (-\omega) \oplus \omega$. (In other words, denoting the projections of $M \times M$ to its two factors as π_1 and π_2, $\Omega = -\pi_1^*\omega + \pi_2^*\omega$.) For any map $\phi\colon M \to M$ we can consider the graph $G_\phi = \{(x,\phi(x))|x \in M\} \subset M \times M$. The projection onto the first factor gives a diffeomorphism $G_\phi \cong M$. For $(x,y) \in M \times M$ we have an obvious direct sum decomposition $T(x,y)M \times M = T_xM \oplus T_yM$, and with respect to this decomposition the tangent spaces to G_ϕ are given by $T_{(x,\phi(x))}G_\phi = \{(v,\phi_*v)|v \in T_xM\}$.

For two elements $(v,\phi_*v),(w,\phi_*w) \in T_{(x,\phi(x))}G_\phi$ we find

$$\Omega_{(x,\phi(x))}((v,\phi_*v),(w,\phi^*w)) = -\omega_x(v,w) + \omega_{\phi(x)}(\phi_*v,\phi_*w) = -\omega_x(v,w) + (\phi^*\omega)_x(v,w).$$

From this it follows that G_ϕ is a Lagrangian submanifold of $(M \times M,\Omega)$ if and only if $\phi^*\omega = \omega$. Thus we have a new source of examples of Lagrangian submanifolds, namely graphs of symplectomorphisms. Of course this includes the case $\phi = 1_M$, in which case the graph is the diagonal $\Delta = \{(x,x)|x \in M\}$.

Now observe that the fixed points x of a map $\phi\colon M \to M$ are in one-to-one correspondence with the points (x,x) of the intersection $G_\phi \cap \Delta$. A way of understanding the Lefschetz fixed point theorem in the case of a compact oriented manifold is that the Lefschetz number $L_\phi = \sum_i (-1)^i \mathrm{tr}(\phi_*\colon H_i(M) \to H_i(M))$ is equal (up to sign) to the intersection number of the homology classes of G_ϕ and Δ in $M \times M$; hence if $L(\phi) \neq 0$ then G_ϕ and Δ must intersect, and if all of these intersections are transverse then there must be at least $|L(\phi)|$ of them. Of course if ϕ is homotopic to the identity (as it will be in the cases discussed below) then $L(\phi)$ simplifies to the Euler characteristic $\chi(M)$.

Now let us suppose that $\phi = \phi_H^1$ for some Hamiltonian $H\colon [0,1] \times M \to \mathbb{R}$. It's easy to see that if we define $\hat{H}\colon [0,1] \times M \times M \to \mathbb{R}$ by $\hat{H}(t,x,y) = H(t,y)$ then the Hamiltonian flow of $\hat{H}$ (using the symplectic form Ω on $M \times M$) will be given by $\phi_{\hat{H}}^t(x,y) = (x,\phi_H^t(y))$. Thus in this situation the Lagrangian submanifold G_ϕ is Hamiltonian isotopic to Δ. **If** the ϕ_H^t are C^1 small enough[25] then we can apply Corollary 6.10 with $L = \Delta$ to conclude that, under a Lagrangian tubular neighborhood map F that identifies a neighborhood of Δ in $M \times M$ with a neighborhood of the zero section in $T^*\Delta$, sending Δ to itself, the graph G_ϕ is sent to the image of some exact one-form.

[25]or, slightly more generally, if ϕ is C^1-small enough and the ϕ_H^t are C^0-small enough

Corollary 6.11. *If $\phi_H^t\colon M \to M$ ($t \in [0,1]$) is a Hamiltonian isotopy of a compact symplectic manifold (M,ω) that remains sufficiently C^1-close to the identity and $\phi = \phi_H^1$, then there is a smooth function $S\colon M \to \mathbb{R}$ such that the fixed locus* $\mathrm{Fix}(\phi) := \{x|\phi(x) = x\}$ *is equal to the critical locus* $\mathrm{Crit}(S) = \{x|dS_x = 0\}$. *Moreover, if G_ϕ is transverse to Δ, then S can be taken to be a Morse function.*

Proof. Using the obvious identification of $T^*\Delta$ with T^*M, we have seen that $F(G_\phi)$ is the image of an exact one-form in T^*M, so we can take $S\colon M \to \mathbb{R}$ to be a smooth function such that $\Gamma_{dS} = F(G_\phi)$. F sends the intersection points (x,x) of G_ϕ with Δ (corresponding to the fixed points of ϕ) to the zeros x of the one-form dS, *i.e.* to the critical points of S. So indeed $\mathrm{Fix}(\phi) = \mathrm{Crit}(S)$. Diffeomorphisms preserve transversality, so if G_ϕ is transverse to Δ then Γ_{dS} is transverse to the zero section M, and it is straightforward to check that the latter condition is equivalent to S being a Morse function (*i.e.*, to the Hessian of S at each critical point of S being a non-degenerate quadratic form). □

(Note that another way of saying that G_ϕ is transverse to Δ is that, for each $x \in \mathrm{Fix}(\phi)$, the derivative $d\phi_x\colon T_xM \to T_xM$ does not have 1 as an eigenvalue. In this case ϕ is called **non-degenerate**.)

A strong version of the Arnol'd conjecture would be that Corollary 6.11 holds for any Hamiltonian diffeomorphism ϕ without any C^1-closeness hypothesis. More often the conjecture is phrased as the statement that the minimal number of fixed points of a Hamiltonian diffeomorphism of (M,ω) is equal to the minimal number of critical points of a smooth function on M, and that if one restricts to non-degenerate Hamiltonian diffeomorphisms this bound can be improved to the minimal number of critical points of a Morse function on M.[26]

This is still not the version of the Arnol'd conjecture usually treated in the literature (though see [RO99] which proves the first clause if $\pi_2(M) = \{0\}$); rather most work involves showing that the number of fixed points is greater than or equal to some other quantity that also serves as a lower bound for the number of critical points. This is easiest to explain in the non-degenerate/Morse case: given a Morse function S on a compact smooth manifold M, one can construct a cell decomposition having one cell for each critical point of S; since the homology of the cellular chain complex of this cell decomposition is the singular homology of M it follows that S has at least as many critical points as $\sum_i \dim H_i(M;F)$ for any field F. So the resulting version of the Arnol'd conjecture is that for a non-degenerate Hamiltonian diffeomorphism the minimal number of critical points is $\sum_i \dim H_i(M;F)$. This typically stronger than the bound one gets from the Lefschetz fixed point theorem, namely $\left|\sum_i (-1)^i \dim H_i(M;F)\right|$. For instance if M is the torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ the Lefschetz fixed point theorem gives a lower bound of 0 while (this version of) the Arnol'd conjecture give a lower bound of 2^{2n}. The example of the torus also shows that it is essential to restrict to Hamiltonian diffeomorphisms instead of more general symplectomorphisms: a translation of T^{2n} is a symplectomorphism but has no fixed points at all.

This version of the Arnol'd conjecture motivated the definition of Lagrangian Floer homology [F88]; as applied to the symplectic fixed point problem, the idea is to, analogously to the Morse cell decomposition, construct a chain complex whose generators are in bijection with the fixed points of a given non-degenerate Hamiltonian diffeomorphism and then show that the homology of the chain complex is isomorphic to the homology of M. Floer first used this to prove the conjecture in the case that $\pi_2(M) = \{0\}$ over the field $F = \mathbb{Z}/2\mathbb{Z}$; later work by a long sequence of authors eventually led to a proof without any topological hypothesis on M (other than compactness) at least if one works over $F = \mathbb{Q}$.

[26]Note that if the Hamiltonian diffeomorphism ϕ is generated by a *time-independent* Hamiltonian $H\colon M \to \mathbb{R}$ then the latter statement is immediate, since the Hamiltonian vector field X_H vanishes at all critical points of H and so all such points will be fixed under the flow. So the subtlety here is the possible time-dependence of H, but Corollary 6.11 shows that the conclusion still holds at least under a C^1-smallness assumption.

6.2. **Liouville vector fields.** As we have seen, giving a symplectic structure ω on a smooth manifold M gives a way of measuring (by integration) signed areas of two-dimensional oriented submanifolds of M, and moreover gives a way of generating flows (via Hamiltonian or more generally symplectic vector fields) that preserve this notion of area. When the closed form ω is exact, one can ask what sort of geometric content is conveyed by the choice of a one-form λ such that $d\lambda = \omega$. One answer to this is that the choice of λ distinguishes a way of "dilating" the manifold M, as follows:

Definition 6.12. Let (M, λ) be an exact symplectic manifold. The **Liouville vector field** of (M, ω) is the vector field X uniquely characterized by the property that

$$\iota_X d\lambda = \lambda.$$

By Exercise 4.5, the flow $\psi^{X,t}$ of X (if it exists) then satisfies $\psi^{X,t*}d\lambda = e^t d\lambda$. So if S is a compact oriented two-dimensional submanifold (possibly with boundary) of X,

$$\text{Area}_{d\lambda}(\psi^{X,t}(S)) := \int_{\psi^{X,t}(S)} d\lambda = \int_S \psi^{X,t*}d\lambda = e^t \text{Area}_{d\lambda}(S),$$

i.e. the flow of X dilates areas of surfaces at an exponential rate (and a similar remark applies to volumes of bounded open sets in M, using $(d\lambda)^n$ instead of $d\lambda$).

Example 6.13. *Let $M = \mathbb{R}^{2n}$ and $\lambda = \frac{1}{2}\sum_j (p_j dq_j - q_j dp_j)$. So $d\lambda = \sum_j dp_j \wedge dq_j$, and it's not hard to check that the Liouville vector field is given by*

$$X = \frac{1}{2}\sum_j (p_j \partial_{p_j} + q_j \partial_{q_j}).$$

Thus in this case X points radially outward from the origin, with magnitude equal to $\frac{1}{2}$ times the distance to the origin. The flow $\psi^{X,t}$ is easily computed, being obtained by solving the ODE system

$$\dot{p}_j = \frac{1}{2}p_j$$
$$\dot{q}_j = \frac{1}{2}q_j$$

with arbitrary initial conditions. So $\psi^{X,t}(\vec{p}, \vec{q}) = (e^{t/2}\vec{p}, e^{t/2}\vec{q})$.

Example 6.14. *Let Q be a smooth manifold and consider T^*Q with its canonical one-form λ_{can}. So if $U \subset Q$ is the domain of a coordinate chart with coordinates $q_1, \ldots, q_n$, then λ_{can} restricts to $T^*Q|_U$ as $\sum_j p_j dq_j$. Hence the Liouville field X is given on $T^*Q|_U$ by $\sum_j p_j \partial_{p_j}$. Thus X points radially outward on each fiber T_q^*Q, and its flow preserves these fibers: we have $\psi^{X,t}(p, q) = (e^t p, q)$.*

In Example 6.13, and also in Example 6.14 in the case that Q is compact, even though the whole manifold M is noncompact we can see M as obtained from a compact manifold with boundary W by using the flow of the vector field to "inflate" W to be larger and larger. Specifically we could take W equal to a closed ball centered at the origin in Example 6.13, or a unit disk cotangent bundle (*i.e.* the set of $(p, q) \in T^*Q$ with $|p|_g \leq 1$,as measured by some Riemannian metric g on Q) in Example 6.14; however there are other choices of W that would yield qualitatively the same picture (*e.g.* one could use an ellipsoid rather than a ball). A reasonable and somewhat flexible requirement to impose is that the Liouville vector field X point outward from W along ∂W; this way flowing along X tends to expand W everywhere consistently with the "inflation" picture (in particular, as one can check, in this case $s < t \Rightarrow \psi^{X,s}(W) \subset \psi^{X,t}(W)^\circ$).

Focusing on the role of ∂W as opposed to W we introduce the following notions:

Definition 6.15. (i) Let (M,λ) be an exact symplectic manifold. A codimension-one smooth submanifold $Y \subset M$ is said to be a **restricted contact type hypersurface** of (M,λ) if the Liouville vector field X is transverse to Y (*i.e.*, for every $y \in Y$, $X(y) \notin T_yY$).

(ii) If (M,ω) is a symplectic manifold and $Y \subset M$ is a codimension-one smooth submanifold, Y is said to be a **contact type hypersurface** if there is a neighborhood U of Y in M and a one-form λ on U with $d\lambda = \omega|_U$ such that Y is a restricted contact type hypersurface of (U,λ).

In particular one can speak of contact type hypersurfaces, but not of restricted contact type hypersurfaces, of symplectic manifolds (M,ω) where ω is not exact.

Definition 6.16. Let Y be a smooth manifold of odd dimension $2n-1$. A one-form $\alpha \in \Omega^1(Y)$ is said to be a **contact form** if $\alpha \wedge (d\alpha)^{n-1}$ is nowhere-vanishing.

The relation between these definitions, and the reason that they share a word in common, is:

Proposition 6.17. *If (M,λ) is an exact symplectic manifold and $Y \subset M$ is a codimension-one smooth smooth submanifold, then Y is a restricted contact type hypersurface of (M,λ) if and only if the one-form $\alpha = \lambda|_Y$ is a contact form.*

Proof. Fix $y \in Y$; it suffices to show that $X(y) \notin T_yY$ if and only if $\lambda_y \wedge (d\lambda)_y^{n-1}$ is non-vanishing, considered as a top-degree form on the tangent space T_yY.

Now Y, having codimension one, is coisotropic, so for each $y \in Y$ the subspace $(T_yY)^{d\lambda}$ of T_yM is contained in T_yY and is one-dimensional. Let $R \in T_yY$ be any nonzero element of $(T_yY)^{d\lambda}$, so $(T_yY)^{d\lambda} = \text{span}\{R\}$, and hence by Corollary 2.12

$$\text{span}\{R\}^{d\lambda} = T_yY.$$

Given this last fact, we see that

$$X(y) \notin T_yY \Leftrightarrow X(y) \notin \text{span}\{R\}^{d\lambda} \Leftrightarrow d\lambda_y(X,R) \neq 0 \Leftrightarrow \lambda_y(R) \neq 0,$$

the last equivalence using the definition of the Liouville field X. So we just need to show that $\lambda_y(R) \neq 0$ iff $\lambda_y \wedge (d\lambda)_y^{n-1}$ is nonzero as a top-degree form on T_yY.

Since R spans $(T_yY)^{d\lambda} \le T_yY$, we can choose a basis $\{R, e_1, \ldots, e_{2n-2}\}$ for T_yY whose first entry is R. Now a top-degree alternating form on a finite-dimensional vector space is nonzero iff it is nonzero when evaluated on one (and hence any) basis, so it suffices to show that $\lambda_y(R) \neq 0$ iff $\lambda_y \wedge (d\lambda)_y^{n-1}(R, e_1, \ldots, e_{2n-2}) \neq 0$.

Because $R \in (T_yY)^{d\lambda}$, we have

$$\lambda_y \wedge (d\lambda)_y^{n-1}(R, e_1, \ldots, e_{2n-2}) = \lambda_y(R)(d\lambda)_y^{n-1}(e_1, \ldots, e_{2n-2}).$$

(All of the other terms that would ordinarily appear in the expansion of the wedge product involve a factor of the form $(d\lambda)_y(R, e_k)$ which is zero by the definition of R.) So certainly if $\lambda_y(R) = 0$ then also $\lambda_y \wedge (d\lambda)_y^{n-1}(R, e_1, \ldots, e_{2n-2}) = 0$. To establish the converse we need to check that $(d\lambda)_y^{n-1}(e_1, \ldots, e_{2n-2}) \neq 0$. As follows readily from Proposition 2.3, this is equivalent to the statement that the alternating bilinear form $(d\lambda)_y$ is non-degenerate when restricted to the subspace $\text{span}\{e_1, \ldots, e_{2n-2}\}$ of T_yY. But this follows readily from the fact that $\text{span}\{e_1, \ldots, e_{2n-2}\}$ is complementary to $(T_yY)^{d\lambda} = \text{span}\{R\}$: if $0 \neq v \in \text{span}\{e_1, \ldots, e_{2n-2}\}$ then $v \notin (T_yY)^{d\lambda}$ so there is $w \in T_yY$ with $(d\lambda)_y(v,w) \neq 0$, and then for some $c \in \mathbb{R}$ we will have $w - cR \in \text{span}\{e_1, \ldots, e_{2n-2}\}$ with $(d\lambda)_y(v, w-cR) = (d\lambda)_y(v,w) \neq 0$. $\square$

6.3. **Contact forms and structures.** Proposition 6.17 leads us to the study of contact forms on (necessarily odd-dimensional) manifolds, as these arise naturally when considering boundaries of certain kinds of regions in exact symplectic manifolds. As stated in Definition 6.16, a contact form on a $(2n-1)$-dimensional smooth manifold is a one-form α with $\alpha \wedge (d\alpha)^{n-1}$ nowhere vanishing. This definition may initially seem hard to interpret; perhaps the following helps:

Proposition 6.18. *Let Y be a $(2n-1)$-dimensional smooth manifold and let α be any one-form on Y. Then α is a contact form iff, for each $y \in Y$, the alternating bilinear form $(d\alpha)_y$ is non-degenerate when restricted to the subspace* $\ker \alpha_y$ *of T_yY.*

(Of course, $(d\alpha)_y$ has no chance of being non-degenerate on all of T_yY, since alternating bilinear forms on odd-dimensional vector spaces are always degenerate by Proposition 2.3.)

Proof. If there were some nonzero element $v \in \ker \alpha_y$ such that $\iota_v(d\alpha)_y = 0$ then we would have, for any $w_1, \dots, w_{2n-2} \in T_yY$, $\left(\alpha_y \wedge (d\alpha)_y^{n-1}\right)(v, w_1, \dots, w_{2n-2}) = 0$ (since including v into either α or $d\alpha$ would give zero). If we choose $w_1, \dots, w_{2n-2}$ so that $\{v, w_1, \dots, w_{2n-2}\}$ is a basis for T_yY then the fact that $\alpha_y \wedge (d\alpha)_y^{n-1}(v, w_1, \dots, w_{2n-2}) = 0$ proves that α is not a contact form. This suffices to prove forward implication.

Conversely suppose that α is a contact form and let $y \in Y$. Since $\alpha_y \wedge (d\alpha)_y^{n-1}$ is nonzero, the linear functional α_y is nonzero, so its kernel is a codimension-one subspace of T_yY. Choose a basis $\{v, w_1, \dots, w_{2n-2}\}$ for T_yY such that $\{w_1, \dots, w_{2n-2}\}$ is a basis for $\ker \alpha_y$. Since $\alpha_y(w_j) = 0$ for all j we have

$$0 \neq \alpha_y \wedge (d\alpha)_y^{n-1}(v, w_1, \dots, w_{2n-2}) = \alpha_y(v)(d\alpha)_y^{n-1}(w_1, \dots, w_{2n-2}).$$

Thus $(d\alpha)_y|_{\ker \alpha_y}$ is an alternating bilinear form on a $2n-2$ dimensional vector space whose top exterior power $(d\alpha)_y^{n-1}$ is nonzero, which implies (using Proposition 2.3, for instance) that $(d\alpha)_y|_{\ker \alpha_y}$ is non-degenerate. $\square$

Proposition 6.18 indicates a special role for the kernel of a contact form; indeed this is often treated as the more fundamental object:

Definition 6.19. A **contact structure** on a $(2n-1)$-dimensional smooth manifold Y is a $(2n-2)$-dimensional subbundle $\xi \subset TY$ such that Y is covered by open sets U for which there exists $\alpha_U \in \Omega^1(U)$ such that $\xi|_U = \ker \alpha_U$ and $\alpha_U \wedge (d\alpha_U)^{n-1}$ is nowhere zero.

For most contact structures that are studied in practice, one can just take $U = M$ and write $\xi = \ker \alpha$ where α is a contact form. Whether this is possible for a given ξ amounts to the algebraic-topological question of whether ξ, as a subbundle of TY, is co-orientable, *i.e.* whether the one-dimensional bundle $\frac{TY}{\xi}$ is orientable. Since the set-theoretic complement of a codimension-one subspace of a vector space has two connected components, the set-theoretic complement of the whole subbundle ξ in TY will have either one or two components. ξ will be coorientable iff $TY \setminus \xi$ has two components, and in this case a coorientation of ξ (*i.e.* an orientation of $\frac{TY}{\xi}$) amounts to designating one of these components as the "positive" one and the other as the "negative" one.

If ξ is coorientable we can (in many ways) write $\xi = \ker \alpha$ for some one-form $\alpha \in \Omega^1(Y)$: for instance we could choose a coorientation for ξ and a Riemannian metric g on Y and let $\alpha_y(v) = g_y(v, X)$ where X is the unique positive (with respect to the chosen coorientation) vector in the g-orthogonal complement of ξ_y such that $g_y(X, X) = 1$.

Definition 6.20. If ξ is a contact structure on Y, a one-form α is said to be a "contact form for ξ" if $\xi = \ker \alpha$. If additionally ξ is cooriented, α is said to be a "positive contact form for ξ" if $\xi = \ker \alpha$ and α evaluates positively on all vectors in $TY \setminus \xi$ that are positive with respect to the coorientation.

As phrased, the above definition does not explicitly say that α is a contact form in the original sense that $\alpha\wedge(d\alpha)^{n-1}$ never vanishes, but this does follow: we know that Y is covered by open sets U on which there exist contact forms (in the original sense) α_U with $\ker\alpha_U = \ker\alpha = \xi$. Two linear functionals on the same vector space (such as $(\alpha_U)_y$ and α_y on T_yY) have the same kernel iff one is a nonzero scalar multiple of the other; as the point y varies through U we see that there is some $f\colon U \to \mathbb{R}\setminus\{0\}$ such that $\alpha = f\alpha_U$. Then $d\alpha = df\wedge\alpha_U + f d\alpha_U$ as two-forms on U; however if we restrict these two-forms to $\xi = \ker\alpha = \ker\alpha_U$ we get $d\alpha|_\xi = f d\alpha_U|_\xi$. Hence $d\alpha$ is nondegenerate on ξ iff $d\alpha_U$ is nondegenerate on ξ, so since $\xi = \ker\alpha = \ker\alpha_U$ Proposition 6.18 shows that the fact that α_U is a contact form implies the same for the restriction of α to U. Such sets U cover M, so indeed $\alpha\wedge(d\alpha)^{n-1}$ is nowhere zero on M.

Remark 6.21. It is immediate from Definition 6.20 that if α is one contact form for a contact structure ξ then the other contact forms for ξ are those of form $f\alpha$ where $f\colon M\to\mathbb{R}\setminus\{0\}$ is an arbitrary smooth function, and that if α is one positive contact form for a cooriented contact structure ξ then the other positive contact forms for ξ are those of form $f\alpha$ where $f\colon M\to(0,\infty)$ is an arbitrary smooth function.

Example 6.22. *Suppose (M,λ) is an exact symplectic manifold, say of dimension $2n-2$. We can take $Y = \mathbb{R}\times M$ with contact form $\alpha = dz - \lambda$ (with z denoting the coordinate on $\mathbb{R}$; we're suppressing the notation for the pullback of λ via the projection $\mathbb{R}\times M\to M$). To see that this is a contact form just note that $d\alpha = -d\lambda$, so $\alpha\wedge(d\alpha)^{n-1} = (-1)^{n-1}dz\wedge(d\lambda)^{n-1}$ which is nowhere zero since $d\lambda$ is non-degenerate (so $(d\lambda)^{n-1}$ is nonwhere zero on TM).*

As a special case we could take $M = \mathbb{R}^{2n-2}$ and $\lambda = \frac{1}{2}\sum_j(p_jdq_j - q_jdp_j) = \sum_j \frac{1}{2}r_j^2 d\theta_j$ (in polar coordinates), giving a contact form $dz - \frac{1}{2}\sum_j r_j^2 d\theta_j$ on $\mathbb{R}^{2n-1}$. Or we could take $M = T^*Q$ with its canonical one-form λ_{can}, giving $Y = \mathbb{R}\times T^*Q$ with $\alpha = dz - \lambda_{can}$. The space $\mathbb{R}\times T^*Q$ is also known as the "one-jet space" J^1Q of Q. In general, for $k\in\mathbb{N}$, the k-jet of a smooth function $f\colon Q\to\mathbb{R}$ at a point $x\in Q$ consists of the data of the derivatives of f from order zero through k at x. So the one-jet of f at x can be expressed as the element $(f(x),(df)_x,x)\in J^1Q = \mathbb{R}\times T^*Q$. Letting x vary gives a section $j^1f\colon Q\to J^1Q$, and from the definition of λ_{can} it is not hard to check that the image of this section is a Legendrian *submanifold of J^1Q,* i.e. *an $(n-1)$-dimensional submanifold on which the form α vanishes identically, or said differently a manifold of the maximal possible dimension whose tangent space is contained in the contact structure* $\ker\alpha$. *This is complementary to the statement that the image of df is a Lagrangian submanifold of the cotangent bundle T^*Q.*

Let's say a bit more about the geometric meaning of a codimension-one subbundle ξ of TY being a contact structure. At least locally, we can write $\xi = \ker\alpha$ where, by Proposition 6.18, $d\alpha$ is non-degenerate when restricted to ξ. Suppose that V and W are two vector fields (defined on some open set) such that for each y we have $V(y),W(y)\in\xi_y$. Now one has the following identity of functions on the common domain of α, V, W ([Wa, Proposition 2.25(f)]):

$$d\alpha(V,W) = V(\alpha(W)) - \alpha(W,V) - \alpha([V,W]).$$

But the functions $\alpha(V)$ and $\alpha(W)$ vanish identically so this simplifies to $d\alpha(V,W) = -\alpha([V,W])$.

Suppose that we have a small two-dimensional disk D embedded in Y in such a way that $TD\subset\xi$. We could then choose V and W above in such a way that $T_yD = \mathrm{span}\{V(y),W(y)\}$ for each $y\in D$. But then $[V,W](y)$ would belong to T_yD for all $y\in D$ (the commutator of V and W could be computed on D first and then pushed forward), so we'd get $\alpha_y([V,W]) = 0$, and hence $d\alpha_y(V,W) = 0$.

So far we have not used that ξ was a contact structure—we have just used that ξ can be written locally as $\ker\alpha$ for some one-form α, which is true for any codimension-one subbundle $\xi\subset TY$. That ξ is a contact structure—so that $d\alpha|_\xi$ is non-degenerate—means that $d\alpha_y(V,W)$ is nonzero for many choices of V and W, which by the previous paragraph means that for many choices of $v = V(y), w = W(y) \in \xi_y$ it is *impossible* to embed a disk having tangent vectors v and w at y in such a way that the disk is tangent everywhere to the subbundle ξ. (More precisely, it means that for any nonzero vector $v \in \xi$ there exists a vector $w \in \xi$ such that v and w do not both lie in the tangent space to any such disk.) The opposite to a contact structure on a $(2n-1)$-dimensional manifold Y is (the tangent space to) an codimension-one foliation, in which case each set in an open cover of Y is filled up by parallel copies of $\mathbb{R}^{2n-2}$ that are tangent to ξ. Applying the reasoning in the previous paragraph to two-dimensional disks in these copies of $\mathbb{R}^{2n-2}$ shows that, if ξ is the tangent space to a foliation, then $d\alpha$ vanishes identically on $\xi = \ker\alpha$. Frobenius' theorem ([Wa, Theorem 1.60]) implies that, conversely, if $d\alpha$ vanishes on ξ then ξ is tangent to a foliation; in this case ξ is said to be integrable. Contact structures (where, again, $d\alpha$ is non-degenerate on ξ instead of being zero on it) are accordingly sometimes described as "maximally nonintegrable" hyperplane fields.

6.3.1. *The Reeb vector field.* Specifying a contact form on an odd-dimensional manifold singles out a special vector field:

Proposition-Definition 6.23. *If α is a contact form on a smooth manifold Y, there is a unique vector field $R = R_\alpha$ on Y, called the **Reeb vector field** of α, characterized the equations*

$$\iota_R d\alpha = 0$$
$$\alpha(R) \equiv 1.$$

Proof. For each $y \in Y$ the vector space $T_yY^{d\alpha} = \{v \in T_yY | d\alpha(v,w) = 0 \text{ for all } w \in T_yY\}$ is nontrivial since Y is odd-dimensional, and it has trivial intersection with the codimension-one subspace $\ker\alpha_y$ by Proposition 6.18. Thus $\dim T_yY^{d\alpha} = 1$, and α restricts nontrivially to $T_yY^{d\alpha}$. Thus for each $y \in Y$ there is a unique $R(y) \in T_yY$ such that $\alpha_y(R_y) = 1$. (If v is an arbitrary generator for $T_yY^{d\alpha}$ we will have $R(y) = \frac{v}{\alpha_y(v)}$.) Letting y vary through Y gives the desired vector field R. (The smooth dependence of R on y follows readily from the smoothness of α.) □

Note that R_α depends in a somewhat delicate way on α, not just on the contact structure $\xi = \ker\alpha$. If we replace α by a different contact form $f\alpha$ (where $f\colon Y \to \mathbb{R}\setminus\{0\}$) for ξ, then we will have $\iota_{R_\alpha} d(f\alpha) = \iota_{R_\alpha}(df \wedge \alpha + f d\alpha) = -df$, whereas $\iota_{R_{f\alpha}} d(f\alpha) = 0$ by definition; this makes it rather non-obvious how to go from R_α to $R_{f\alpha}$—if f is nonconstant it will certainly not suffice to multiply R_α by a scalar-valued function.

Let us give two somewhat general examples of Reeb vector fields:

Example 6.24. *As in Example 6.22 let $Y = \mathbb{R}\times M$ and $\alpha = dz - \lambda$ for some exact symplectic manifold (M,λ). Then $d\alpha = -d\lambda$, and $R = \partial_z$ evidently satisfies the equations defining the Reeb vector field.*

Example 6.25. *Let (M,λ) be an exact symplectic manifold and suppose that Y is a restricted contact type hypersurface (Definition 6.15). Assume moreover that Y is expressed as a level set $Y = H^{-1}(\{c\})$ for some smooth function $H\colon M \to \mathbb{R}$, with c a regular value for H. This implies that we have $T_yY = \ker dH_y$ for all $y \in Y$. By Proposition 6.17 $\alpha = \lambda|_Y$ is a contact form on M. Thus the Reeb vector field R for α is characterized by the properties that that, for all y, $R(y) \in (\ker dH_y)^{d\lambda}$ and $\alpha(R) = 1$. Now as in Proposition 5.9, another vector field on Y which lies in $(\ker dH_y)^{d\lambda}$ for all $y \in Y$ is the Hamiltonian vector field X_H of the function H of which Y is a level set: indeed if $v \in \ker dH_y$ then $0 = dH_y(v) = -d\lambda_y(X_H, v)$. Moreover the fact that c is a regular value of H implies that, for all $y \in Y$, dH_y and hence also $X_H(y)$ is nonzero. Thus for all $y \in Y$, $R(y)$ and $X_H(y)$ are nonzero* *elements of the one-dimensional vector space* $(\ker dH_y)^{d\lambda}$*. Hence we have*

$$R = \frac{1}{\lambda(X_H)} X_H|_Y$$

Note that R is determined as soon as Y is given, while there are many different possible choices of H with Y regular level set of H which have different associated functions $\lambda(X_H)$ *(for instance replacing H by 2H does not change the fact that Y is a regular level set of H, but multiplies* $\lambda(X_H)$ *by two).*

The **Weinstein conjecture** asserts that, if α is a contact form on a compact smooth manifold Y, then the Reeb vector field R_α has periodic orbits. Example 6.24 shows that one cannot do without the compactness assumption. The Weinstein conjecture has been proven for many classes of Y, *e.g.* for compact contact type hypersurfaces in $\mathbb{R}^{2n}$ [Vi87] and for arbitrary compact 3-manifold, but the general case remains open. In the case that Y is a contact type hypersurface in a symplectic manifold (M, ω) this is somewhat complementary to statements like the Arnol'd conjecture which concern fixed points of the time-one map of a Hamiltonian flow—in the Arnol'd conjecture case one is looking for points located anywhere in the symplectic manifold that come back to themselves after time one, whereas in the Weinstein conjecture case one is prescribing the energy H of the point but allowing the elapsed time for the point to come back to itself to be arbitrary.

Example 6.26. *An instructive family of examples is provided by the boundaries of ellipsoids in* $\mathbb{R}^{2n} = \mathbb{C}^n$*. Fix positive numbers* $a_1, \ldots, a_n$ *and let*

$$Y = \{\vec{z} \in \mathbb{C}^n | \sum_{j=1}^n a_j |z_j|^2 = 1\}.$$

This is readily seen to be a restricted contact type hypersurface with respect to the one-form $\lambda = \frac{1}{2}\sum_j (p_j dq_j - q_j dp_j)$ *on* $\mathbb{C}^n$ *(as seen in Example 6.13, the Liouville vector field points radially outward and so is evidently transverse to Y). As in Example 5.7 one can simplify calculations using polar coordinates; letting* $\rho_j = \frac{1}{2}(p_j^2 + q_j^2)$ *and letting* θ_j *be the standard angular polar coordinate on the jth copy of* $\mathbb{C}$*, one has* $\lambda = \sum_j \rho_j d\theta_j$*,* $d\lambda = \sum_j d\rho_j \wedge d\theta_j$*, and* $Y = H^{-1}(\{1\})$ *where* $H = \sum_j 2a_j\rho_j$*. Hence* $dH = \sum_j 2a_j d\rho_j$*, and* $X_H = \sum_j 2a_j \partial_{\theta_j}$*, whence*

$$R = \frac{1}{\lambda(X_H)} X_H|_Y = \frac{1}{\sum_j 2a_j\rho_j} \sum_j 2a_j \partial_{\theta_j} = \sum_j 2a_j \partial_{\theta_j}$$

since by definition $\sum_j 2a_j\rho_j$ *is equal to* 1 *everywhere on Y. Thus the Reeb flow on Y is given by*

$$\psi^{R,t}(z_1, \ldots, z_n) = \left(e^{2ia_1 t} z_1, \ldots, e^{2ia_n t} z_n\right).$$

Motivated by the Weinstein conjecture we look for periodic orbits of this flow. One way of getting a periodic orbit is for all of the $e^{2ia_j t}$ *to be equal to* 1 *for some t,* i.e. *for each* $a_j t \in \pi\mathbb{Z}$*. So for instance for a sphere, where all* a_j *are equal,* every *orbit will be periodic with period* $\frac{\pi}{a_j}$*. More generally though it may not be possible to find a single time t for which all* $a_j t$ *are integers; in particular if* $\frac{a_j}{a_k} \notin \mathbb{Q}$ *for* $j \neq k$ *then the factors* $e^{2ia_j t}$ *and* $e^{2ia_k t}$ *will never be equal. However there is a way for this to be compatible with* $(z_1, \ldots, z_n)$ *being a periodic point for the flow of R, namely perhaps one or both of* z_j *and* z_k *is zero.*

The full general description of the periodic orbits for the Reeb flow on Y is that $(z_1, \ldots, z_n)$ *lies on a periodic orbit of period t iff* $a_j t \in \pi\mathbb{Z}$ *for all j such that* $z_j \neq 0$*. In particular if all ratios* $\frac{a_j}{a_k}$ *between coefficients are irrational then the unique way of finding a periodic orbit for the Reeb flow on Y is to take all entries in* $(z_1, \ldots, z_n)$ *equal to zero except for one of them, say* z_j*. This yields an orbit contained in the jth coordinate plane of* $\mathbb{C}^n$*. The n possible choices of j yield n different periodic orbits, and there* *is a conjecture that this is the smallest number of periodic Reeb orbits that a restricted contact type hypersurface in $\mathbb{R}^{2n}$ can have.*

The main point of this example is that changing the parameters a_j changes very basic features of the Reeb flow on the hypersurface Y: all orbits are periodic if the a_j are all equal, whereas only finitely many periodic orbits exist if the $\frac{a_j}{a_k}$ are irrational for all $j \neq k$.

In the foregoing we have fixed the one-form λ and varied the hypersurface Y, but we could also do the reverse, taking Y to always be the unit sphere and taking $\lambda = \sum_j \frac{1}{a_j}\rho_j d\theta_j$. One again finds that $R = \sum_j 2a_j \partial_{\theta_j}$, so the Reeb flow is just as described in the rest of this example.

6.3.2. *Gray stability.* We now turn to a stabiliy theorem for contact structures, analogous to stability results for symplectic structures such as Corollary 4.8. If α is a contact form on Y and α' is a contact form on Y', then by definition a **strict contactomorphism** from (Y, α) to (Y', α') is a diffeomorphism $\phi\colon Y \to Y'$ such that $\phi^*\alpha' = \alpha$. A **contactomorphism** is a weaker notion in that it depends only on the corresponding contact *structures* $\xi = \ker\alpha$ and $\xi' = \ker\alpha'$: a diffeomorphism $\phi\colon Y \to Y'$ is a contactomorphism from (Y, ξ) to (Y', ξ') iff $\phi_*\xi = \xi'$. Since the one-forms on Y having kernel equal to $\ker\alpha$ are precisely those one-forms given by $f\alpha$ where $f\colon Y \to \mathbb{R}\setminus\{0\}$, if $\xi = \ker\alpha$ and $\xi' = \ker\alpha'$ then a diffeomorphism $\phi\colon Y \to Y'$ is a contactomorphism iff $\phi^*\alpha' = f\alpha$ for some $f\colon Y \to \mathbb{R}\setminus\{0\}$.

The sort of stability result we hope to prove concerns a situation where we have a smooth one-parameter family of contact forms α_t on a compact smooth manifold Y and we would like to say (perhaps under an additional hypothesis) that they are all equivalent in some sense. Without any additional hypotheses it would not be reasonable to expect the various (Y, α_t) to be *strictly* contactomorphic, since if they were then the volumes $\int_Y \alpha_t \wedge (d\alpha_t)^{n-1}$ would all be the same, which need not be the case (as can be seen by, say, taking $\alpha_t = e^t\alpha_0$). One could perhaps imagine imposing the hypothesis that all $\int_Y \alpha_t \wedge (d\alpha_t)^{n-1}$ are all equal to each other (maybe this could be viewed as a contact analogue of the requirement in Corollary 4.8 that the symplectic forms all represent the same cohomology class) but this still wouldn't work for a somewhat more subtle reason: a strict contactomorphism $(Y, \alpha) \to (Y', \alpha')$ would send the Reeb vector field of α to the Reeb orbit for α', and so would send periodic orbits of the Reeb flow on (Y, α) to periodic orbits of the Reeb flow on (Y', α'). But Example 6.26 shows that it is possible for a contact form to have all of its Reeb orbits periodic even though nearby contact forms have only finitely many periodic orbits. Thus one can form a path of contact forms such that different points on the path have different Reeb dynamics.

It does turn out to be possible to obtain this type of stability result if one accepts the equivalence being one of contact structures, not (necessarily) contact forms:

Theorem 6.27 (Gray stability)**.** *Let Y be a compact manifold equipped with a smooth one-parameter family of contact forms α_t ($0 \leq t \leq 1$), and write $\xi_t = \ker\alpha_t$. Then there is a smooth family $\psi_t\colon Y \to Y$ such that $\psi_0 = 1_Y$ and each ψ_t is a contactomorphism from (Y, ξ_0) to (Y, ξ_t).*

Proof. We will set ψ_t equal to the flow $\psi^{\mathbb{V},t}$ for some time-dependent vector field $\mathbb{V} = (V_t)_{t\in[0,1]}$; since Y is compact there are no concerns about the existence of the flow. Then automatically $\psi_0 = 1_Y$, and it suffices to choose $\mathbb{V}$ so that $\psi_t^*\alpha_t = f_t\alpha_0$ for some nowhere zero family of functions f_t to be determined. Using Proposition 4.4 (and the multivariable chain rule) we find that

$$\frac{d}{dt}(\psi_t^*\alpha_t) = \psi_t^*\left(d\iota_{V_t}\alpha_t + \iota_{V_t}d\alpha_t + \frac{d\alpha_t}{dt}\right). \tag{21}$$

Since each α_t is a contact form, the restriction of $d\alpha_t$ to ξ_t is non-degenerate. Hence there is a unique vector field V_t such that $V_t(y) \in (\xi_t)_y$ for all $y \in Y$ that satisfies the equation

$$\iota_{V_t} d\alpha_t\big|_{\xi_t} = -\left.\frac{d\alpha_t}{dt}\right|_{\xi_t}.$$

The fact that $V_t(y) \in (\xi_t)_y$ for all y implies that the function $\iota_{V_t}\alpha_t$ is identically zero. Thus, if we use the V_t to generate ψ_t, the expression in parentheses on the right-hand side of (21) will be a one-form on Y whose restriction to ξ_t vanishes identically; hence this expression can be written as $g_t\alpha_t$ for some smooth function $g_t\colon Y \to \mathbb{R}$. We thus have an equation of one-forms $\frac{d}{dt}(\psi_t^*\alpha_t) = \psi_t^*(g_t\alpha_t)$, *i.e.*

$$\frac{d}{dt}(\psi_t^*\alpha_t) = (g_t \circ \psi_t)\psi_t^*\alpha_t.$$

In other words, for any $y \in Y$ and $v \in T_yY$ the real-valued function $p(t) := (\psi_t^*\alpha_t)_y(v)$ obeys the first-order ODE $\dot{p}(t) = g_t(\psi_t(y))p(t)$. The unique solution to this equation (for a given initial condition $p(0)$) is $p(t) = e^{\int_0^t g_s(\psi_s(y))ds}p(0)$. In other words, for any $y \in Y$ and $v \in T_yY$ we have

$$(\psi_t^*\alpha_t)_y(v) = e^{\int_0^t g_s(\psi_s(y))ds}(\alpha_0)_y(v).$$

So the desired conclusion $\psi_t^*\alpha_t = f_t\alpha_0$ holds with $f_t(y) = e^{\int_0^t g_s(\psi_s(y))ds}$. □

6.3.3. *Symplectization.* Let ξ be a cooriented contact structure on Y. The most canonical way of defining the **symplectization** of (Y,ξ) is:

$$S(Y,\xi) = \left\{(p,y) \in T^*Y | p|_{\xi_y} = 0,\ p(v) > 0 \text{ for positive elements } v \in T_yY \setminus \xi_y\right\}$$

(recall that the coorientability of ξ means that $TY \setminus \xi$ has two connected components, and that a choice of coorientation designates one of these as positive). It is not hard to see that $S(Y,\xi)$ is a smooth submanifold of T^*Y having dimension equal to $\dim Y + 1$. The canonical one-form λ_{can} on T^*Y thus restricts to a one-form on $S(Y,\xi)$, and we will see momentarily that this one-form makes $S(Y,\xi)$ into an exact symplectic manifold. (This latter fact depends on ξ being a contact structure, not just an arbitrary hyperplane field.)

Although the above definition is canonically determined by the cooriented contact structure ξ, making a noncanonical choice facilitates computations. More specifically we can choose a positive contact form α for ξ. So for each $y \in Y$, $\alpha_y \in T_y^*Y$ has $\ker\alpha_y = \xi_y$ and α_y evaluates positively on positive elements of $T_yY \setminus \xi_y$. Hence if $(p,y) \in S(Y,\xi)$ then $p = s\alpha_y$ for some $s > 0$. Accordingly we get diffeomorphism

$$\begin{aligned}\Phi\colon \mathbb{R}_+ \times Y &\to S(Y,\xi)\\ (s,y) &\mapsto (s\alpha_y, y).\end{aligned}$$

We will now calculate $\Phi^*\lambda_{can}$. For $(s,y) \in \mathbb{R}\times Y$ the tangent space $T_{(s,y)}(\mathbb{R}_+\times Y)$ splits as a direct sum $\mathrm{span}\{\partial_s\} \oplus T_yY$. Recall that $(\lambda_{can})_{(p,y)}(v) = p(\pi_* v)$ where $\pi\colon T^*Y \to Y$ is the bundle projection. In particular $\pi \circ \Phi(s,y) = y$. So for $a\partial_s + v \in T_{(s,y)}(\mathbb{R}\times Y)$ where $a \in \mathbb{R}$ and $v \in T_yY$ we have

$$\begin{aligned}(\Phi^*\lambda_{can})_{(s,y)}(a\partial_s + v) &= (\lambda_{can})_{(s\alpha_y,y)}(\Phi_*(a\partial_s + v)) = s\alpha_y(\pi_*\Phi_*(a\partial_s + v))\\ &= s\alpha_y(v).\end{aligned}$$

This shows that

$$\Phi^*\lambda_{can} = s\alpha \in \Omega^1(\mathbb{R}_+ \times Y) \tag{22}$$

where s is the coordinate on $\mathbb{R}_+$ and we suppress notation for the pullback of α under the projection $\mathbb{R}_+ \times Y \to Y$. Hence

$$\Phi^* d\lambda_{can} = d\Phi^*\lambda_{can} = ds \wedge \alpha + s d\alpha.$$

Now if R is the Reeb vector field of α and $\xi = \ker\alpha$, we can decompose $T(\mathbb{R}_+ \times Y)$ as $\mathrm{span}\{\partial_s, R\} \oplus \xi$. The two-form $ds \wedge \alpha$ is non-degenerate on $\mathrm{span}\{\partial_s, R\}$ and vanishes on ξ, while by Proposition 6.18 $s d\alpha$ is non-degenerate on ξ and vanishes on $\mathrm{span}\{\partial_s, R\}$, in view of which $\Phi^* d\lambda_{can}$ is non-degenerate. So $d\lambda_{can}$ restricts non-degenerately to $S(Y,\xi)$, *i.e.* $(S(Y,\xi), \lambda_{can})$ is an exact symplectic manifold. The diffeomorphism Φ (constructed using a choice of positive contact form α) identifies $(S(Y,\xi), \lambda_{can})$ with $(\mathbb{R}_+ \times Y, s\alpha)$. Yet another way that the symplectization is expressed can then be obtained by using logarithms to replace $\mathbb{R}$ by $\mathbb{R}_+$. If we replace s by the coordinate $t = \log s$, so $s = e^t$, then the symplectization becomes identified with the exact symplectic manifold $(\mathbb{R} \times Y, e^t\alpha)$.

Recall that the Liouville vector field of (T^*Y, λ_{can}) is the vector field that points radially outward within each fiber of the cotangent bundle, with proportionality constant 1. (Locally the vector field is given by $\sum_j p_j \partial_{p_j}$.) This vector field is tangent to $S(Y,\xi)$, and then the equation $\iota_X d\lambda_{can} = d\lambda_{can}$ is inherited from T^*Y, so the Liouville vector field for $S(Y,\xi)$ is the same fiberwise-radial vector field as for T^*Y.

If one prefers to work in one of the more explicit models of the symplectization, since $d(s\alpha) = ds \wedge \alpha + s d\alpha$ it's not hard to see that the vector field $s\partial_s$ obeys the condition required of the Liouville vector field on $(\mathbb{R}_+ \times Y, s\alpha)$. In the model $(\mathbb{R} \times Y, e^t\alpha)$, since $d(e^t\alpha) = e^t(dt \wedge \alpha + d\alpha)$ the Liouville vector field is simply given by ∂_t.

At least part of the symplectization turns up anywhere that we find a contact manifold Y as a restricted contact type hypersurface in a symplectic manifold:

Proposition 6.28. *Let (M,λ) be an exact symplectic manifold with (possibly empty) boundary and Liouville vector field X, let Y be a restricted contact type hypersurface of M, and let $\alpha = \lambda|_Y$. Suppose that $I \subset \mathbb{R}$ is an interval with the property that, for all $t \in I$ and $y \in Y$, the time-t flow of y under the Liouville flow X is well-defined. Then the map*

$$\begin{aligned} \phi\colon\, I \times Y &\to M \\ (t,y) &\mapsto \psi^{X,t}(y) \end{aligned}$$

obeys $\phi^*\lambda = e^t\alpha$. In particular if the Liouville field X on M is complete and if Y and M are both connected while $M \setminus Y$ has two connected components then (M,λ) contains an exact-symplectic embedded copy of the symplectization $(S(Y,\ker\alpha), \lambda_{can})$.

Proof. First observe that $\iota_X\lambda = \iota_X\iota_X d\lambda = 0$ and so $L_X\lambda = d\iota_X\lambda + \iota_X d\lambda = 0 + \lambda = \lambda$. Thus, just as in Exercise 4.5, we have $\psi^{X,t*}\lambda = e^t\lambda$. Decomposing $T_{(t,y)}(\mathbb{R} \times Y) = \mathrm{span}\{\partial_t\} \oplus T_yY$ and evaluating $\phi^*\lambda$ on elements of each summand we find

$$(\phi^*\lambda)_{(t,y)}(\partial_t) = \lambda_{\psi^{X,t}(y)}(\phi_*\partial_t) = \lambda_{\psi^{X,t}(y)}(X) = 0$$

and, for $v \in T_yY$,

$$(\phi^*\lambda)_{(t,y)}(v) = \lambda_{\psi^{X,t}(y)}(\psi^{X,t}_* v) = (\psi^{X,t*}\lambda)_y(v) = e^t\lambda_y(v) = e^t\alpha_y(v).$$

So indeed $\phi^*\lambda = e^t\alpha$.

For the last sentence of the proposition, that the Liouville vector field is complete means that we can take $I = \mathbb{R}$, and then we can use the identification of $(\mathbb{R} \times Y, e^t\alpha)$ with $(S(Y,\ker\alpha), \lambda_{can})$ discussed above the proposition to view ϕ as a map $S(Y,\ker\alpha) \to M$ that pulls back λ to λ_{can}. It remains to argue that, in this case, ϕ is an embeddding. Now the fact that X is transverse to the hypersurface Y readily implies that ϕ is a submersion, and hence that it is an open and continuous map, so it will be a homeomorphism to its image provided that it is injective. If we had distinct elements $(s,y),(t,z) \in \mathbb{R} \times Y$ with $\phi(s,y) = \phi(t,z)$ and (without loss of generality) $s < t$, we would then have $y = \psi^{X,t-s}(z)$. But X is transverse to Y, and since Y is connected and $M \setminus Y$ has two connected components, X must point into the same one of the two connected components, say M_0, of $M \setminus Y$ at all points of Y. This implies that we must have $\psi^{X,u}(z) \in M_0$ for all $u > 0$ (if not, at the infimal u_0 for which the statement was false X would point out of M_0 at $\psi^{X,u_0}(z)$ instead of into it), contradicting that $y = \psi^{X,t-s}(z)$. □

6.4. **Morse theory and Weinstein handles.**

6.4.1. *Recollections from Morse theory.* We now turn to some connections between Morse theory and the topology of some symplectic and contact manifolds. To start, we recall some aspects of Morse theory just at the level of smooth manifolds. Let M be a $2n$-dimensional[27] smooth manifold, and consider a Morse function $f\colon M \to \mathbb{R}$. Thus f is a smooth function, and for each critical point $p \in M$ for f the Hessian matrix at p (*i.e.* the square matrix of second partial derivatives of f at p, constructed from local coordinates around p) is invertible. This condition is easily seen to be independent of the coordinates used; if one chooses a Riemannian metric and uses this to define covariant derivatives of vector fields then the condition is can equivalently be phrased as saying that the endomorphism of T_pM given by $v \mapsto \nabla_v(\nabla f)$ is invertible. In particular each critical point p of a Morse function is an isolated critical point—if we travel in any direction v from p the gradient of f will change from zero. In fact, the *Morse Lemma* [Mi, Lemma 2.2] shows that around each critical point p one can find coordinates $(x_1,\ldots,x_{2n-k},y_1,\ldots,y_k)$ and positive numbers $a_1,\ldots,a_{2n-k},b_1,\ldots,b_k > 0$ in terms of which f is given by

$$f(x_1,\ldots,x_{2n-k},y_1,\ldots,y_k) = \sum_{j=1}^{2n-k} a_j x_j^2 - \sum_{j=1}^{k} b_j y_j^2 + f(p). \tag{23}$$

The number k—which more intrinsically is the number of negative eigenvalues of the Hessian matrix at p—is called the *index* of f at p. (Clearly we could rescale the coordinates x_j, y_j above so as to make all a_j, b_j equal to 1, but in the symplectic setting described later it will be more convenient to not do this.)

The utility of Morse theory in topology comes from the fact that (under suitable assumptions) it is relatively easy to see how the level sets $\{x \in M | f(x) = t\}$ and sublevel sets $\{x \in M | f(x) \le t\}$ depend on the real number t. We will assume that M is compact, or more generally that f is bounded below and proper (preimages of compact sets are compact). Since critical points of f are isolated either assumption implies that for each $t \in \mathbb{R}$ the sublevel set $\{f \le t\}$ will contain only finitely many critical points.

The main tool for understanding relations between the sublevel sets of f is the gradient flow of f (*i.e.* the flow of the gradient vector field ∇f, which is defined with the help of an auxiliary Riemannian metric g on M by the formula $g(\nabla f, \cdot) = df$). We will sketch the relevant arguments here; see [Mi, Chapter 3] for a fuller treatment. Write ϕ_t for the flow $\psi^{\nabla f,t}$, and observe first that for each x,

$$\frac{d}{dt} f(\phi_t(x)) = df_{\phi_t(x)}(\nabla f) = \|\nabla f(\phi_t(x))\|^2. \tag{24}$$

This is nonnegative, and is zero only for those t, x such that $\nabla f(\phi_t(x)) = 0$. Moreover if $\nabla f(\phi_t(x)) = 0$ then $\phi_{s+t}(x) = \phi_s(\phi_t(x)) = \phi_t(x)$ for all s, and in particular for $s = -t$. So in this case $x = \phi_t(x)$. In other words, $\frac{d}{dt} f(\phi_t(x)) = \|\nabla f(\phi_t(x))\|^2 > 0$ except in the case that x is a critical point of f, in which case $\phi_t(x) = x$ for all t.

[27] The only reason that we are taking he dimension of M to be even here is that we will eventually apply this to the case where M is a symplectic manifold.

For $c \in \mathbb{R}$ and $\epsilon > 0$ we consider how the sublevel set $\{f \leq c-\epsilon\}$ compares to the sublevel set $\{f \leq c+\epsilon\}$. The simplest case is that the interval $[c-\epsilon, c+\epsilon]$ contains no critical values of f (*i.e.* that $f^{-1}[c-\epsilon, c+\epsilon]$ contains no critical points). Then $\|\nabla f\|^2$ is bounded below by a positive constant δ on $f^{-1}[c-\epsilon, c+\epsilon]$. This allows one to construct a diffeomorphism $\{f = c-\epsilon\} \cong \{f = c+\epsilon\}$ by sending $x \in \{f = c-\epsilon\}$ to $\phi_{\tau(x)}(x)$ where $\tau(x) > 0$ has the property that $f(\phi_{\tau(x)}(x)) = c+\epsilon$. (One can see using (24) that this number exists, is unique, and is bounded above by $\frac{\epsilon}{\delta}$.); the implicit function theorem can be used to show that τ is smooth as a function of x.) Moreover this diffeomorphism extends to a diffeomorphism $\{f \leq c-\epsilon\} \cong \{f \leq c+\epsilon\}$: one can first extend τ suitably to a function $\{f \leq c-\epsilon\} \to [0, \infty)$ in such a way that it vanishes outside a small neighborhood of $\{f = c-\epsilon\}$ and is nondecreasing along gradient flowlines, and once this is done use the same formula $x \mapsto \phi_{\tau(x)}(x)$.

What the above shows is that, as $t \in \mathbb{R}$ varies, the topology of the sublevel sets $\{f \leq t\}$ changes only when t passes through a critical value of f. Each such change can be understood as follows. Let us assume for simplicity that $f^{-1}[c-\epsilon, c+\epsilon]$ contains only a single critical point p, with $f(p) = c$, and that $\epsilon > 0$ is small. Outside of a suitable small neighborhood U of p, we will still have a lower bound on $\|\nabla f\|^2$ and so (if we arrange U so that its boundary is preserved by the gradient flow, as is possible) we get a diffeomorphism $\{f \leq c-\epsilon\} \setminus U \cong \{f \leq c+\epsilon\} \setminus U$ by following gradient flowlines just as in the previous case. However inside a neighborhood of p this can't be expected to work because a point $x \in \{f = c-\epsilon\}$ might have the property that $f(\phi_t(x))$ never grows as large as $c+\epsilon$ no matter how large t is; more specifically it might happen that $\phi_t(x) \to p$ as $t \to \infty$.

To understand what happens in U we work in a coordinate patch $(\vec{x}, \vec{y})$ given by the Morse Lemma, so

$$f(\vec{x}, \vec{y}) = \sum_{j=1}^{2n-k} a_j x_j^2 - \sum_{j=1}^{k} b_j y_j^2 + c$$

where all $a_j, b_j > 0$ and k is the index of p. For convenience in what follows let $\|\vec{x}\|^2 = \sum_{j=1}^{2n-k} a_j x_j^2$ and $\|\vec{y}\|^2 = \sum_{j=1}^{k} b_j y_j^2$. Thus

$$\{f \leq c \pm \epsilon\} \cap U = \{(\vec{x}, \vec{y}) | \|\vec{x}\|^2 - \|\vec{y}\|^2 \leq \pm\epsilon\}.$$

Also assume that the restriction to U of our Riemannian metric is the standard Euclidean metric in this coordinate chart. Then, within U, we have

$$\nabla f = \sum_j 2a_j x_j \partial_{x_j} - \sum_j 2b_j \partial_{y_j}, \tag{25}$$

and the gradient flow is given in U by

$$\phi_t(x_1, \ldots, x_{2n-k}, y_1, \ldots, y_k) = \left(e^{2a_1 t} x_1, \ldots, e^{2a_{2n-k} t} x_{2n-k}, e^{-2b_1 t} y_1, \ldots, e^{-2b_k t} y_k\right).$$

See Figure 1 for a depiction of the sublevel sets $\{f \leq c-\epsilon\} \cap U$ and $\{f \leq c+\epsilon\} \cap U$ and the gradient flow in the case that $k = 1, 2n = 2$.

The gradient flow of f evidently expands $\vec{x}$ and contracts $\vec{y}$; points of $\{f = c-\epsilon\} \cap U$ where $\vec{x} \neq \vec{0}$ will flow up to $\{f = c+\epsilon\}$. However points where $\vec{x} = \vec{0}$ will retain this property under the gradient flow, and will converge as $t \to \infty$ to the origin (*i.e.* to the unique critical point of f in the neighborhood, which has value c under f). The locus of points in $U \cap \{f = c-\epsilon\}$ which fail to flow up to $\{f = c+\epsilon\}$ under the gradient flow is the sphere $\{(0, \vec{y}) | \|\vec{y}\|^2 = \epsilon\} \cong S^{k-1}$. If one adds to $U \cap \{f = c-\epsilon\}$ a small neighborhood of the disk $\{(0, \vec{y}) | \|\vec{y}\|^2 \leq \epsilon\} \subset \{\vec{0}\} \times \mathbb{R}^k$ (in Figure 1, this disk is depicted in red, and its neighborhood in green) then the boundary of the resulting union does map diffeomorphically via the gradient flow to $U \cap \{f = c+\epsilon\}$. Combining this with the diffeomorphism

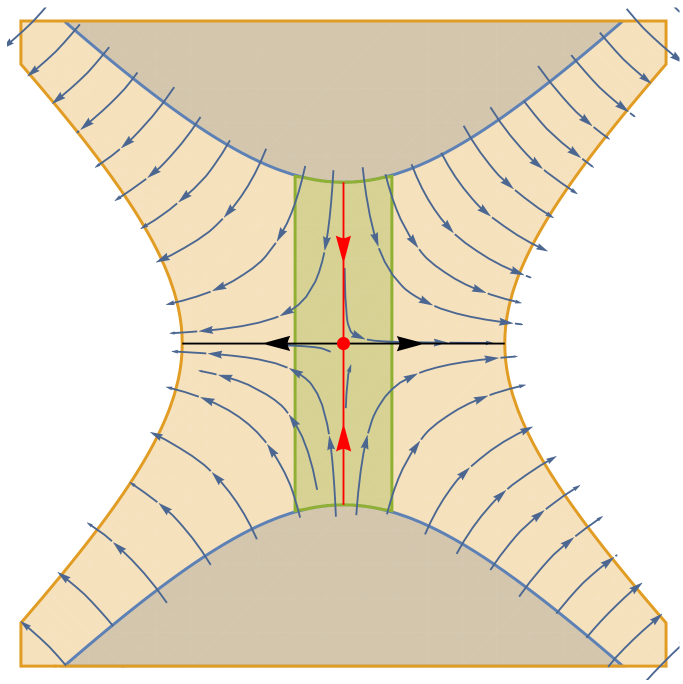

FIGURE 1. Gradient flow and sublevel sets near a Morse critical point.

given by the gradient flow outside of U leads to the conclusion that, if $f^{-1}[c-\epsilon, c+\epsilon]$ contains a unique critical point p, with critical value c and index k, then $\{f \leq c+\epsilon\}$ is obtained from $\{f \leq c-\epsilon\}$ by the attachment of a $2n$-dimensional k-handle (a copy of $D^{2n-k}\times D^k$), to the boundary $\{f = c-\epsilon\}$, with the attaching sphere $\{\vec{0}\}\times S^{k-1}$ being the locus of points in $\{f = c-\epsilon\}$ whose images under the gradient flow converges as $t\to\infty$ to the critical point p. (See [Mi, Theorem 3.3] for a full argument.)

The upshot is that, as $t\in\mathbb{R}$ increases, the sublevel sets $\{f\leq t\}$ change only when t passes through a critical value, and this change amounts to the addition of a k-handle where k is the index of the corresponding critical point. The attaching sphere for this k-handle (say associated to the critical point p) is the intersection of the *descending manifold*

$$D(p) = \{x\in M | \lim_{t\to\infty}\phi_t(x) = p\}$$

with $\{f = f(p)-\epsilon\}$ for a small $\epsilon > 0$. The corresponding level sets (boundaries of the sublevel sets) $\{f = f(p)-\epsilon\}$ and $\{f = f(p)+\epsilon\}$ are related by framed surgery on this attaching sphere. (Remove a neighborhood $D^{2n-k}\times S^{k-1}$, glue in $S^{2n-k-1}\times D^k$).

6.4.2. *When the gradient field is a Liouville field.* The first observation that connects the above discussion to symplectic geometry is that, more often than might be expected, the Liouville vector field X of an exact symplectic manifold (M,λ) can be identified as the gradient vector field of some function. For instance the Liouville vector field on $(\mathbb{R}^{2n}, \frac{1}{2}\sum_j (p_j dq_j - q_j dp_j))$ (computed in in Example 6.13) works out to be the gradient with respect to the standard metric on $\mathbb{R}^{2n}$ of the function $(\vec{p},\vec{q})\mapsto \frac{1}{4}(\|\vec{p}\|^2+\|\vec{q}\|^2)$, and in Example 6.14 the Liouville vector field on T^*Q is the gradient (with respect to an appropriate metric) of $(p,q)\mapsto\frac{1}{2}|p|^2$. In each of these examples the function of which the Liouville vector field is the gradient might be accused of being somewhat topologically uninteresting since its only critical points are minima, but the following shows that other behavior can arise.

Example 6.29. *Let $k \in \{0, \dots, n\}$ and endow $\mathbb{R}^{2n}$ the one-form*

$$\lambda_{k,n} = \sum_{k=1}^{k}(2p_j dq_j + q_j dp_j) + \sum_{j=k+1}^{n}\left(\frac{1}{2}p_j dq_j - \frac{1}{2}q_j dp_j\right).$$

Thus $d\lambda_{k,n}$ is the standard symplectic form $\sum_j dp_j \wedge dq_j$, regardless of the choice of k. The associated Liouville field is then

$$X = \sum_{j=1}^{k}(2p_j\partial_{p_j} - q_j\partial_{q_j}) + \sum_{j=k+1}^{n}(\frac{1}{2}p_j\partial_{p_j} + \frac{1}{2}q_j\partial_{q_j}).$$

This exactly coincides with (25) if we identify $q_1, \dots, q_k$ with $y_1, \dots, y_k$, $p_1, \dots, p_k, p_{k+1}, q_{k+1}, \dots, p_n, q_n$ with $x_1, \dots, x_{2n-k}$ and set $a_1 = \cdots = a_k = 1$, $a_{k+1} = \cdots = a_{2n-k} = \frac{1}{4}$, and $b_1 = \cdots = b_k = \frac{1}{2}$. In other words X is the gradient (with respect to the standard metric on $\mathbb{R}^{2n}$) of the function

$$f_{k,n}\colon (\vec{p}, \vec{q}) \mapsto \sum_{j=1}^{k}(p_j^2 - \frac{1}{2}q_j^2) + \sum_{j=k+1}^{n}(\frac{1}{4}p_j^2 + \frac{1}{4}q_j^2).$$

This shows that the local picture near a critical point of a Morse function, as described in the previous subsection, can be realized in a case where the gradient vector field is a Liouville field at least when the index k of the critical point is at most half the dimension of the manifold. (We required $k \le n$ in the previous example, whereas the discussion in the previous subsection would have allowed k to be as large as $2n$.) There is a corresponding global notion, as follows:

Definition 6.30.
(a) A **Liouville domain** is a compact exact symplectic manifold with boundary (W, λ) such that the Liouville vector field X of λ points outward along ∂W.
(b) A **Weinstein domain** is a Liouville domain (W, λ) together with a Morse[28] function $f\colon W \to \mathbb{R}$ such that, with respect to some metric on W, the gradient vector field of f is equal to the Liouville vector field of (W, λ).

If (W, λ, f) is a Weinstein domain, then since $X = \nabla f$ points outward along ∂W and W is compact the flow ϕ_t of X is defined for all $t \le 0$. For any critical point p of f, say with index k, one has a "descending manifold"

$$D(p) = \{x \in W | \phi_t(x) \text{ exists for all } t > 0 \text{ and } \lim_{t\to\infty}\phi_t(x) = p\}.$$

The intersection of this manifold with a small neighbohood of p is a k-dimensional disk D_k as in Figure 1, and the entire descending manifold can be obtained as the increasing union $\cup_{N>0}\phi_{-N}(D_k)$; from this one can see that $D(p)$ is diffeomorphic to $\mathbb{R}^k$.

Proposition 6.31. *If (W, λ, f) is a $2n$-dimensional Weinstein domain and p is an index-k critical point of f, then $\lambda|_{D(p)} = 0$. Consequently $k \le n$.*

Thus the fact that we needed to take $k \le n$ in Example 6.29 was not a coincidence.

Proof. (Sketch) As in the previous subsection, one can show that coordinates $y_1, \dots, y_k$ can be chosen on $D(p)$ such that the restriction of ϕ_t to $D(p)$ is (at least C^1-close to) $(y_1, \dots, y_k) \mapsto (e^{-b_1 t}y_1, \dots, e^{-b_k t}y_k)$ where $b_1, \dots, b_k > 0$. From this it follows that, for tangent vector $v \in TD(p)$ to the descending manifold, one has $|\phi_{t*}v| \to 0$ as $t \to \infty$. Thus if $\beta \in \Omega^1(D(p))$, then for all

[28] Some references allow f to be somewhat more general than Morse.

$v \in T_yD(p)$ we have $(\phi_t^*\beta)_y(v) = \beta_{\phi_t(y)}(\phi_{t*}v) = 0$ (since $\phi_t(y) \to p$ and $|\phi_{t*}v| \to 0$. Thus each $\beta \in \Omega^1(D(p))$ has $\phi_t^*\beta \to 0$ as $t \to \infty$.

On the other hand since ϕ_t is the flow of the Liouville vector field of λ one has $\frac{d}{dt}\phi_t^*\lambda = \lambda$ and hence $\phi_t^*\lambda = e^t\lambda$. Now apply the previous paragraph with $\beta = \lambda|_{D(p)}$ to find that $\lim_{t\to\infty} e^t\lambda|_{D(p)} = 0$ and hence $\lambda|_{D(p)} = 0$.

For the final statement of the proposition, just note that $D(p)$ is a k-dimensional submanifold of the $2n$-dimensional manifold W on which the symplectic form $d\lambda$ vanishes, so $k \leq n$ since isotropic submanifolds have dimension at most half the dimension of the ambient manifold. □

Note that in a Weinstein domain (W, λ, f), if c is any regular value of f which is less than $\min_{\partial W} f$ then the Liouville field X will be nonvanishing along the submanifold $\{f = c\}$ and will point in direction of increasing f; thus the manifold with boundary $\{f \leq c\}$ will be a Liouville domain and $\lambda|_{\{f=c\}}$ will be a contact form. If we apply this in the setting where c is slightly less than $f(p)$ for some index-k critical point p, then it is not hard to see that $D(p) \cap \{f = c\}$ is a copy of S^{k-1}, and by Proposition 6.31 λ will vanish on the tangent space to this copy of S^{k-1} in the $(2n-1)$-dimensional contact manifold $(\{f = c\}, \ker\lambda)$. There is also a way of going in the opposite direction—from a suitable sphere in the boundary of a symplectic manifold to a Morse function on a larger symplectic manifold—which we describe next.

7. Constructions of symplectic and contact manifolds

This final section will present some methods of constructing symplectic or contact forms on various kinds of smooth manifolds. The first method builds on what we have just done in Section 6.4.

7.1. **Weinstein surgery.**

Assume now that (W, λ) is a $2n$-dimensional exact symplectic manifold having restricted contact type boundary ∂W along which the Liouville field points outward. (If W is compact we could more concisely say that (W, λ) is a Liouville domain, but compactness of W won't be needed anywhere in the following discussion.) Suppose we have an *isotropic* sphere $\Lambda \subset \partial W$, *i.e.* an embedded copy of S^{k-1} such that $\lambda|_\Lambda = 0$. Under an additional topological hypothesis that will soon be explained, we will now sketch a construction of a new exact symplectic manifold with restricted contact type boundary $(W(\Lambda), \lambda')$ which, at the topological level, can be described as the result of adding a k-handle along Λ; thus $\partial W(\Lambda)$ can be regarded as obtained from ∂W by surgery along the sphere Λ.

Before describing the additional topological hypothesis we make some observations about properties of $(k-1)$-dimensional isotropic submanifolds Λ of $(2n-1)$-dimensional contact manifolds (Y, ξ). (Assume $\xi = \ker\lambda$ for some contact form λ on Y; again isotropic means that $\lambda|_\Lambda = 0$.) The isotropy condition can be phrased without reference to λ as saying that $T\Lambda \subset \xi$. Now ξ is a symplectic vector bundle over Y using the form $d\lambda$, and $d\lambda|_\Lambda = 0$, so for each $x \in \Lambda$, $T_x\Lambda$ is a $(k-1)$-dimensional isotropic subspace of the $(2n-2)$-dimensional symplectic vector space $(\xi_x, d\lambda)$, whence $k \leq n$. Moreover if J is an almost complex structure on ξ compatible[29] with $d\lambda$, then for each $x \in \Lambda$ one has $T_x\Lambda \cap JT_x\Lambda = \{0\}$, and $T\Lambda \oplus JT\Lambda$ is a symplectic subbundle of $\xi|_\Lambda$. This symplectic subbundle has a symplectic orthogonal complement $(T\Lambda \oplus JT\Lambda)^{d\lambda|_\xi}$, and using Proposition 2.11 one can identify this symplectic orthogonal complement with the quotient bundle $\frac{T\Lambda^{d\lambda|_\xi}}{T\Lambda}$, which carries a natural symplectic vector bundle structure using the construction in Exercise 2.10. The condition that will allow us to perform Weinstein surgery is that this symplectic vector bundle

[29]The proof that compatible almost complex structures exist on general symplectic vector bundles is the same as that of Theorem 3.4, which was stated in the case that the symplectic vector bundle happened to be a tangent bundle.

$\frac{T\Lambda^{d\lambda|_\xi}}{T\Lambda}$ (called in [We91] the **symplectic subnormal bundle**) be symplectically trivial. Note that in the case that $k = n$ (in which case the isotropic submanifold $\Lambda \subset Y$ is said to be **Legendrian**) one has $T\Lambda = T\Lambda^{d\lambda}$ and so the condition is obviously satisfied.

A key tool for our surgery operation will be the following strong version of relative Moser stability for isotropic submanifolds of contact hypersurfaces of symplectic manifolds. We refer to [We91] for the proof, which should be comprehensible to someone familiar with the arguments of Section 4.3.

Proposition 7.1. [We91, Proposition 4.2] *Suppose we are given two exact symplectic manifolds $(W_0, \lambda_0), (W_1, \lambda_1)$ with restricted contact type hypersurfaces $Y_0 \subset W_0, Y_1 \subset W_1$ containing compact isotropic submanifolds $\Lambda_0 \subset Y_0$ and $\Lambda_1 \subset Y_1$. If $f : \Lambda_0 \to \Lambda_1$ is a diffeomorphism that is covered by an isomorphism between the symplectic subnormal bundles Λ_0 and Λ_1, then there are neighborhoods U_0 of Λ_0 in W_0 and U_1 of Λ_1 in W_1 and a diffeomorphism $F\colon U_0 \to U_1$ such that $F^*\lambda_1 = \lambda_0$, $F(U_0 \cap Y_0) = U_1 \cap Y_1$, and $F|_{\Lambda_0} = f$.*

We can now explain the Weinstein surgery construction. Suppose that (W, λ) is an exact symplectic manifold with boundary whose Liouville field points outward along ∂W, and suppose that Λ is an isotropic $(k-1)$-dimensional sphere in ∂W having trivial symplectic subnormal bundle. The key point is that another example of such a setup is provided by the exact symplectic manifold $\left(W_1 := \{f_{k,n} \leq -\epsilon\}, \lambda_{k,n}\right)$ from Example 6.29, with the isotropic sphere $\Lambda_1 \subset \{f_{k,n} = -\epsilon\}$ being given by the locus where $p_j = 0$ for all j, $q_j = 0$ for $j > k$, and $\frac{1}{2}\sum_{j=1}^k q_j^2 = \epsilon$. Indeed it's easy to see that the symplectic subnormal bundle to Λ_1 is trivialized by the frame $\{\partial_{p_{k+1}}, \partial_{q_{k+1}}, \ldots, \partial_{p_n}, \partial_{q_n}\}$. So (perhaps after replacing $\partial W, \partial W_1, \Lambda, \Lambda_1$ by their images under the time $-\epsilon$ flows of the respective Liouville vector fields for some small $\epsilon > 0$ so that we are working in the interiors of symplectic manifolds rather than their boundaries) we can find a diffeomorphism $F\colon U_0 \to U_1$ between neighborhoods of Λ and Λ_1 as in Proposition 7.1. In particular $F^*\lambda_{k,n} = \lambda$, in consequence of which F_* sends the Liouville vector field of λ to that of $\lambda_{k,n}$.

By using F to glue U_0 to U_1 (and hence also λ to $\lambda_{k,n}$) we can make a new manifold $W(\Lambda)$ as the union of W together with a certain subset V of $\mathbb{R}^{2n}$. Namely, following Section 6.4.1 as applied to Example 6.29 let us write $\|\vec{x}\|^2 = \sum_{j=1}^k p_j^2 + \frac{1}{4}\sum_{j=k+1}^n (p_j^2 + q_j^2)$ and $\|y\|^2 = \frac{1}{2}\sum_{j=1}^k q_j^2$; then we set W equal to a small neighborhood of $\{\|x\|^2 = 0, \|y\|^2 \leq \epsilon\} \cup \{\|y\|^2 = 0, \|x\|^2 \leq \epsilon\}$ in $\mathbb{R}^{2n}$, such that (cf. Figure 1) the Liouville vector field is positively transverse to that part of the boundary of this neighborhood that is not contained in the gluing region U_1. (The isotropic sphere Λ appears as the locus $\{\|x\|^2 = 0, \|y\|^2 = \epsilon\}$, and the gluing region is a small neighborhood of this sphere.) After smoothing corners appropriately, the result is a new exact symplectic manifold $(W(\Lambda), \lambda')$ which smoothly can be regarded as the result of adding a k-handle attached to Λ; the new boundary $\partial W(\Lambda)$ is obtained from the old one by removing a neighborhood of Λ and inserting a neighborhood of the $(2n-k-1)$-dimensional belt sphere $\{\|x\|^2 = \epsilon, \|y\|^2 = 0\} \subset \{\|x\|^2 - \|y\|^2 = \epsilon\}$. By construction the new Liouville vector field is again positively transverse to the boundary; thus if (W, λ) is a Liouville domain so is $(W(\Lambda), \lambda_{k,n})$. By an appropriate modification of the function $f_{k,n}$ from Example 6.29 one can see that if the original manifold (W, λ) has the structure of a Weinstein domain then so too does $(W(\Lambda), \lambda_{k,n})$.

As a basic but broad family of examples, let us suppose that $(W, \lambda) = (B^4, \frac{1}{2}\sum_j (p_j dq_j - q_j dp_j))$; this is a Weinstein domain (as noted earlier we can use $f = \frac{1}{4}(\|\vec{p}\|^2 + \|\vec{q}\|^2)$) and so any isotropic $S^{k-1} \cong \Lambda \subset S^3 = \partial B^4$ having trivial symplectic subnormal bundle we can attach a k-handle and obtain a new four dimensional Weinstein domain $B^4(\Lambda)$ whose boundary is the result of surgery along Λ. (In particular it follows that this surgery along Λ admits contact forms.) Now in earlier notation the value of n in this case is 2, so the only possible values of k are 1 and 2, so the condition on the triviality of the symplectic subnormal bundle is automatically satisfied: if $k = 1$ this is because every bundle over S^0 is trivial, and if $k = 2 = n$ this is because the symplectic subnormal bundle then has rank 0. The $k = 1$ case amounts to attaching a one-handle to two points on S^3, yielding the four-manifold $S^1 \times B^3$ with boundary $S^1 \times S^2$; iterating the construction one gets contact forms on $\#^m(S^1 \times S^2)$ for any m.

The $k = 2$ (so $\dim \Lambda = 1$) case is more interesting, showing that a suitable surgery along any *Legendrian knot* Λ in S^3 admits contact forms. (Moreover working inductively component by component the same evidently applies to Legendrian links, since the Weinstein surgery procedure on one component will only modify the manifold and contact form on a neighborhood of that component.) Topologically, specifying a surgery on a link requires one to choose a framing, which can be specified by choosing a nonvanishing normal vector field to the link. If one inspects the construction described above one can see that the associated nonvanishing normal vector field can be taken to span the complement of $T\Lambda$ inside the contact structure ξ on S^3 (this is called the "Legendrian framing").

While a typical embedding of S^1 into S^3 will not be Legendrian, many Legendrian embeddings (in particular, ones representing the same knot class as any given embedding) can be constructed. To see this, using a suitable contact version of Darboux's theorem one can identify the contact structure in some ball containing (a scaled copy of) the knot with the contact structure $\ker(dz - ydx)$ on $\mathbb{R}^3$. An embedding $t \mapsto (x(t), y(t), z(t))$ into this region will be Legendrian iff $z'(t) = y(t)x'(t)$ everywhere; at points where $x'(t) \neq 0$ this just says that $y = \frac{dz}{dx}$. So one can take an appropriate projection of the knot into the xz-plane, and then (try to) convert it into a Legendrian embedding into $\mathbb{R}^3 \subset S^3$ by solving for y as $y = \frac{dz}{dx}$. (One should take care that at any crossings the strand with larger slope goes over the strand with smaller slope, as can be arranged by a simple local modification.) The only difficulty is at vertical tangencies of the projection (where $x'(t) = 0$) but these can be handled by replacing such tangencies with cusps with $x(t) = t^2 + a$ and $z(t) = t^3 + b$ for suitable a and b; then $y(t) = \frac{3t}{2}$ indeed solves $z'(t) = y(t)x'(t)$.

In this way one can use Weinstein surgery to make certain surgeries on any link into contact manifolds which moreover are (strongly) fillable in the sense that they are boundaries of Liouville (in fact Weinstein) domains. While any compact three-manifold can be obtained as surgery on a link, and every compact three-manifold admits contact structures ([Ma71]), it is not the case that every compact three-manifold admits fillable contact structures ([Li98]); this is consistent with the above because the framings needed to realize an arbitrary three manifold as surgery on a link may not be realizable as Legendrian framings. However if one adds to one's allowed surgery tools a variant of the Weinstein surgery where the Liouville field points into the boundary instead of out of it (this variant is usually called contact $(+1)$-surgery, as opposed to the original Weinstein construction which is contact (-1)-surgery), then it was shown in [DG04] that any compact contact three-manifold can be obtained from S^3 by a combination of contact (-1) and $(+1)$ surgeries.

7.2. **Complex submanifolds.** Here is one way of finding new examples of symplectic manifolds from old ones:

Proposition 7.2. *Let (M, ω) be a symplectic manifold and let J be an ω-tame almost complex structure*[30] *on M. Suppose that $Z \subset M$ is a submanifold which is J-complex in the sense that $J(TZ) = TZ$. Then $(Z, \omega|_Z)$ is a symplectic manifold.*

Proof. One has $d(\omega|_Z) = (d\omega)|_Z = 0$ (restriction of forms to Z is the same as pullback by the inclusion, and d commutes with pullback). Given that J maps TZ to itself, if $0 \neq v \in T_xZ$ for some $x \in Z$, we have $Jv \in T_xZ$ and by definition $\omega(v, Jv) > 0$, proving the nondegeneracy of $\omega|_Z$. Thus $\omega|_Z$ is a closed non-degenerate two-form, and the conclusion follows from Darboux's theorem. □

[30] *i.e.*, for each $x \in M$ the complex structure J_x on the vector space T_xM is ω_x-tame in the sense of Definition 2.24

Example 7.3. *The standard complex structure J_0 (multiplication by i) is ω_0-tame for the standard symplectic structure ω_0 on $\mathbb{C}^n = \{\vec{x} + i\vec{y}\}$. Suppose that $F\colon \mathbb{C}^n \to \mathbb{C}^k$ is any holomorphic function and that $\vec{a}$ is a regular value for F. By the implicit function theorem, $Z := F^{-1}(\vec{a})$ is then a submanifold of $\mathbb{C}^n$, with tangent spaces $T_{\vec{z}}Z = \ker dF_{\vec{z}}$. Now the statement that F is holomorphic amounts to the statement that $F_* \circ J_0 = J_0 \circ F_*$ and so $\ker F_{*\vec{z}}$ is mapped to itself by J_0. Thus Proposition 7.2 shows that any regular level set of a holomorphic function on $\mathbb{C}^n$ is a symplectic submanifold of $\mathbb{C}^n$.*

The examples supplied by Example 7.3 can have many different topological features; however except for the zero-dimensional ones they are never compact (sketch of proof: apply a version of the maximum principle to the restriction of each coordinate function z_j to Z). Thus far in these notes we have had few examples of compact symplectic manifolds (without boundary), and we will start to remedy that now.

As noted after Theorem 3.7, if ω is any volume form on a compact two-manifold Σ then (Σ, ω) is a symplectic manifold. If instead we are given an almost complex structure J on Σ, then one can construct a volume form ω such that J is ω-tame: choose a Riemannian metric g on Σ and let $\omega(v, w) = g(Jv, w)$. For instance we could take $\Sigma = \mathbb{C}P^1$ with its standard complex structure. Now products of symplectic manifolds $(M_1, \omega_1), (M_2, \omega_2)$ are still symplectic with respect to the obvious "product" form $\pi_1^*\omega_1 + \pi_2^*\omega_2$, so by induction we see for instance that $(\mathbb{C}P^1)^n$ can be made into a symplectic manifold, with a form that moreover tames the obvious product complex structure on $(\mathbb{C}P^1)^n$. Then Proposition 7.2 shows that any smooth complex subvariety of $(\mathbb{C}P^1)^n$ has the structure of a symplectic manifold; as in Example 7.3 these are fairly diverse topologically but now they are compact. I won't develop this further, because one gets a richer and more widely-studied set of examples from smooth complex subvarieties of $\mathbb{C}P^n$, which can be viewed as symplectic manifolds using the construction to be explained next.

7.3. **Complex projective space, projective manifolds, and Kähler manifolds.** We will now construct a natural symplectic form, the so-called Fubini-Study form ω, on complex projective space $\mathbb{C}P^n$ for any n. Recall that by definition

$$\mathbb{C}P^n = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\vec{z} \sim \lambda\vec{z} \text{ for } \lambda \in \mathbb{C}^*} = \frac{S^{2n+1}}{\vec{z} \sim e^{i\theta}\vec{z} \text{ for } \theta \in [0, 2\pi]}.$$

In particular we have a diagram

$$\begin{tikzcd} S^{2n+1} \arrow[r, hook, "j"] \arrow[d, "p"'] & \mathbb{C}^{n+1} \\ \mathbb{C}P^n \end{tikzcd} \tag{26}$$

where j is the inclusion and p is the quotient projection. Note that p is a submersion. Write a general element of $\mathbb{C}P^n$ as $[\vec{z}]$ where $\vec{z} \in S^{2n+1}$; the only non-uniqueness in this specification is that $[e^{i\theta}\vec{z}] = [\vec{z}]$ for all $e^{i\theta} \in S^1$.

For any $\vec{z} \in S^{2n+1}$, because p is a submersion (so $p_{*\vec{z}}$ is surjective) the kernel of $p_{*\vec{z}}$ will be one-dimensional. Since p is constant along the fiber $\{e^{i\theta}\vec{z} | e^{i\theta} \in S^1\}$, this one-dimensional kernel will then be equal to the tangent space at $\vec{z}$ to this fiber which (viewing tangent vectors to S^{2n+1} as vectors in $\mathbb{C}^{n+1}$) is spanned by the vector $i\vec{z}$.

Restrict the standard Hermitian metric $h_0 = g_0 + i\omega_0$ to $T_{\vec{z}}S^{2n+1} \subset \mathbb{C}^{n+1}$. The g_0-orthogonal complement of $\ker(p_{*\vec{z}}) = \text{span}\{i\vec{z}\}$ inside $T_{\vec{z}}S^{2n+1}$ is mapped isomorphically by $p_{*\vec{z}}$ to $T_{[\vec{z}]}\mathbb{C}P^n$. So we can define a skew-symmetric bilinear form $\omega_{[\vec{z}]}$ on $T_{[\vec{z}]}\mathbb{C}P^n$ by the requirement that

$$\text{For all } v, w \in \ker(p_{*\vec{z}})^{\perp_{g_0}} \subset T_{\vec{z}}S^{2n+1},\ \omega_{[\vec{z}]}(p_*v, p_*w) = \omega_0(v, w). \tag{27}$$

Proposition 7.4. *The above definition is independent of the choice of $\vec{z}$ such that $p(\vec{z}) = [\vec{z}]$,* i.e. *we have $\omega_{[e^{i\theta}\vec{z}]} = \omega_{[\vec{z}]}$ for all $e^{i\theta} \in S^1$.*

Proof. Let $R_\theta\colon S^{2n+1} \to S^{2n+1}$ be the map defined by $\vec{z} \mapsto e^{i\theta}\vec{z}$. By construction $p \circ R_\theta = p$, so the derivative $R_{\theta*}$ maps $\ker(p_{*\vec{z}})$ to $\ker(p_{*e^{i\theta}\vec{z}})$. Also $R_{\theta*}$ preserves the Hermitian inner product h_0, and hence preserves its real and imaginary parts g_0 and ω_0. Since R_θ preserves g_0, the fact that $R_{\theta*}$ maps $\ker(p_{*\vec{z}})$ to $\ker(p_{*e^{i\theta}\vec{z}})$ implies that it likewise maps $\ker(p_{*\vec{z}})^{\perp_{g_0}}$ to $\ker(p_{*e^{i\theta}\vec{z}})^{\perp_{g_0}}$. So using that $R_{\theta*}$ also preserves ω_0 and that the restriction of $p_{*e^{i\theta}\vec{z}}$ to $\ker(p_{*e^{i\theta}\vec{z}})^{\perp_{g_0}}$ is an isomorphism, we find that if $v, w \in \ker(p_{*\vec{z}})^{\perp_{g_0}}$ then $R_{\theta*}v, R_{\theta*}w$ are the unique elements of $\ker(p_{*e^{i\theta}\vec{z}})^{\perp_{g_0}}$ that map under p_* to p_*v and p_*w, and that $\omega_0(R_{\theta*}v, R_{\theta*}w) = \omega_0(v, w)$. Thus

$$\omega_{[\vec{z}]}(p_*v, p_*w) = \omega_0(v, w) = \omega_0(R_{\theta*}v, R_{\theta*}w) = \omega_{[e^{i\theta}\vec{z}]}(p_*v, p_*w).$$

Since p_* restricts surjectively to $\ker(p_{*\vec{z}})$ this proves that $\omega_{[\vec{z}]} = \omega_{[e^{i\theta}\vec{z}]}$. □

In view of Proposition 7.4 we see that (27) validly defines a two-form on $\mathbb{C}P^n$.

To say more about the properties of this two-form we now note that, for $\vec{z} \in S^{2n+1}$, the tangent space $T_{\vec{z}}S^{2n+1}$ is the g_0-orthogonal complement of the vector $\vec{z} \in S^{2n+1}$. Thus the subspace $\ker(p_{*\vec{z}})^{\perp_{g_0}}$ can be rewritten as $\{\vec{z}, i\vec{z}\}^{\perp_{g_0}}$ (now considered as a subspace of $\mathbb{C}^{n+1}$ rather than of $T_{\vec{z}}S^{2n+1}$). Since $\omega_0(\vec{z}, \cdot) = g_0(i\vec{z}, \cdot)$ and $\omega_0(i\vec{z}, \cdot) = -g_0(\vec{z}, \cdot)$ we could equivalently say that

$$\ker(p_{*\vec{z}})^{\perp_{g_0}} = \{\vec{z}, i\vec{z}\}^{\perp_{\omega_0}}. \tag{28}$$

Corollary 7.5. *The two-form ω on $\mathbb{C}P^n$ is closed, non-degenerate, and satisfies $p^*\omega = j^*\omega_0$ where p and j are as in (26).*

Proof. Since p_* projects $\ker(p_{*\vec{z}})^{\perp_{g_0}}$ isomorphically to $T_{[\vec{z}]}\mathbb{C}P^n$, by (27) the non-degeneracy of ω is equivalent to the non-degneracy of the restriction of ω_0 to each $\ker(p_{*\vec{z}})^{\perp_{g_0}}$. But this follows by (28), since $\mathrm{span}\{\vec{z}, i\vec{z}\}$ is a symplectic subspace of $(\mathbb{C}^{n+1}, \omega_0)$, whence so is its ω_0-orthogonal complement $\ker(p_{*\vec{z}})^{\perp_{g_0}}$.

We now prove the identity $p^*\omega = j^*\omega_0$ of two-forms on S^{2n+1}. A general element of $T_{\vec{z}}S^{2n+1}$ can be written as $v + ai\vec{z}$ where $v \in \ker(p_{*\vec{z}})$ and $a \in \mathbb{R}$, and we have $p_*(v + ai\vec{z}) = p_*v$. Given two such elements $v + ai\vec{z}$, $w + bi\vec{z}$ we find based on (27) that

$$(p^*\omega)_{\vec{z}}(v + ai\vec{z}, w + bi\vec{z}) = \omega_{[\vec{z}]}(v, w) = \omega_0(v, w) = \omega_0(v + ia\vec{z}, w + ib\vec{z}),$$

where the vanishing of the additional terms in the expansion of $\omega_0(v + ia\vec{z}, w + ib\vec{z})$ follows from (28) and skew-symmetry. So indeed $p^*\omega = j^*\omega_0$.

Since ω_0 is closed and d commutes with pullback, this implies that $p^*d\omega = 0$, *i.e.* that $d\omega$ vanishes on any triple of vectors p_*u, p_*v, p_*w. But p_* is surjective, so this means that $d\omega = 0$, proving that ω is closed. □

We have now shown that the Fubini-Study form ω from (27) is indeed a symplectic form on $\mathbb{C}P^n$. To make further contact with complex geometry let us show that the standard complex structure J on $\mathbb{C}P^n$ is compatible with ω. Of course this requires first saying what J is. As was mentioned in Section 3.2, a complex (as opposed to just almost complex) structure on a manifold M is obtained by specifying an atlas on that manifold having holomorphic transition functions, and then defining an endomorphism of TM by setting it equal to multiplication by i on $\mathbb{C}^n$ when expressed in terms of any of the charts in the atlas. (The holomorphicity of the transition functions implies that this definition is independent of the chart.) On $\mathbb{C}P^n$ one has an atlas given by the open sets

$$U_j = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n | z_j \neq 0\}$$

and corresponding charts

$$\phi_j([z_0:\cdots:z_n]) = \left(\frac{z_0}{z_j},\ldots,\frac{z_{j-1}}{z_j},\frac{z_{j+1}}{z_j},\ldots,\frac{z_n}{z_j}\right).$$

If $\pi\colon \mathbb{C}^{n+1}\setminus\{0\}$ is the quotient projection, evidently the map $\phi_j\circ\pi$ is holomorphic (on its domain of definition, namely $\pi^{-1}(U_j)=\{z_j\neq 0\}$); consequently the complex structure J on $\mathbb{C}P^n$ defined via the atlas has the property that $\pi_*J_0 = J\pi_*$.

Now let $\vec{z}\in S^{2n+1}\subset\mathbb{C}^{n+1}\setminus\{0\}$. Note that the standard complex structure J_0 on $\mathbb{C}^{n+1}$ maps $\ker(p_{*\vec{z}})^{\perp_{g_0}} = \{\vec{z}, i\vec{z}\}^{\perp_{g_0}}$ to itself, and that π_* coincides with $p_* = (\pi|_{S^{2n+1}})_*$ on this subspace. So the fact that $\pi_*J_0 = J\pi_*$ means that, for $v\in\ker(p_{*\vec{z}})^{\perp_{g_0}}$, we have $p_*J_0v = Jp_*v$. But then, again recalling (27), the fact that J_0 is compatible with ω_0 on $\ker(p_{*\vec{z}})^{\perp_{g_0}}$ (which follows from the fact that this is a complex subspace of $\mathbb{C}^{n+1}$) immediately implies that J is compatible with ω on $T_{[\vec{z}]}\mathbb{C}P^n$.

We sum the previous discussion up as follows:

Corollary 7.6. *The Fubini-Study form $\omega\in\Omega^2(\mathbb{C}P^n)$, defined by (27), is a symplectic form, and the standard complex structure J on $\mathbb{C}P^n$ is ω-compatible.*

Combined with Proposition 7.2, this shows that if a smooth manifold M is diffeomorphic to a smooth complex (with respect to the standard complex structure) submanifold of $\mathbb{C}P^n$ for some n, then M admits a symplectic form (namely the pullback of the Fubini-Study form by the diffeomorphism. Complex submanifolds of $\mathbb{C}P^n$ have been heavily studied since the 19th century; a typical way of making one is to choose homogeneous[31] polynomials $f_1,\ldots,f_k\in\mathbb{C}[z_0,\ldots,z_n]$ and let $Z=\{[z_0:\cdots:z_n]\in\mathbb{C}P^n | f_j(z_0,\ldots,z_n)=0 \text{ for } j=1,\ldots,k\}$. (The homogeneity of the polynomials implies that the condition for membership in Z is independent of which representative $(z_0,\ldots,z_n)$ of the equivalence class $[z_0:\cdots:z_n]$ one chooses.) Under an appropriate condition on the polynomials one can see from the implicit function theorem that this is a smooth submanifold of $\mathbb{C}P^n$, and similarly to Example 7.3 it is in this case a J-complex, and hence symplectic, submanifold. One notable difference between the symplectic and complex viewpoints is that if one varies the coefficients of the polynomials f_j, then as long as one avoids coefficients for which the attempted appeal to the implicit function theorem fails, the resulting submanifolds will all remain symplectomorphic by an application of the Moser trick (or, alternatively, of Proposition 7.12 below). On the other hand they will usually not be isomorphic as complex manifolds, reflecting the existence of sometimes-high-dimensional moduli spaces parametrizing isomorphism classes of complex manifolds even when topological data are fixed.

As mentioned at the end of Section 3.2 (with slightly different notational conventions), a **Kähler manifold** is a triple (M,ω,J) where (M,ω) is symplectic and J is an ω-compatible complex (not just almost complex) structure. In particular if M is a smooth complex submanifold of $\mathbb{C}P^n$ then $(M,\omega|_M,J|_{TM})$ is a Kähler manifold by Corollary 7.6 and Proposition 7.2, where now ω is the Fubini-Study form and J is the standard complex structure on TM. Not every Kähler manifold can be realized in this way; the easiest way of seeing this is that, because $H^2(\mathbb{C}P^n;\mathbb{R})$ is one-dimensional, if ω' is a symplectic form on M which is the pullback of ω by some embedding into $\mathbb{C}P^n$, the integrals of ω' over two-cycles in M would all be integer multiples of a common value; in a Kähler manifold having $b_2>1$ this often won't be the case—the simplest example is a product $\frac{\mathbb{C}}{\mathbb{Z}^2}\times\frac{C}{a\mathbb{Z}^2}$ for some irrational number a.

But this might just be because we chose the wrong form as our Kähler form ω', or perhaps the wrong complex structure J'. To account for this possibility, let's stop regarding them as fixed and fix only the compact smooth manifold M. We will say that a compact smooth manifold M is

[31] *i.e.* every monomial has the same total degree, so $z_0^2z_2+z_1^3$ is homogeneous, $z_0^2+z_1$ is not

projectivizable if it is diffeomorphic to a complex submanifold of $\mathbb{C}P^n$, and Kählerian if there are ω', J' on M such that (M, ω', J') is Kähler. As noted in the previous paragraph, projectivizable manifolds are Kählerian. The converse holds if $\dim_{\mathbb{R}} M = 2$ (because there are complex curves in $\mathbb{C}P^3$ of arbitrary genus) and if $\dim_{\mathbb{R}} M = 4$ (using the Kodaira classification of complex surfaces from the 1960's). However in all even real dimensions greater than or equal to 8, a construction of non-projectivizable Kählerian manifolds is given in [Vo04].

For some time in the early history of symplectic geometry it was thought that if (M, ω) is a compact symplectic manifold then M might have to be Kählerian. We'll soon explain why this is false, which obviously would require having a way of telling that some smooth manifold is not Kählerian. An obstruction is provided by Hodge theory, a topic that is beyond the scope of these notes; this theory yields a decomposition of the (complexified) de Rham cohomology groups of a Kähler manifold (M, ω, J) as

$$H^k(M; \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(M) \quad \text{where} \quad \dim H^{p,q}(M) = \dim H^{q,p}(M).$$

(Roughly speaking, $H^{p,q}(M)$ is spanned by classes of certain differential forms that can be written as combinations of expressions $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, and complex conjugation then interchanges $H^{p,q}(M)$ with $H^{q,p}(M)$.) In the case that k is odd every $H^{p,q}(M)$ has either $p > q$ or $p < q$, and these can be matched up with each other, and so one obtains that $b_k(M) = 2\sum_{p<q, p+q=k} \dim H^{p,q}(M)$. Thus the odd-index Betti numbers of any Kählerian manifold are even. So, contrapositively, if a compact manifold M has an odd-index Betti number that is odd, then M must not be Kählerian. In 1976, Thurston [Th76] gave an example of a compact symplectic four-manifold (M, ω) with $b_1(M) = 3$, which thus isn't Kählerian, using a general construction to which we'll turn next.

7.4. **Fiber bundles.** Recall that a smooth fiber bundle with fiber F over a base B consists of a smooth map $\pi \colon E \to B$ between two smooth manifolds such that B is covered by open sets U each having the property that $\pi^{-1}(U)$ is diffeomorphic to $U \times F$ by a diffeomorphism that identifies $\pi|_{\pi^{-1}(U)}$ with the projection $U \times F \to U$. I'll often denote the fiber $\pi^{-1}(\{b\})$ over a point $b \in B$ as F_b.

Proposition 7.7. *Suppose that $\pi \colon E \to B$ is a smooth fiber bundle and that $\Omega \in \Omega^2(E)$ is a two-form such that for each $b \in B$, $\Omega|_{F_b}$ is non-degenerate. Then for each $e \in E$ with $\pi(e) = b$, we have*

$$T_e E = T_e F_b \oplus T_e F_b^{\Omega}$$

where

$$T_e F_b^{\Omega} = \{v \in T_e E | (\forall w \in T_e F_b)(\Omega(v, w) = 0)\}.$$

Moreover the derivative π_* of π at e restricts to $T_e F_b^{\Omega}$ as an isomorphism to $T_b B$.

Proof. The non-degeneracy of $\Omega|_{F_b}$ implies that $T_e F_b \cap T_e F_b^{\Omega} = \{0\}$. So $\dim T_e E \geq \dim T_e F_b + \dim T_e F_b^{\Omega}$. On the other hand $T_e F_b^{\Omega}$ is the kernel of the map $T_e E \to (T_e F_b)^*$ defined by $v \mapsto \iota_v \omega|_{T_e F_b}$, and hence $\dim T_e F_b^{\Omega} \geq \dim T_e E - \dim T_e F_b$. These two inequalities together imply that $\dim T_e E = \dim T_e F_b + \dim T_e F_b^{\Omega}$, so since $T_e F_b \cap T_e F_b^{\Omega} = \{0\}$ we indeed have $T_e E = T_e F_b \oplus T_e F_b^{\Omega}$.

For the final statement just note that $T_e F_b = \ker(\pi_{*e})$ and π_{*e} surjects to $T_b B$, so the fact that $T_e F_b^{\Omega}$ is a complement to $T_e F_b$ implies that π_{*e} restricts to it as an isomorphism to $T_b B$. □

Thus if $\pi \colon E \to B$ and Ω are as in Proposition 7.7, at each $e \in E$ the full tangent space $T_e E$ splits up as the sum of a "vertical" tangent space $T_e F_b$ and a "horizontal" tangent space $T_e F_b^{\Omega}$. The vertical tangent space is specified just by the fiber bundle π; the additional information provided by Ω gives us a distinguished horizontal space (from among the many vector space complements to $T_e F_b$ in $T_e E$).

7.4.1. *Symplectic structures on fiber bundles.* The relatively simple observation leading to Thurston's example is the following:

Proposition 7.8. *Let $\pi\colon E \to B$ be a smooth fiber bundle with E (and hence also B) compact, and suppose that $\Omega \in \Omega^2(E)$ is a* ***closed*** *2-form such that, for each $b \in B$, $\Omega|_{\pi^{-1}(\{b\})}$ is non-degenerate. Suppose also that ω_B is a symplectic structure on B. Then for all sufficiently large real numbers K, the two-form $\Omega_K = \Omega + K\pi^*\omega_B$ is a symplectic structure on E.*

(The reason that we need to add $K\pi^*\omega_B$ is that we lack control over the behavior of Ω on the horizontal subspaces $T_eF_b^{\Omega}$, as will be reflected in the proof.)

Proof. Choose an ω_B-compatible almost complex structure J_B on B, and for any $b \in B$ and $v \in T_bB$ define $|v| = \sqrt{\omega(v, Jv)}$; by the definition of ω_B-compatibility this defines a norm on T_bB. If $\pi(e) = b$ and we write $F_b = \pi^{-1}(\{b\})$, then since (by Proposition 7.7) π_{*e} maps $T_eF_b^{\Omega}$ isomorphically to T_bB, the set of $v \in T_eF_b^{\Omega}$ such that $|\pi_*v| = 1$ is a sphere in $T_eF_b^{\Omega}$. Given this and the assumed compactness of E, the set

$$\{(e, v, w) | e \in E, v, w \in T_eF_b^{\Omega}, |\pi_*v| = |\pi_*w| = 1\}$$

is compact (in the subspace topology induced by $E \times TE \times TE$), and so there is some $C \geq 0$, independent of e, such that

$$|\Omega(v, w)| \leq C \text{ whenever } v, w \in T_eF_b^{\Omega},\ |\pi_*v| = |\pi_*w| = 1.$$

But again because π_* maps $T_eF_b^{\Omega}$ isomorphically to T_bB, arbitrary nonzero vectors $v, w \in T_eF_b^{\Omega}$ can be rescaled so that their images under π_* have norm one. Accounting for such rescalings we see that,

$$|\Omega(v, w)| \leq C|\pi_*v||\pi_*w| \text{ for all } v, w \in T_eF_b^{\Omega}, \tag{29}$$

again with C independent of e and b.

We shall show that the conclusion of the proposition holds for all $K > C$. Clearly Ω_K is closed, so we just need to show that, for all nonzero $v \in T_eE$, there is $w \in T_eE$ with $\Omega_K(v, w) \neq 0$.

Suppose first that $0 \neq v \in T_eF_b^{\Omega}$. Choose w to be the unique vector in $T_eF_b^{\Omega}$ with the property that $\pi_*w = J\pi_*v$. By the definition of ω_B-compatibility and of $|\cdot|$ we then have $|\pi_*w| = |\pi_*v|$, so $|\pi_*v||\pi_*w| = \omega_B(\pi_*, J\pi_*v) = \omega_B(\pi_*v, \pi_*w)$. Hence by (29),

$$\Omega_K(v, w) = \Omega(v, w) + K\pi_*\omega_B(v, w) \geq -C|\pi_*v|\pi_*w| + K\omega_B(\pi_*v, \pi_*w) = (K - C)|\pi_*v||\pi_*w| > 0$$

since we assume $K > C$ and since π_*v and π_*w are nonzero.

Having dispensed with the (harder) case that $v \in T_eF_b^{\Omega}$, we now suppose that $v \notin T_eF_b^{\Omega}$. Then by Proposition 7.7 we can write $v = v_1 + v_2$ where $0 \neq v_1 \in T_eF_b$ and $v_2 \in T_eF_b^{\Omega}$. But Ω is non-degenerate on F_b so we can find $w \in T_eF_b$ with $\Omega(v_1, w) \neq 0$. Of course $\Omega(v_2, w) = 0$ by definition of $T_eF_b^{\Omega}$. Moreover $\pi_*w = 0$. Hence (for any value of K) $\Omega_K(v, w) = \Omega(v_1, w) \neq 0$. $\square$

We can now explain Thurston's example. Begin with the torus $F = \mathbb{R}^2/\mathbb{Z}^2$, with symplectic form $\omega = dx \wedge dy$ induced from the standard symplectic form on $\mathbb{R}^2$. Define $\phi\colon F \to F$ by $\phi([x, y]) = [x + y, y]$, *i.e.* ϕ is the map induced on F by the linear map of $\mathbb{R}^2$ represented by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; this is a diffeomorphism with inverse $[x, y] \mapsto [x - y, y]$, and we have $\phi^*\omega = \omega$ since $d(x + y) \wedge dy = dx \wedge dy$. Now form the mapping torus

$$Y_\phi = \frac{\mathbb{R} \times F}{(t + 2\pi, [x, y]) \sim (t, \phi([x, y]))}.$$

Projection to the t coordinate gives a fiber bundle $\pi\colon Y_\phi \to S^1$, and because ϕ is a symplectomorphism there is a well-defined 2-form $\tilde{\omega}$ on Y_ϕ defined by the properties that $\iota_{\partial_t}\tilde{\omega} = 0$ and

$\tilde{\omega}$ restricts to each $\{t\} \times F$ coincides with ω. Said differently, $\tilde{\omega}$ is defined by the property that $p^*\tilde{\omega} = dx \wedge dy \in \Omega^2(\mathbb{R} \times F)$ where $p\colon \mathbb{R} \times F \to Y_\phi$ is the quotient projection. Since p is a submersion and $dx \wedge dy$ is closed it follows from this that $d\tilde{\omega}$ is closed.

Thurston's example is then $S^1 \times Y_\phi$. This is the total space of a fiber bundle $\Pi\colon S^1 \times Y_\phi \to S^1 \times S^1$ (given by the identity on the first factor and by π on the second), and if Ω is the pullback of $\tilde{\omega}$ by the projection $S^1 \times Y_\phi \to Y_\phi$ then Ω evidently satisfies the hypotheses of Proposition 7.8. Hence if we add to Ω a large multiple of $\Pi^*\omega_B$ where ω_B is any symplectic form on $S^1 \times S^1$ then the result will be a symplectic form on $S^1 \times Y_\phi$.

Exercise 7.9. *Show that $H_1(S^1 \times Y_\phi) \cong \mathbb{Z}^3$.*

By Exercise 7.9 and results from Hodge theory, $S^1 \times Y_\phi$ cannot be equipped with a Kähler structure, even though we have just shown that it has a symplectic structure (and hence an almost Kähler structure as discussed in Section 3.2). It turns out that $S^1 \times Y_\phi$ does admit a complex structure; in fact it is generally known as the "Kodaira–Thurston manifold," and Kodaira considered it before Thurston did in the context of the classification of complex surfaces. However this complex structure cannot be made compatible with a symplectic form, as again follows from the fact that $b_1(S^1 \times Y_\phi) = 3$. There also exist T^2-bundles over T^2 having $b_1 = 2$ that have a symplectic structure but do not have a complex structure, see [FGG88].

In the case of the Kodaira–Thurston manifold we were able to construct a two-form Ω for use in Proposition 7.8 rather directly. In other examples this may not be so easy, but the following gives a simple criterion as long as the fiber is two-dimensional.

Proposition 7.10. *Let $\pi\colon E \to B$ be a smooth fiber bundle whose fibers $F_b = \pi^{-1}(\{b\})$ are compact orientable surfaces and whose base B is connected. Suppose that $c \in H^2(E;\mathbb{R})$ such that $\langle c, [F_b]\rangle \neq 0$ where $[F_b]$ denotes the image of the fundamental class of F_b under the inclusion of F_b into E. Then there is a closed 2-form $\Omega \in \Omega^2(E)$ such that the de Rham cohomology class of Ω is equal to c and such that, for each $b \in B$, $\Omega|_{F_b}$ is non-degenerate.*

Remark 7.11. From the definition of a fiber bundle it's easy to check that, given $b_0 \in B$, the set of $b \in B$ with $[F_b] = \pm[F_{b_0}]$ is both open and closed, so the connectedness of B implies that $\langle c, [F_b]\rangle \neq 0$ for one b if and only if $\langle c, [F_b]\rangle \neq 0$ for all b. Also, by the universal coefficient theorem, if we just specify the fiber bundle $\pi\colon E \to B$ then the existence of a cohomology class c as in the proposition is equivalent to the class $[F_b]$ having infinite order in $H_2(E;\mathbb{Z})$.

Proof. Fix $b_0 \in B$ and let $F = \pi^{-1}(\{b_0\})$. Note that c determines an orientation on each fiber $F_b = \pi^{-1}(\{b\})$ by the requirement that the integral over F_b of a two-form representing c be positive rather than negative.

Cover B by open sets U_α, which (refining the cover if necessary) we assume to be contractible, such that for each α it holds that $E_\alpha := \pi^{-1}(U_\alpha)$ is diffeomorphic to $U_\alpha \times F$ by a diffeomorphism that identifies $\pi|_{E_\alpha}$ with the projection $U_\alpha \times F \to U_\alpha$; composing with an orientation-reversing diffeomorphism of F if necessary we may assume that these diffeomorphisms $E_\alpha \to U_\alpha \times F$ restrict to each F_b for $b \in U_\alpha$ as an orientation-preserving diffeomorphism $F_b \to \{b\} \times F$. (We are using that U_α is contractible—really just that is connected—so that if this condition holds for one $b \in U_\alpha$ then it holds for all $b \in U_\alpha$.)

Since $\langle c, [F]\rangle \neq 0$ there is a symplectic form ω on F whose class in $H^2(F;\mathbb{R})$ is equal to $c|_F$. Let Ω_α denote the pullback of ω to E_α via the composition of the fiber bundle trivialization $E_\alpha \cong U_\alpha \times F$ and the projection $U_\alpha \times F \to F$. Then for $b \in U_\alpha$ we have $\int_{F_b} \Omega_\alpha = \langle c, [F_b]\rangle$, so since the inclusion $F_b \hookrightarrow E_\alpha$ is a homotopy equivalence it follows that $[\Omega_\alpha] = c|_{E_\alpha}$. Clearly for all $b \in B$, $\Omega_\alpha|_{F_b}$ is a symplectic form

Choose any closed form $\Psi \in \Omega^2(E)$ representing c in de Rham cohomology. Then since $[\Omega_\alpha] = c|_{E_\alpha}$, we can write $\Psi|_{E_\alpha} = \Omega_\alpha + d\tau_\alpha$ for some $\tau_\alpha \in \Omega^1(E_\alpha)$. Let $\{\chi_\alpha\}$ be a (locally finite) partition of unity subordinate to the cover $\{U_\alpha\}$ of B; this readily implies that $\{\chi_\alpha \circ \pi\}$ is a locally finite partition of unity subordinate to the cover $\{E_\alpha\}$ of B. Now define

$$\Omega = \Psi - \sum_\alpha d\left((\chi_\alpha \circ \pi)\tau_\alpha\right)$$

(implicitly we are extending $(\chi_\alpha \circ \pi)\tau_\alpha$ by zero outside of E_α. Clearly Ω is a closed form representing the same de Rham cohomoloy class as Ψ, namely c. It remains to show that Ω is non-degenerate when restricted to any fiber. For this we first note that, if $v \in T_eF_b$ where $\pi(e) = b$ and $b \in U_\alpha$, then $d(\chi_\alpha \circ \pi)(v) = 0$ since $\pi_* v = 0$. So

$$d\left((\chi_\alpha \circ \pi)\tau_\alpha\right)|_{F_b} = (\chi_\alpha \circ \pi)d\tau_\alpha|_{F_b}$$

(as the other term in the Leibniz rule vanishes). So since $\sum_\alpha(\chi_\alpha \circ \pi) = 1$ we see that, for any $b \in B$,

$$\Omega_{F_b} = \sum_\alpha(\chi_\alpha \circ \pi)(\Psi - d\tau_\alpha)|_{F_b} = \sum_\alpha(\chi_\alpha \circ \pi)\Omega_\alpha|_{F_b}.$$

But the $\Omega_\alpha|_{F_b}$ are all volume forms on F_b that induce the same orientation, so the above convex combination of them is likewise a volume form on F_b that induces this orientation. So indeed Ω restricts non-degenerately to F_b for all $b \in B$. □

Thus fiber bundles with symplectic bases and fibers which are homologically essential surfaces admit symplectic forms. It is necessary to require the fibers to be homologically essential: consider $M = S^1 \times S^3$ and define $\pi\colon M \to S^2$ to be independent of the first factor, and then determined by the Hopf fibration $S^3 \to S^2$. This is clearly a T^2-bundle over S^2, but since it is diffeomorphic to $S^1 \times S^3$ which has $b_2 = 0$ it cannot be symplectic. Of course that $b_2 = 0$ also means the fibers cannot be homologically essential, and it's easy to see that they aren't since the Hopf circles in S^3 are contractible.

7.4.2. *Parallel transport.* If $\pi\colon E \to B$ is a smooth fiber bundle over a connected base B and if $b_0, b_1 \in B$ then one can show that the fibers F_{b_0}, F_{b_1} are diffeomorphic by combining local trivializations defined over a chain of open subsets of B that connect b_0 to b_1. A somewhat more geometric way of obtaining relationships between the different fibers is to use a *connection* $\mathscr{H}$ on the fiber bundle, which for our purposes we will define to be a smoothly-varying choice, for all $e \in E$, of subspace $H_e \le T_eE$ which is "horizontal" in the sense that $T_eE = T_eF_{\pi(e)} \oplus H_e$. As in Proposition 7.7, in this case for each $e \in E$ the derivative π_{*e} restricts to H_e as an isomorphism to $T_{\pi(e)}B$. Thus, for $b \in B$, a vector $v \in B$ has unique "horizontal lifts" $v^\# \in H_e$ for all $e \in F_b$ (namely $v\#$ is the unique vector in H_e with $\pi_{*e}v^\# = v$). Likewise a time-dependent vector field $\mathbb{V} = (V_t)$ on B has a horizontal lift to a time-dependent vector field $\mathbb{V}^\# = (V_t^\#)$ defined everywhere on e.

Given a smooth path $\gamma\colon [0, T] \to B$, subject to the usual caveats about existence of ODE solutions (which certainly cause no problems if E is compact), one can use a connection $\mathscr{H}$ as above to define a **parallel transport map** $P_\gamma^{\mathscr{H}}\colon F_{\gamma(0)} \to F_{\gamma(T)}$ as follows. Choose a time-dependent vector field $\mathbb{V} = (V_t)$ on B such that $V_t(\gamma(t)) = \gamma'(t)$ for all t, and set $P_\gamma^{\mathscr{H}} = \psi^{\mathbb{V}^\#,T}|_{F_{\gamma(0)}}$. To see that this indeed maps $F_{\gamma(0)}$ to $F_{\gamma(T)}$, just note that if $e \in F_{\gamma(0)}$ we have

$$\frac{d}{dt}\left(\pi \circ \psi^{\mathbb{V}^\#,t}(e)\right) = \pi_*\left(V_t^\#(\psi^{\mathbb{V}^\#,t}(e))\right) = V_t\left(\pi \circ \psi^{\mathbb{V}^\#,t}(e)\right),$$

so that $\eta(t) := \pi \circ \psi^{\mathbb{V}^\#,t}(e)$ is, like γ, a solution to $\frac{d\eta}{dt} = V_t(\eta(t))$ with $\eta(0) = \gamma(0)$, whence $\eta(t) = \gamma(t)$ for all t. So $\psi^{\mathbb{V}^\#,t}$ maps $F_{\gamma(0)}$ to $F_{\gamma(t)}$ for all t, and in particular $P_\gamma^{\mathscr{H}} = \psi^{\mathbb{V}^\#,T}$ maps $F_{\gamma(0)}$

to $F_{\gamma(T)}$. It is not hard to check that $P_{\gamma}^{\mathscr{H}}$ depends only on γ, not on the choice of time-dependent vector field $\mathbb{V}$ obeying $V_t(\gamma(t)) = \gamma'(t)$ for all t.

Assuming that the relevant ODE's have solutions so that P_{γ} is well-defined, it's easy to see that it is a diffeomorphism from $F_{\gamma(0)}$ to $F_{\gamma(T)}$: an inverse is given by $P_{\bar{\gamma}}$ where by definition $\bar{\gamma}(t) = \gamma(T-t)$.

The above discussion applies to a general connection on a smooth fiber bundle; Proposition 7.7 indicates a context where this becomes relevant to symplectic geometry. Namely, if $\Omega \in \Omega^2(E)$ has non-degenerate restriction to every fiber of $\pi\colon E \to B$, then we get a connection $\mathscr{H}$ by setting $H_e = T_e F_{\pi(e)}^{\Omega}$. This just follows from linear algebra, but if we furthermore assume the two-form Ω this leads to an important consequence for the geometry of parallel transport:

Proposition 7.12. *Let $\pi\colon E \to B$ be a smooth fiber bundle with $F_b = \pi^{-1}(\{b\})$ for each $b \in B$, and suppose that $\Omega \in \Omega^2(E)$ is a two-form such that $d\Omega = 0$ and $\Omega|_{F_b}$ is non-degenerate for all b. Define a connection $\mathscr{H}$ on e by $H_e = T_e F_{\pi(e)}^{\Omega}$, and suppose that $\gamma\colon [0,T] \to B$ is a smooth path such that the parallel transport $P_{\gamma}^{\mathscr{H}}$ is well-defined. Then $P_{\gamma}^{\mathscr{H}}\colon F_{\gamma(0)} \to F_{\gamma(T)}$ is a* ***symplectomorphism*** *between the symplectic manifolds $(F_{\gamma(0)}, \Omega|_{F_{\gamma(0)}})$ and $(F_{\gamma(T)}, \Omega|_{F_{\gamma(T)}})$.*

Proof. With notation as in the start of this subsection, if for each $b \in B$ we denote by $i_b\colon F_b \to E$ the inclusion of the fiber over b, we have $P_{\gamma}^{\mathscr{H}} = \psi^{\mathbb{V}^{\#},T} \circ i_{\gamma(0)}$. More generally the map $\psi^{\mathbb{V}^{\#},t} \circ i_{\gamma(0)}$ is a diffeomorphism from $F_{\gamma(0)}$ to $F_{\gamma(t)}$; for $t = 0$ this map is the identity. We find that

$$\begin{aligned} \frac{d}{dt}\left((\psi^{\mathbb{V}^{\#},t} \circ i_{\gamma(0)})^*\Omega\right) &= i_0^*\left(\frac{d}{dt}\psi^{\mathbb{V}^{\#},t*}\Omega\right) \\ i_0^*\psi^{\mathbb{V}^{\#},t*}\left(d\iota_{V_t^{\#}}\Omega + \iota_{V_t^{\#}} d\Omega\right) &= d\left((\psi^{\mathbb{V}^{\#},t} \circ i_{\gamma(0)})^*(\iota_{V_t^{\#}}\Omega)\right). \end{aligned}$$

(In the last equality we used that $d\Omega = 0$.) But the one-form $(\psi^{\mathbb{V}^{\#},t} \circ i_{\gamma(0)})^*(\iota_{V_t^{\#}}\Omega)$ on $F_{\gamma(0)}$ is zero: the image of any vector in $TF_{\gamma(0)}$ under $(\psi^{\mathbb{V}^{\#},t} \circ i_{\gamma(0)})_*$ is tangent to $F_{\gamma(t)}$, and $\iota_{V_t^{\#}}\Omega$ vanishes on such a vector because by the definition of our connection $V_t^{\#} \in TF_{\gamma(t)}^{\Omega}$.

Thus $\frac{d}{dt}\left((\psi^{\mathbb{V}^{\#},t} \circ i_{\gamma(0)})^*\Omega\right) = 0$, so since $\psi^{\mathbb{V}^{\#},t} \circ i_{\gamma(0)} = 1_{F_{\gamma(0)}}$ and $\psi^{\mathbb{V}^{\#},t} \circ i_{\gamma(T)}$ is our diffeomorphism $P_{\gamma}^{\mathscr{H}}\colon F_{\gamma(0)} \to F_{\gamma(T)}$ it follows that this diffeomorphism is a symplectomorphism with respect to the symplectic structures given by restricting Ω. □

Of course the symplectomorphism $P_{\gamma}^{\mathscr{H}}\colon F_{\gamma(0)} \to F_{\gamma(T)}$ depends on the path γ from $\gamma(0)$ to $\gamma(T)$. It should be fairly clear that (under the hypotheses of Proposition 7.12) a fixed-endpoint homotopy of paths γ yields a symplectic isotopy between the associated parallel transports. In fact one can show that this isotopy is a Hamiltonian isotopy (see [MS, Theorem 6.4.1]).

7.5. **Lefschetz fibrations and open books.** While Propositions 7.8 and 7.10 can be used to show that four-dimensional oriented fiber bundles over oriented surfaces carry symplectic forms, a much broader class of manifolds is covered by a weaker analogue of a fiber bundle called a Lefschetz fibration. In general if X is an oriented $2n$-dimensional smooth manifold and Σ is an oriented surface, a Lefschetz fibration $f : X \to \Sigma$ is smooth map having only finitely many critical points, such that for each critical point p there are orientation-preserving, complex coordinates around p and $f(p)$ in terms of which f is given by the map $(z_1, \ldots, z_n) \mapsto \sum_j z_j^2$. At least if the fibers are compact, the restriction of f to the preimage of the complement of the critical values will then be a smooth fiber bundle; however the fibers over critical values will be singular. If $n = 2$ the nature of the singularities in the critical fibers are fairly easy to understand: near the critical point the fiber looks like $\{z_1^2 + z_2^2 = 0\} \subset \mathbb{C}^2$, and the fact that $z_1^2 + z_2^2$ factors as $(z_1 + iz_2)(z_1 - iz_2)$ implies that a neighborhood of the critical point in its fiber is homeomorphic to the union of two-transversely intersecting complex coordinate lines (copies of $\mathbb{R}^2$) in $\mathbb{C}^2 \cong \mathbb{R}^4$.

A modest extension of the proofs of Propositions 7.8 and 7.10 (see [GS, Theorem 10.2.18]) shows that if X is a compact oriented four-manifold with a Lefschetz fibration $f: X \to \Sigma$ whose fibers have infinite order in $H_2(X;\mathbb{Z})$, then X admits symplectic forms that make the smooth fibers into symplectic manifolds.

7.5.1. *The model Lefschetz critical point.* Before discussing the more global theory of Lefschetz fibrations we should understand more about the behavior of a Lefschetz fibration near its critical points, so we consider the function

$$f: \mathbb{C}^n \to \mathbb{C}$$
$$(z_1, \ldots, z_n) \mapsto \sum_j z_j^2.$$

Note that for all $t \in \mathbb{C}^*$ the fiber F_t over t is diffeomorphic to F_1; a diffeomorphism $F_1 \to F_t$ is given by choosing $s \in \mathbb{C}^*$ such that $s^2 = t$ and sending $\vec{z} \to s\vec{z}$. It's easy to see from essentially this same argument that the restriction of f to $f^{-1}(\mathbb{C}^*)$ is a fiber bundle, even though the fibers are noncompact. (A local trivialization over a suitable subset $U \subset \mathbb{C}^*$ is given by letting $s: U \to \mathbb{C}^*$ be a branch of the square root function and sending $\vec{z} \in f^{-1}(U)$ to $(f(\vec{z}), s(f(\vec{z}))^{-1}\vec{z}) \in U \times F_1$.)

Moreover there is a two-form $\Omega \in \Omega^2(\mathbb{C}^n)$ whose restriction to the total space $f^{-1}(\mathbb{C}^*)$ of this fiber bundle is suitable for use in Proposition 7.12, namely the standard symplectic form $\Omega = \sum_j dx_j \wedge dy_j$ where we write $z_j = x_j + iy_j$. That Ω restricts non-degenerately to each $F_t = \{\sum_j z_j^2 = t\}$ for $t \in \mathbb{C}^*$ is an instance of Example 7.3. So the $(F_t, \Omega|_{F_t})$ are symplectic manifolds, and at least assuming that the relevant ODE solutions exist a path γ in $\mathbb{C}^*$ from t_0 to t_1 gives rise to a symplectomorphism $F_{t_0} \to F_{t_1}$.

We will see now that these fibers are symplectomorphic to a known manifold; given our conventions it turns out to be most convenient to do this for the fiber F_{-1}.

Proposition 7.13. *There is a symplectomorphism $\Phi: F_{-1} \to T^*S^{n-1}$, such that the preimage of the zero-section of T^*S^{n-1} is $\{\vec{z} \in F_{-1} | \mathrm{Re}(\vec{z}) = \vec{0}\}$.*

Proof. Write a general element of $\mathbb{C}^n$ as $\vec{z} = \vec{x} + i\vec{y}$ where $\vec{x}, \vec{y} \in \mathbb{R}^n$. We then have

$$f(\vec{z}) = \sum_j z_j^2 = \sum_j \Big((x_j^2 - y_j^2) + 2ix_jy_j \Big) = (\|\vec{x}\|^2 - \|\vec{y}\|^2) + 2i\vec{x} \cdot \vec{y}.$$

Thus

$$F_{-1} = f^{-1}(\{-1\}) = \left\{ \vec{x} + i\vec{y} \in \mathbb{C}^n \,\middle|\, \|\vec{x}\|^2 - \|\vec{y}\|^2 = -1,\, \vec{x} \cdot \vec{y} = 0 \right\}.$$

In particular any element of F_{-1} has nonzero imaginary part $\vec{y}$.

Now for $\vec{q} \in S^{n-1}$ the tangent space T_qS^{n-1} is the orthogonal complement of $\{\vec{q}\}$ in $\mathbb{R}^n$, and the linear functionals on $T_{\vec{q}}S^{n-1}$ are precisely those maps given by dot product with some element of $T_{\vec{q}}S^{n-1}$. This yields an identification

$$T^*S^{n-1} = \{(\vec{p}, \vec{q}) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, \|\vec{q}\| = 1,\, \vec{p} \cdot \vec{q} = 0\},$$

under which the canonical one-form λ_{can} becomes identified with the restriction to the above subset of the form $\sum_j p_j dq_j$ on $\mathbb{R}^n \times \mathbb{R}^n$. Using these identifications we now define the promised map $\Phi: F_{-1} \to T^*S^{n-1}$:

$$\Phi(\vec{x} + i\vec{y}) = \left(\|\vec{y}\|\vec{x}, \frac{\vec{y}}{\|\vec{y}\|} \right).$$

Since $\|\vec{y}\| \neq 0$ for $\vec{x} + i\vec{y} \in F_{-1}$ this map is well-defined. If $\vec{p} = \|\vec{y}\|\vec{x}$ where $\|\vec{x}\|^2 - \|\vec{y}\|^2 = -1$ then $\|\vec{y}\|$ can be recovered from $\|\vec{p}\|$ by the formula $\|\vec{y}\|^2 = \frac{1}{2}(1 + \sqrt{1 + 4\|\vec{p}\|^2})$; from this it is straightforward to construct an inverse for Φ, so Φ is a diffeomorphism. Clearly $\Phi(\vec{x} + i\vec{y})$ is in the zero section iff $\vec{x} = \vec{0}$, proving the last clause of the proposition. To show that Φ is a symplectomorphism we find $\Phi^*\lambda_{can}$ (and then take d of the result):

$$\begin{aligned}\Phi^*\lambda_{can} &= \sum_j \Phi^* p_j dq_j = \sum_j \|\vec{y}\| x_j d\left(\frac{y_j}{\|\vec{y}\|}\right) \\ &= \sum_j \frac{\|\vec{y}\|}{\|\vec{y}\|} x_j dy_j + \left(\sum_j x_j y_j\right)\|\vec{y}\| d\left(\frac{1}{\|\vec{y}\|}\right) = \sum_j x_j dy_j\end{aligned}$$

since elements of F_{-1} have $\sum_k x_j y_j = 0$. Thus $\Phi^* d\lambda_{can} = d\Phi^*\lambda_{can} = \sum_j dx_j \wedge dy_j$, and Φ is indeed a symplectomorphism. $\square$

Now let's consider the connection induced by Ω on $f^{-1}(\mathbb{C}^*)$ and its associated parallel transport maps, so we need to understand the horizontal subspaces at general points $\vec{z} = \vec{x} + i\vec{y} \in f^{-1}(\mathbb{C}^*)$. By definition these horizontal subspaces are the Ω-orthogonal complements to the tangent spaces to the fibers. These can be related to something more familiar by bringing into play the standard complex structure J_0 and the standard inner product g_0, using the identities

$$\Omega(\vec{v}, \vec{w}) = g_0(J_0\vec{v}, \vec{w}) = -g(\vec{v}, J_0\vec{w}).$$

The first equality shows that $\vec{v}$ lies in the Ω-orthogonal complement to the tangent space to the fiber iff $J_0\vec{v}$ lies in the g_0-orthogonal complement to the tangent space to the fiber. But as noted earlier the tangent spaces to the fibers are preserved by J_0, so the second equality shows that $J_0\vec{v}$ lies in the g_0-orthogonal complement to the tangent space to the fiber iff $\vec{v}$ does. In other words the horizontal subspaces are the orthogonal complements—with respect to the standard dot product—of the tangent spaces to the fibers of f.

But, as one learns (at least in simple cases) in multivariable calculus, the orthogonal complement to the tangent space to a regular level set of a smooth function $g = (g_1, \ldots, g_k)\colon \mathbb{R}^m \to \mathbb{R}^k$ is the span of the gradients $\nabla g_1, \ldots, \nabla g_k$ (one inclusion is easily verified from the directional derivative formula $dg_j(v) = (\nabla g_j)\cdot v$; the reverse inclusion follows from the implicit function theorem). So the horizontal subspaces for our connection on $f\colon f^{-1}(\mathbb{C}^*) \to \mathbb{C}^*$ are spanned by the gradients $\nabla(\mathrm{Re} f)$ and $\nabla(\mathrm{Im} f)$ of the real and imaginary parts $\|\vec{x}\|^2 - \|\vec{y}\|^2$ and $2\vec{x}\cdot\vec{y}$ of f. Evidently

$$\nabla(\mathrm{Re} f) = 2\sum_j (x_j \partial_{x_j} - y_j \partial_{y_j}), \qquad \nabla(\mathrm{Im} f) = 2\sum_j (y_j \partial_{x_j} + x_j \partial_{y_j});$$

in particular these are orthogonal (which should not be surprising; why?).

Radial parallel transport. We'll now consider the parallel transport maps associated to some simple paths in $\mathbb{C}^*$ starting at -1. One interesting question is what happens to the parallel transport as the fibers approach the singular fiber over 0, so for $\epsilon > 0$ consider the path $\gamma_\epsilon\colon [0, 1-\epsilon] \to \mathbb{C}^*$ given by $\gamma(t) = t - 1$ from -1 to $-\epsilon$, yielding a parallel transport map $P^{\mathcal{H}}_{\gamma_\epsilon}\colon F_{-1} \to F_{-\epsilon}$; ideally we'd like to say something about the limit of this as $\epsilon \to 0$. This map is given by following the horizontal lift $\partial_x^{\#}$ of the vector field ∂_x; by the above considerations (in particular the fact that $d(\mathrm{Im} f)(\nabla(\mathrm{Re} f)) = 0$) this will be a scalar function times $\nabla(\mathrm{Re} f)$, with the role of the scalar function being to normalize $f_*\partial_x^{\#}$ as ∂_x rather than some multiple thereof. The appropriate function is easily seen to be $\|\nabla(\mathrm{Re} f)\|^{-2} = \frac{1}{4(\|\vec{x}\|^2 + \|\vec{y}\|^2)}$. So the parallel transport along γ_ϵ is given by solving the differential equations

$$\dot{x}_j = \frac{x_j}{2(\|\vec{x}\|^2 + \|\vec{y}\|^2)}$$
$$\dot{y}_j = -\frac{y_j}{2(\|\vec{x}\|^2 + \|\vec{y}\|^2}. \tag{30}$$

on the time interval $[0, 1-\epsilon]$, with arbitrary initial conditions on $F_{-1} = \{\|x\|^2 - \|y\|^2 = -1, \vec{x}\cdot\vec{y} = 0\}$. We can verify that the above equations do have solutions over the desired time-interval by noting that

$$\left|\frac{d}{dt}(\|\vec{x}\|^2 + \|\vec{y}\|^2)\right| = |2x_j\dot{x}_j + 2y_j\dot{y}_j| = \left|\frac{\|\vec{x}\|^2 - \|\vec{y}\|^2}{\|\vec{x}\|^2 + \|\vec{y}\|^2}\right| \leq 1, \tag{31}$$

which both prevents solutions from diverging to ∞ (which ODE theory tells us is the only way existence might fail) and, given that points of F_{-1} have $\|\vec{x}\|^2 + \|\vec{y}\|^2 \geq 1$, prevents the denominators in (30) from reaching zero for $t < 1$.

Now consider how the solutions $\vec{x}(t)+i\vec{y}(t)$ to (30) behave for different kinds of initial conditions $\vec{x}_0 + i\vec{y}_0$. Evidently solutions of (30) have the properties that $\vec{x}(t)$ and $\vec{y}(t)$ stay on the same respective lines through the origin in $\mathbb{R}^n$, with $\vec{x}(t)$ going further from the origin as t increases and $\vec{y}(t)$ going closer to the origin (and if $\vec{x}_0$ or $\vec{y}_0$ is $\vec{0}$ then the same will hold for $\vec{x}(t)$ or $\vec{y}(t)$ for all t). By the calculation in (31), if $\vec{x}_0 \neq \vec{0}$ then we will have $\frac{d}{dt}(\|\vec{x}\|^2 + \|\vec{y}\|^2) > -1$, and so there will be no problem in continuing the solution to (30) to time $t = 1$, at which time $\vec{x}(t) + i\vec{y}(t)$ will lie at a nonzero point on the singular fiber F_0. (Note that the only point on $F_0 = \{\sum_j z_j^2 = 0\}$ having $\vec{x} = \vec{0}$ is the origin.)

On the other hand if $\vec{x}_0 = \vec{0}$ then $\vec{x}(t) = \vec{0}$ for all t and so the calculation in (31) together with the fact that in this case $\|\vec{y}_0\|^2 = 1$ shows that $\|\vec{x}(t)\|^2 + \|\vec{y}(t)\|^2 = 1 - t$. So the equation (30) would be ill-defined at $t = 1$; as $t \to 1^-$ the solution would however behave simply: it would just converge to the origin, *i.e.* to the singular point of F_0.

The upshot is that the limit as $\epsilon \to 0$ of parallel transport along the radial path γ_ϵ from -1 to $-\epsilon$ is a map $F_{-1} \to F_0$ which restricts to a diffeomorphism from $F_{-1} \setminus \{\vec{x} = \vec{0}\}$ to $F_0 \setminus \{\vec{0}\}$, but which contracts $F_{-1} \cap \{\vec{x} = \vec{0}\}$ to the origin. Note that $F_{-1} \cap \{\vec{x} = \vec{0}\}$ is a Lagrangian sphere in F_{-1}; indeed it is identified by Proposition 7.13 with the zero section of T^*S^{n-1}. This sphere (an analogue of which exists in a more general context) is called the *vanishing cycle* associated to the path $t \mapsto t-1$ from -1 to 0; the union of its images under the parallel transport maps $P^{\mathcal{H}}_{\gamma_\epsilon}$ (together with $\vec{0}$) is called the "Lefschetz thimble" of the vanishing cycle and is an n-dimensional Lagrangian disk (in this case $\{\vec{x} = \vec{0}, \vec{y} \leq 1\}$) in $\mathbb{C}^n$ with boundary on the vanishing cycle (which in turn is a Lagrangian sphere in F_{-1}).

Parallel transport around a circle. Next we consider the parallel transport map associated to a loop around the critical point; we'll use the unit circle, oriented counterclockwise and based at -1, so $\gamma(t) = -e^{it}$ for $t \in [0, 2\pi]$. The result of this will be a symplectomorphism $P^{\mathcal{H}}_\gamma$ from F_{-1} to itself (equivalently, by Proposition 7.13, a symplectomorphism from T^*S^{n-1} to itself).

Letting θ denote the usual polar coordinate, we will need to find the horizontal lift of ∂_θ at a general point $\vec{z}$ with $f(\vec{z}) = -e^{it}$ and $t \in [0, 2\pi]$ arbitrary. When $t = 0$ we have $\partial_\theta = -\partial_y$, so similar reasoning as in the previous case shows that

$$\begin{aligned}(\partial_\theta^\#)_{\vec z} &= \frac{-\nabla(\mathrm{Im} f)}{\|\nabla(\mathrm{Im} f)\|^2} = -\sum_j \frac{y_j\partial_{x_j} + x_j\partial_{y_j}}{2(\|\vec x\|^2 + \|\vec y\|^2)} \\ &= -\frac{i\vec z^*}{2\|\vec z\|^2} \quad (\text{if } f(\vec z) = -1)\end{aligned}$$

where we use the tautological identification of $T_{\vec z}\mathbb{C}^n$ with $\mathbb{C}^n$ and we use $*$ for complex conjugation (a bar would interfere with the vector symbol). More generally, we can use the fact that multiplication by $e^{it/2}$ gives a diffeomorphism from F_{-1} to $F_{-e^{it}}$ which (because it preserves Ω on $\mathbb{C}^n$, and hence maps orthogonal complements to orthogonal complements) maps $\partial_\theta^\#$ at an arbitrary vector $\vec w$ to $\partial_\theta^\#$ at $e^{it/2}\vec w$. So if $f(\vec z) = e^{it}$ then we can apply the above calculation with $\vec z$ replaced by $e^{-it/2}\vec z$ to see that

$$\begin{aligned}(\partial_\theta^\#)_{\vec z} &= e^{it/2}(\partial_\theta^\#)_{e^{-it/2}\vec z} = e^{it/2}\frac{-i(e^{-it/2}\vec z)^*}{2\|\vec z\|^2} \\ &= \frac{e^{it}\vec z^*}{2\|\vec z\|^2}. \quad \text{if } f(\vec z) = -e^{it}.\end{aligned}$$

So finding the map $P_\gamma^{\mathcal{H}}\colon F_{-1} \to F_{-1}$ amounts to solving the equation

$$\dot{\vec z}(t) = \frac{-ie^{it}\vec z(t)^*}{2\|\vec z(t)\|^2}$$

on the time interval $[0, 2\pi]$ with arbitrary initial conditions on F_{-1}. To approach this, let $\vec w(t) = e^{-it/2}\vec z(t)$. Then we find that $\vec w(t)$ satisfies an autonomous equation:

$$\dot{\vec w}(t) = -\frac{i}{2}e^{-it/2}\vec z(t) + e^{it/2}\frac{-ie^{it}\vec z^*(t)}{2\|\vec z\|^2} = -\frac{i}{2}\left(\vec w(t) + \frac{\vec w(t)^*}{\|\vec w(t)\|^2}\right). \tag{32}$$

Since $\vec z(t)$ is meant to lie in the fiber $F_{-e^{it}}$, we would expect $\vec w(t)$ to lie in the fiber F_{-1} for all t; this is true for $t = 0$ since we are taking the initial condition for $\vec z(t)$ to be on F_{-1}, and we see that

$$\frac{d}{dt}(f(\vec w(t))) = 2\sum_j w_j(t)\dot w_j(t) = -i\sum_j\left(w_j(t)^2 + \frac{|w_j(t)|^2}{\|\vec w(t)\|^2}\right) = -i\left(f(\vec w(t)) + 1\right).$$

But $f(\vec w(0)) = -1$, and (using the uniqueness theorem for ODE's) the only solution g to $\dot g = -i(g+1)$ having $g(0) = -1$ is $g(t) = -1$. So indeed $f(\vec w(t)) = \sum_j w_j(t)^2 = -1$ for all t.

Additionally, we claim that a solution $\vec w(t)$ to (32) having $\vec w(0) \in F_{-1}$ has the property that $\|\vec w(t)\|^2$ is independent of t. Indeed

$$\begin{aligned}\frac{d}{dt}\|\vec w(t)\|^2 &= 2\mathrm{Re}\left(\sum_j \bar w_j(t)\dot w_j(t)\right) = -\mathrm{Re}\left(i\sum_j \bar w_j(t)\left(w_j(t) + \frac{\bar w_j(t)}{\|\vec w(t)\|^2}\right)\right) \\ &= -\mathrm{Re}\left(i\left(\|\vec w(t)\|^2 + \frac{\overline{f(\vec w(t))}}{\|\vec w(t)\|^2}\right)\right) = 0\end{aligned}$$

since we have already shown that $f(\vec w(t)) = -1$ for all t, so the final expression above is the real part of an imaginary number.

The fact that $\|\vec w(t)\|^2$ is independent of t makes (32) simpler to solve; for any given initial condition $\vec w(0) \in F_{-1}$ we can replace $\|\vec w(t)\|^2$ in (32) by the constant $c := \|\vec w(0)\|^2$ and then (32) just becomes the linear equation

$$\dot{\vec{w}}(t) = -\frac{i}{2}\left(\vec{w}(t) + \frac{1}{c}\vec{w}(t)^*\right).$$

Note that $c \geq 1$ since $f(\vec{w}(0)) = \sum_j w_j(0)^2 = -1$, and $c = 1$ iff $\mathrm{Re}\vec{w}(0) = \vec{0}$.

Let us write $\vec{w}(t) = \vec{x}(t) + i\vec{y}(t)$ where $\vec{x}$ and $\vec{y}$ are real-valued; our equation is then

$$\dot{\vec{x}}(t) = \frac{1}{2}\left(1 - \frac{1}{c}\right)\vec{y}(t)$$
$$\dot{\vec{y}}(t) = \frac{1}{2}\left(1 + \frac{1}{c}\right)\vec{x}(t).$$

If $\vec{x}(0) = \vec{0}$, so that $c = 1$, the only solution to this system is the constant solution $\vec{w}(t) = \vec{w}(0)$. If $\vec{x}(0) \neq \vec{0}$, so that $c > 1$, then one finds the solution to be

$$\vec{x}(t) = \cos(at)\vec{x}(0) + \sqrt{\frac{c-1}{c+1}}\sin(at)\vec{y}(0)$$
$$\vec{y}(t) = -\sqrt{\frac{c+1}{c-1}}\sin(at)\vec{x}(0) + \cos(at)\vec{y}(0)$$

where

$$a = \frac{1}{2}\sqrt{1 - \frac{1}{c^2}}, \quad c = \|\vec{x}(0)\|^2 + \|\vec{y}(0)\|^2. \tag{33}$$

Recall that $\vec{x}(t)$ and $\vec{y}(t)$ are the real and imaginary parts of $\vec{w}(t) = e^{-it/2}\vec{z}(t)$; what we are ultimately trying to compute is the parallel transport map $F_{-1} \to F_{-1}$ that sends $\vec{z}(0)$ to $\vec{z}(2\pi)$, *i.e.* that sends $\vec{w}(0)$ to $-\vec{w}(2\pi)$. A formula for this map can be extracted from the above, but it is more informative to interpret this map as being defined on the cotangent bundle T^*S^{n-1} using the symplectomorphism $(\vec{x}, \vec{y}) \mapsto \left(\|\vec{y}\|\vec{x}, \frac{\vec{y}}{\|\vec{y}\|}\right)$ from Proposition 7.13. Writing elements of T^*S^{n-1} as $(\vec{p}, \vec{q})$ as in the proof of Proposition 7.13, under our symplectomorphism we have

$$\|\vec{p}\|^2 = \|\vec{x}\|^2\|\vec{y}\|^2 = \frac{1}{4}\left((\|\vec{x}\|^2 + \|\vec{y}\|^2) - (\|\vec{x}\|^2 - \|\vec{y}\|^2)\right) = \frac{1}{4}(c^2 - 1)$$

and so by (33) $a = \frac{1}{2}\frac{\|\vec{p}\|}{\sqrt{\|\vec{p}\|^2+1}}$. Making appropriate additional substitutions, one find that the parallel transport map $\vec{w}(0) \mapsto -\vec{w}(2\pi)$ is identified by the symplectomorphism Φ in the proof of Proposition 7.13 to the map $\Psi\colon T^*S^{n-1} \to T^*S^{n-1}$ defined by

$$\Psi(\vec{p}, \vec{q}) = \left(\cos\left(\pi\left(1 + \frac{\|\vec{p}\|}{\sqrt{\|\vec{p}\|^2+1}}\right)\right)\vec{p} + \sin\left(\pi\left(1 + \frac{\|\vec{p}\|}{\sqrt{\|\vec{p}\|^2+1}}\right)\right)\|\vec{p}\|\vec{q},\right.$$
$$\left.\cos\left(\pi\left(1 + \frac{\|\vec{p}\|}{\sqrt{\|\vec{p}\|^2+1}}\right)\right)\vec{q} - \sin\left(\pi\left(1 + \frac{\|\vec{p}\|}{\sqrt{\|\vec{p}\|^2+1}}\right)\right)\frac{\vec{p}}{\|\vec{p}\|}\right).$$

(For $\vec{p} = \vec{0}$ the right-hand side should be interpreted as $(\vec{0}, -\vec{q})$, since $\sin(\pi) = 0$ and $\cos(\pi) = -1$.)

To understand this map it is helpful to fix an arbitrary pair of orthogonal unit vectors $\vec{p}_0, \vec{q}_0 \in S^{n-1}$ and consider how $(\vec{p}(s), \vec{q}(s)) := \Psi(s\vec{p}_0, \vec{q}_0)$ depends on $s \in \mathbb{R}$. Evidently both $\vec{p}(s)$ and $\vec{q}(s)$ lie in the plane spanned by $\vec{p}_0$ and $\vec{q}_0$ for all s, and we always have $\vec{p}(s)\cdot\vec{q}(s) = 0$, $\|\vec{q}(s)\| = 1$, $\|\vec{p}(s)\| = |s|$. We find

$$\vec{q}(s) = \cos(g(s))\vec{q}_0 - \sin(g(s))\vec{p}_0 \quad \text{where } g(s) = \pi\left(1 + \frac{s}{\sqrt{s^2+1}}\right). \tag{34}$$

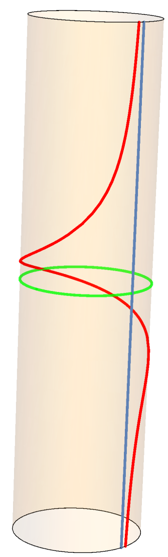

FIGURE 2. The action of Ψ on the cylinder in T^*S^{n-1} where $\vec{p},\vec{q}$ both lie in the plane spanned by $\vec{p}_0,\vec{q}_0$. The set $\{(s\vec{p}_0,\vec{q}_0)|s\in\mathbb{R}\}$ is shown in blue, its image in red, and the intersection of the cylinder in the zero section in green.

A similar formula can be given for $\vec{p}(s)$; more geometrically one can characterize it in terms of $\vec{p}_0$ and $\vec{q}(s)$ by the facts that it has norm $|s|$ and is orthogonal to $\vec{q}(s)$ in the plane spanned by $\vec{p}_0$ and $\vec{q}_0$, and that for $s\neq 0$ the orientation induced by $(\vec{p}(s),\vec{q}(s))$ is the same as that induced by $(s\vec{p}_0,\vec{q}_0)$ (in particular this orientation reverses when s crosses zero). Now the function g in (34) is increasing with $\lim_{s\to-\infty}g(s)=0$, $\lim_{s\to\infty}g(s)=2\pi$, and $g(0)=\pi$. Hence as s increases from very negative to very positive, the vector $\vec{q}(s)$ in the $\vec{p}_0\vec{q}_0$-plane makes almost one full rotation, with $\lim_{s\to\pm\infty}\vec{q}(s)=\vec{q}_0$, and with the orientation being counterclockwise with respect to the orientation of the plane given by taking $(\vec{p}_0,\vec{q}_0)$ as an oriented basis.

The locus of $(\vec{p},\vec{q})\in T^*S^{n-1}$ such that $\vec{p},\vec{q}$ both lie in the $\vec{p}_0\vec{q}_0$-plane forms a symplectic submanifold of T^*S^{n-1}, symplectomorphic to $\mathbb{R}\times S^1$ (or more naturally to the cotangent bundle of the unit circle in the $\vec{p}_q\vec{q}_0$-plane). Figure 2 depicts the action of Ψ on $\{(s\vec{p}_0,\vec{q}_0)|s\in\mathbb{R}\}$ under this symplectomorphism, illustrating that Ψ acts on $\mathbb{R}\times S^1$ via a positive[32] Dehn twist along the zero section $\{0\}\times S^1$.

As $\vec{p}_0,\vec{q}_0$ vary through pairs of orthogonal unit vectors, the description of $\vec{q}(s)$ in (34), together with the facts that $\vec{p}(s)$ is orthogonal to $\vec{q}(s)$ with norm $|s|$ and that $(\vec{p}(s),\vec{q}(s))$ spans the same oriented plane as $(s\vec{p}_0,\vec{q}_0)$, is enough to completely describe the diffeomorphism Ψ. A version Ψ_g of Ψ as in (34) for a general increasing function $g\colon\mathbb{R}\to[0,2\pi]$ with $g(s)\to 0$ as $s\to-\infty$, $g(s)\to 2\pi$

[32]Whether a Dehn twist along a curve in a surface is positive or negative is dictated by an orientation of the surface (an orientation of the curve is not required); in our case the orientation of the surface is determined by the symplectic form restricted from T^*S^{n-1}. A positive Dehn twist is characterized by the rule that if a neighborhood of the curve is identified with a standard cylinder in $\mathbb{R}^3$ oriented via an outward pointing unit normal, then a transversal arc to the curve should twist according to the right-hand rule, with thumb corresponding to the velocity of the arc and curled fingers corresponding to the direction of rotation.

as $s \to \infty$, and $g(0) = \pi$ is slightly more general than what is usually called a **model Dehn twist** on T^*S^{n-1}. To get the usual notion one should require that instead $g(s) = 2\pi$ for $s \gg 0$ and $g(s) = 0$ for $s \ll 0$ (rather than just converging to these values in the limit), which will result in Ψ_g being compactly supported; by a modification of the symplectic form Ω outside of a small neighborhood of the critical point one can arrange for the parallel transport around the unit circle to be given by such a Ψ_g with g supported in an arbitrarily small neighborhood of the zero section (cf. [Se03, Lemma 1.10]).

More globally, if $f : X \to \Sigma$ is a Lefschetz fibration and X is equipped with a symplectic form whose restrictions to neighborhoods of the critical points of f are of the type just described, and if t_0 is a critical value of f and t is a regular value that is close to t, then a path from t to t_0 determines a vanishing cycle for each critical point p in the singular fiber F_{t_0}; each such vanishing cycle is a Lagrangian sphere $L_p \subset F_t$ (identified in appropriate coordinates with $\{\mathrm{Re}\vec{z} = 0\} \subset \{\sum z_j^2 = -1\}$, or equivalently with the zero section in T^*S^{n-1}), and the parallel transport from F_t to itself associated to a loop going counterclockwise (with respect to the orientation on Σ) around t_0 will be a composition of positive Dehn twists around the various (disjoint) vanishing cycles L_p for critical points $p \in F_{t_0}$. Here by a positive Dehn twist around L_p we mean a map which is given by identifying a neighborhood of L_p in F_t with a neighborhood of the zero section in T^*S^{n-1} via Theorem 4.20, and applying a model Dehn twist with support in this neighborhood.

While this description required t to be close enough to t_0 to apply our earlier local description, this isn't really necessary in the sense that if t' is an arbitrary regular value of f we obtain a symplectomorphism $F_{t'} \to F_t$ by parallel transport along an arc connecting t' to t and missing the (finitely many) critical values. Since parallel transport maps compose in the obvious way we again get vanishing cycles in $F_{t'}$ associated to each critical point of f; these are Lagrangian spheres, determined by arcs connecting t' to the corresponding critical values.

7.5.2. *Open book decompositions and contact forms.* Suppose that F is a smooth manifold with boundary ∂F, and that $\phi\colon F \to F$ is a diffeomorphism that restricts to the identity on a neighborhood of ∂F. We can then form the mapping torus of ϕ:

$$Y_\phi = \frac{\mathbb{R} \times F}{(\theta + 2\pi, x) \sim (\theta, \phi(x))} = \frac{[0, 2\pi] \times F}{(2\pi, x) \sim (0, \phi(x))};$$

this is a $(\dim F + 1)$-dimensional smooth manifold with boundary equipped with a fiber bundle map $\pi\colon Y_\phi \to S^1$ (sending $[\theta, x]$ to $e^{i\theta}$); since ϕ restricts as the identity to ∂F we see that $\partial Y_\phi = S^1 \times \partial F$. To get a smooth manifold without boundary we can then form the **open book with monodromy** ϕ:

$$OB(Y, \phi) = Y_\phi \cup_\partial (D^2 \times \partial F)$$

(*i.e.* we glue Y_ϕ and $D^2 \times F$ along their common boundary ∂F; the fact that ϕ restricts as the identity to a neighborhood of ∂F—not just to ∂F—ensures that this has a natural smooth structure without requiring any noncanonical smoothings of corners). The **binding** of the open book is the codimension-2 submanifold $B := \{0\} \times \partial F \subset D^2 \subset \partial F \subset OB(Y, \phi)$. We then have a fiber bundle $\hat{\pi}\colon OB(F, \phi) \backslash B \to S^1$, which coincides with π on Y_ϕ and is given on the rest of $OB(Y, \phi) \backslash B$, namely $(D^2 \setminus \{0\}) \times \partial F$, by composing the projection to $D^2 \setminus \{0\}$ with the radial retraction $D^2 \setminus \{0\} \to S^1$. The **pages** of $OB(Y, \phi)$ are the closures of the fibers of $\hat{\pi}$; these form an S^1-parametrized family of copies of F whose interiors are embedded disjointly into $OB(Y, \phi)$ but all of whose boundaries are equal to the binding B.

Open books provide a flexible tool for constructing manifolds; indeed work of Alexander from the 1920's shows that any compact orientable smooth 3-manifold is diffeomorphic to $OB(F, \phi)$ for some orientation-preserving diffeomorphism ϕ of some compact orientable surface with nonempty boundary F, with ϕ equal to the identity on a neighborhood of ∂F.

Moving to the realm of symplectic and contact topology, let us now suppose that F carries a one-form λ such that (F,λ) is a Liouville domain, and that our diffeomorphism $\phi\colon F\to F$ is a symplectomorphism (still restricting to a neighborhood of ∂F as the identity). Note that (up to diffeomorphism) this suffices to account for all of the cases in Alexander's theorem. Indeed a compact oriented surface with nonempty boundary can always be realized as a Liouville (indeed Weinstein) domain by starting with the disk and successively attaching Weinstein one-handles. If (F,λ) is a two-dimensional Liouville domain and $\phi\colon F\to F$ is an orientation-preserving diffeomorphism restricting to the identity near the boundary, then for $\eta=\phi^*\lambda-\lambda$ the two-forms $\omega_t=d\lambda+td\eta=(1-t)d\lambda+t\phi^*d\lambda$ for $t\in[0,1]$ are each symplectic since $d\lambda$ and $\phi^*d\lambda$ induce the same orientation. We can then use the Moser trick to find a time-dependent vector field $\mathbb{V}=(V_t)$ such that $\psi^{\mathbb{V},t*}\omega_t=\omega_0$ for all t. Since F has boundary there would ordinarily be a concern about whether $\psi^{\mathbb{V},t}$ exists since solutions might run off of the boundary, but the construction in Section 4.2 has V_t equal to the unique solution to $\iota_{V_t}\omega_t=-\eta=\lambda-\phi^*\lambda$, and then the fact that ϕ is the identity near the boundary shows that each V_t will have compact support in the interior of F. So the resulting flow $\psi^{\mathbb{V},t}$ does exist for all t and is the identity near ∂F, and we have $\psi^{\mathbb{V},1*}\omega_1=\omega_0$, *i.e.* $(\phi\circ\psi^{\mathbb{V},1})^*d\lambda=d\lambda$. Thus our diffeomorphism ϕ is isotopic by an isotopy supported away from ∂F to a symplectomorphism of $(F,d\lambda)$. Such an isotopy is easily seen to not affect the diffeomorphism type of $OB(F,\phi)$, so Alexander's theorem in fact implies that any compact oriented three-manifold can be written as $OB(F,\phi)$ for some symplectomorphism ϕ of a Liouville domain (F,λ) which is the identity near ∂F.

Thurston and Winkelnkemper used this fact in combination with the following theorem to reprove Martinet's theorem that every orientable three-manifold admits a contact structure:

Theorem 7.14. *([TW75]) If (F,λ) is a Liouville domain and $\phi\colon F\to F$ is a symplectomorphism which is the identity near the boundary, then $OB(F,\phi)$ admits a contact form α. Moreover α can be chosen to have the properties that $d\alpha$ is non-degenerate on the interior of each page of $OB(F,\phi)$, and that α restricts to the binding $\{0\}\times\partial F$ as a constant positive multiple of the contact form $\lambda|_{\partial F}$.*

Proof. We start by putting a contact form on the mapping torus $Y_\phi=\frac{[0,2\pi]\times F}{(2\pi,x)\sim(0,\phi(x))}$. Choose a smooth function $\chi\colon\mathbb{R}\to[0,1]$ such that, for some $\epsilon>0$, $\chi(\theta)=0$ for $\theta<\epsilon$ and $\chi(\theta)=1$ for $\theta>2\pi-\epsilon$. Write $\eta=\phi^*\lambda-\lambda$ and observe that the one-form $\lambda+\chi(\theta)\eta$ on $[0,2\pi]\times F$ descends via the quotient projection to a one-form β_0 on Y_ϕ. This is perhaps best seen by regarding Y_ϕ the quotient space obtained from $(-\epsilon,2\pi+\epsilon)\times F$ by gluing $(2\pi-\epsilon,2\pi+\epsilon)\times F$ to $(-\epsilon,\epsilon)\times F$ by the map $\Phi(\theta,x)=(\theta-2\pi,\phi(x))$. Since $\lambda+\chi(\theta)\eta$ is equal to $\lambda+\eta=\phi^*\lambda$ on $(2\pi-\epsilon,2\pi+\epsilon)\times F$ and to λ on $(-\epsilon,\epsilon)\times F$ we see that Φ pulls back $\lambda+\chi(\theta)\eta$ to itself, which suffices to show that $\lambda+\chi(\theta)\eta$ descends to the quotient.

Now β_0 is probably not a contact form on Y_ϕ, but we claim that, for $K\gg 0$,

$$\beta_K=\beta_0+Kd\theta$$

is a contact form. To see this, note that $d\beta_K=d\beta_0=d\lambda+\chi'(\theta)d\theta\wedge\eta$ (we have used that $d\eta=d(\phi^*\lambda-\lambda)=0$, ϕ being a symplectomorphism) and so, taking the dimension of F to be $2n-2$ so that $\dim Y_\phi=2n-1$,

$$\frac{1}{K}\beta_K\wedge(d\beta_K)^{n-1}=\frac{1}{K}\beta_0\wedge(d\beta_0)^{n-1}+d\theta\wedge(d\lambda)^{n-1}.$$

But $d\theta\wedge(d\lambda)^{n-1}$ is nowhere zero since $d\lambda$ is non-degenerate on the tangent spaces to the fibers of $\pi\colon Y_\phi\to S^1$, and at each point of Y_ϕ the tangent space to the fiber is the kernel of $d\theta$. Hence since

Y_ϕ is compact, $\frac{1}{K}\beta_0\wedge(d\beta_0)^{n-1}+d\theta\wedge(d\lambda)^{n-1}$ will still be nowhere zero if K is taken sufficiently large. This shows that, for $K\gg 0$, $\beta_K\wedge(d\beta_K)^{n-1}$ is nowhere zero, *i.e.* that $\beta_K\in\Omega^1(Y_\phi)$ is a contact form. Note that the restriction of $d\beta_K$ to each fiber $\{\theta\}\times F$ is just the symplectic form $d\lambda$, consistently with the requirements of the last sentence of the theorem.

It remains to extend β_K over the rest of $OB(F,\phi)$, *i.e.* to $D^2\times\partial F$. Let $\alpha_0=\lambda|_{\partial F}$. Using Proposition 6.28, we can identify a neighborhood U of ∂F in F with the set $(1-\delta,1]\times\partial F$ in such a way that $\lambda|_U$ is identified with the one-form $s\alpha_0$, s being the coordinate on $(1-\delta,1]$ (this s corresponds to e^t in Proposition 6.28). Shrinking δ if necessary we may assume that $\phi|_U=1_U$, in which case $S^1\times U$ will be a neighborhood of $S^1\times\partial F=\partial Y_\phi$ in Y_ϕ which is identified with $S^1\times(1-\delta,1]\times\partial F$, and on which the contact form β_K is given by $s\alpha_0+Kd\theta$. Letting D' be the open disk in $\mathbb{C}$ of radius $\sqrt{1+\delta}$, we can then regard $OB(F,\phi)$ as formed from the disjoint union of Y_ϕ and $D'\times\partial F$ by gluing $S^1\times U\cong S^1\times(1-\delta,1]\times\partial F$ to $\{z\in D'||z|\geq 1\}\times\partial F$ by the diffeomorphism $(e^{i\theta},s,x)\mapsto(\sqrt{2-s}e^{i\theta},x)$. This diffeomorphism identfies the function $\rho=\frac{|z|^2}{2}$ on the D^2 factor of $D^2\times F$ with the function $1-\frac{s}{2}$ on the $(1-\delta,1]$ factor of $S^1\times(1-\delta,1]\times\partial F$, and so it identifies our contact form $\beta_K|_{S^1\times U}$ with the one-form $(2-2\rho)\alpha_0+Kd\theta\in\Omega^1((D'\setminus\mathrm{int}(D^2))\times\partial F)$.

So we shall extend the one-form $(2-2\rho)\alpha_0+Kd\theta$ defined near $(\partial D^2)\times\partial F$ to a contact form α' on all of $D^2\times F$, in such a way that $\alpha'|_{\{0\}\times\partial F}$ is a positive constant multiple of α_0 and $d\alpha'$ is nondegenerate on $\{re^{i\theta_0}|0<r\leq 1\}\times\partial F$ for each fixed θ_0 (for these sets are the parts of the interiors of the pages that lie in $D^2\times\partial F$). The form α promised in the statement of the theorem will then be the one that is equal to α' on $D^2\times\partial F$ and to β_K on Y_ϕ.

As an ansatz for α' we use

$$\alpha'=f_1(\rho)\alpha_0+Kf_2(\rho)d\theta \tag{35}$$

for functions $f_1,f_2\colon[0,1/2]\to\mathbb{R}$ to be determined,[33] and equal respectively to $2-2\rho$ and 1 for ρ near $1/2$. Recall that although the one-form $d\theta$ is not defined at the origin of D^2, the one-form $\rho d\theta=xdy-ydx$ extends smoothly as zero over the origin of D^2. Accordingly for ρ near zero we intend to take $f_2(\rho)=\rho$ and $f_1(\rho)>0$ (the latter ensures that the last clause of the theorem will hold).

For any fixed θ_0 we have

$$d\alpha'|_{\{\theta=\theta_0\}}=\left(f_1(\rho)d\alpha_0+f_1'(\rho)d\rho\wedge\alpha_0\right)\big|_{\{\theta=\theta_0\}}.$$

Since (away from $\{\rho=0\}$) $d\alpha_0$ vanishes on the Reeb field R_{α_0} of α_0 and on ∂_ρ and is non-degenerate on $\ker(\alpha_0)$, while $d\rho\wedge\alpha_0$ is non-degenerate on $\mathrm{span}\{R_{\alpha_0},\partial_\rho\}$ and vanishes on $\ker(\alpha_0)$, we thus see that $d\alpha'$ is non-degenerate on each $\{re^{i\theta_0}|0<r\leq 1\}\times\partial F$ iff $f_1(\rho)$ and $f_1'(\rho)$ are both nonzero for all $\rho>0$.

We still need to determine how to ensure that α' as in (35) is a contact form. Since $d\alpha'=f_1(\rho)d\alpha_0+f_1'(\rho)d\rho\wedge\alpha_0+Kf_2'(\rho)d\rho\wedge d\theta$ (where $(d\alpha_0)^{n-1}=0$ since $\dim\partial F=2n-3$) we find

$$(d\alpha')^{n-1}=(n-1)f_1(\rho)^{n-2}d\rho\wedge(f_1'(\rho)\alpha_0+Kf_2'(\rho)d\theta)\wedge(d\alpha_0)^{n-2},$$

from which one obtains

$$\alpha'\wedge(d\alpha')^{n-1}=K(n-1)f_1(\rho)^{n-2}(f_1(\rho)f_2'(\rho)-f_2(\rho)f_1'(\rho))d\rho\wedge d\theta\wedge\alpha_0\wedge(d\alpha_0)^{n-2}.$$

Since $d\rho\wedge d\theta$ is the standard area form on the disk and α_0 is a contact form on ∂F, we conclude that α' is a contact form provided that $f_1(\rho)$ and $f_1(\rho)f_2'(\rho)-f_2(\rho)f_1'(\rho)$ are both nonzero for all $\rho\in[0,1/2]$.

Summing up, α' will satisfy all of the required properties provided that:

[33] recall that $\rho=|z|^2/2$, so $\rho=1/2$ on ∂D^2

- For ρ near $\frac{1}{2}$, $f_1(\rho) = 2-2\rho$ and $f_2(\rho) = 1$;
- For ρ near 0, $f_1(\rho) > 0$ and $f_2(\rho) = \rho$;
- For all ρ, $f_1(\rho)$ and $f_1'(\rho)$ are both nonzero; and
- For all ρ, $f_1(\rho)f_2'(\rho) - f_2(\rho)f_1'(\rho) \neq 0$.

There are many pairs of smooth functions $f_1, f_2\colon [0,1/2] \to \mathbb{R}$ obeying these properties; for instance we could take $f_1(\rho) = 2-2\rho$ for all ρ and have f_2 be any monotone function such that $f(\rho) = \rho$ for ρ near 0 and $f(\rho) = 1$ for ρ near 1. $\square$

A very important converse to Theorem 7.14 was established early this century by Giroux [Gi02, Théorème 10]: If ξ is any cooriented contact structure on a compact manifold V then V can be presented as an open book $OB(F, \phi)$ such that, for some positive contact form α for ξ, $d\alpha$ restricts to the interior of each page as a symplectic form and α restricts to the binding as a contact form. This has been especially useful in when $\dim V = 3$, as it allows many questions about three-dimensional contact topology to be reduced to questions about maps of surfaces.

7.5.3. *Open books as boundaries of Lefschetz fibrations.* Let us now suppose that X is a compact oriented smooth manifold with boundary; there is then a notion of Lefschetz fibration $f : X \to \Sigma$ where Σ is compact oriented surface with boundary; we will only consider the case that $\Sigma = D^2$. The only point to be added to the definition given earlier is that there should be no critical points for f on ∂X. As in the boundaryless case it is a fact that if $f : X \to D^2$ is a Lefschetz fibration then there exist symplectic structures on X restricting symplectically to each regular fiber provided that these fibers represent infinite-order elements of the relative homology $H_2(X, \partial X)$. (The fibers F themselves may or may not have nonempty boundary; in any case there is a fundamental class $[F] \in H_2(F, \partial F) \cong \mathbb{Z}$ and the class we are asking to have infinite order is the image of this class under the inclusion-induced map $H_2(F, \partial F) \to H_2(X, \partial X)$.) The symplectic form can moreover be taken to restrict near the critical points to the form used in Section 7.5.1, making relevant the analysis of parallel transport that was carried out therein.

If $f : X \to D^2$ is a Lefschetz fibration with a symplectic form Ω as above, then any point of $f^{-1}(\partial D^2)$ will lie on the boundary of X (as is easy to see from the fact that f_* is surjective at any such point). In the case that the fibers have no boundary, we will have $\partial X = f^{-1}(\partial D^2)$; if the fibers have nonempty boundary then points on the boundary of some fiber will also be boundary points of X and so we will have a decomposition of ∂X into a "vertical boundary" $f^{-1}(\partial D^2)$ and a "horizontal boundary" which is the union of the boundaries of the fibers.

Let $p_1, \ldots, p_k$ denote the critical values of f, all of which lie in the interior of D^2. If γ is any smooth path in $D^2 \setminus \{p_1, \ldots, p_k\}$ the symplectic form Ω induces a parallel transport map between the fibers[34] of the endpoints of γ. In particular this applies to the path γ that goes counterclockwise around ∂D^2, inducing a symplectomorphism $\phi\colon F_1 \to F_1$ where $F_1 = f^{-1}(\{1\})$. By a further modification of Ω near the intersection of the vertical and horizontal boundaries one can arrange for ϕ to be the identity on a neighborhood of ∂F_1.

By using the parallel transport along the counterclockwise arc from 1 to $e^{i\theta}$ to identify the fibers F_1 and $F_{e^{i\theta}}$ it is straightforward to check that one has a diffeomorphism $f^{-1}(\partial D^2) \cong Y_\phi$ where Y_ϕ is the mapping torus of ϕ. In particular the boundary of a Lefschetz fibration over the disk (or indeed over any compact surface with connected boundary) whose fibers do not have boundary is the mapping torus of the parallel transport (also called the monodromy) around the boundary. To say what this parallel transport is, as in Figure 3 draw arcs η_j from 1 to each critical value $p_j \in \mathrm{int}(D^2)$

[34] For this to work in the case that the fibers have nonempty boundary one needs to modify Ω near the horizontal boundary so that the Ω-orthogonal complements to the tangent spaces to the fibers are tangent to the horizontal boundary, which is always possible.

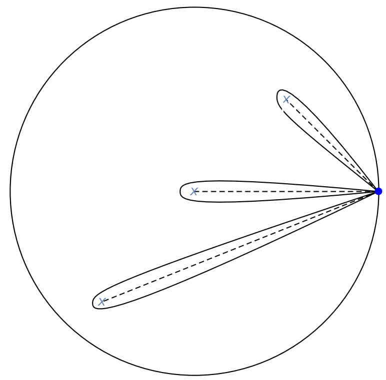

FIGURE 3. The parallel transport along the boundary for a Lefschetz fibration over D^2 is, up to isotopy, the composition of the parallel transport around small loops encircling the critical values, which in turn are positive Dehn twists around vanishing cycles.

such that η_j remains in $D^2 \setminus \{p_1, \ldots, p_k\}$ except at its endpoint, and for each j let γ_j be a loop based at 1 that follows along η_j as indicated in the figure, traveling once counterclockwise around p_j. The arc η_j determines a vanishing cycle in F_1; this is a Lagrangian sphere in F_1 characterized by the fact that parallel transport along η_j collapses it to the critical point in the fiber over p_j. The parallel transport along the loop γ_j will then be a positive Dehn twist around this vanishing cycle. (The above implicitly assumes that there is only one critical point in $f^{-1}(\{p_j\})$; if there are more than one then we'll get a disjoint union of vanishing cycles, one for each critical point, and the parallel transport along γ_j will be the composition of these.)

Traversing all of the γ_j in appropriate order yields a loop based at 1 which is homotopic, in $D^2 \setminus \{p_1, \ldots, p_k\}$, to ∂D^2 oriented counterclockwise. This homotopy induces a Hamiltonian isotopy between our parallel transport map ϕ around ∂D^2 and the composition (in a certain order) of the parallel transports around the γ_j, which themselves are given by positive Dehn twists around the vanishing cycles. In other words, up to Hamiltonian isotopy, the parallel transport around the boundary for a Lefschetz fibration over D^2 is given by composing the positive Dehn twists around the vanishing cycles of the Lefschetz fibration.

If the fibers have no boundary then this completes the description of ∂X: it is the mapping torus of a product of positive Dehn twists in Lagrangian spheres in the fiber F_1 over 1. If the fibers do have boundary then we have only described the vertical boundary as a mapping torus Y_ϕ; we need to add in the horizontal boundary. But as noted already, we can arrange for ϕ to be the identity near ∂F_1, so that a neighborhood of ∂Y_ϕ in Y_ϕ is naturally identified with $S^1 \times V$ where V is a neighborhoof of ∂F_1 in F_1. Now ∂Y_ϕ is the intersection of the vertical and horizontal boundaries, and (because all critical points are in the interior of X) the restriction of F to the horizontal boundary is a fiber bundle over ∂D^2 with fiber ∂F_1; since D^2 is contractible any such bundle can be trivialized as $D^2 \times \partial F_1$. In other words, the decomposition of ∂X into its vertical and horizontal parts amounts to writing ∂X as a union $Y_\phi \cup_\partial (D^2 \times \partial F_1)$, *i.e.* to identifying ∂X as $OB(F_1, \phi)$, where again ϕ is a product of positive Dehn twists around vanishing cycles. By pasting together local constructions this can be reversed, in the sense that if we have an open book $OB(F, \phi)$ where ϕ is a product of positive Dehn twists then we can construct a Lefschetz fibration over D^2 with F equal to the fiber over 1 and with $\partial X = OB(F, \phi)$.

Fairly recently [GP17], it has been shown that, up to deformation, every Weinstein domain X admits the structure of a Lefschetz fibration over D^2; this complements Giroux's earlier result which implies that the boundary of X supports an open book.

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